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1

Logarithm and Its Properties

EXPONENTIAL FUNCTION

If a is any number such that $a > 0$ and $a \neq 1$ then an exponential function is a function in the form $f(x) = a^x$, where a is called the base and x can be any real number. e.g., $f(x) = 2^x$, $g(x) = (4/7)^x$ are exponential functions.

There is a big difference between an exponential function and a polynomial. The function $g(x) = x^3$ is a polynomial. Here, the variable, x , is being raised to some constant power. The function $f(x) = 3^x$ is an exponential function; the variable is the exponent.

Here, the restriction on the base a is that it is not 0 or 1 as $f(x) = 0^x = 0$ and $f(x) = 1^x = 1$ and hence, these are constant functions and won't have many of the same properties that general exponential functions have. Also, we avoid negative numbers as base so that we don't get any complex values out of the function evaluation. For instance, if we allowed $a = -4$ the function would be, $f(x) = (-4)^x$, then we have $f(0.5) = (-4)^{0.5}$ which is a complex number. We only want real numbers to arise from function evaluation and so to make sure of this we require that a is not a negative number.

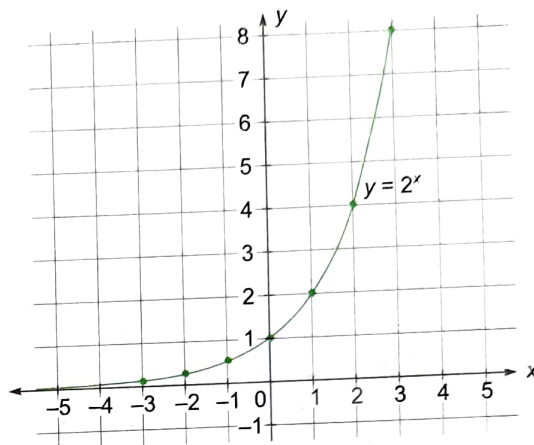
GRAPH OF EXPONENTIAL FUNCTION AND ITS PROPERTIES

Consider exponential function $y = f(x) = 2^x$. Here $f(0) = 1$, and on increasing the value of x from $x = 0$ onwards value of y increases, i.e., $f(1) = 2$, $f(2) = 4$, $f(10) = 1024$ When x approaches to infinity, 2^x approaches to infinity. Thus $f(x)$ is an increasing function. Now, consider some negative values of x , i.e., $f(-1) = 0.5$, $f(-2) = 0.25$, $f(-3) = 0.125$ Thus, graph of the function gets closer to x -axis, but never touches or crosses it, as $2^x > 0, \forall x \in R$. When x approaches to negative infinity, 2^x approaches to 0. This means that there is a horizontal asymptote at the x -axis or $y = 0$.

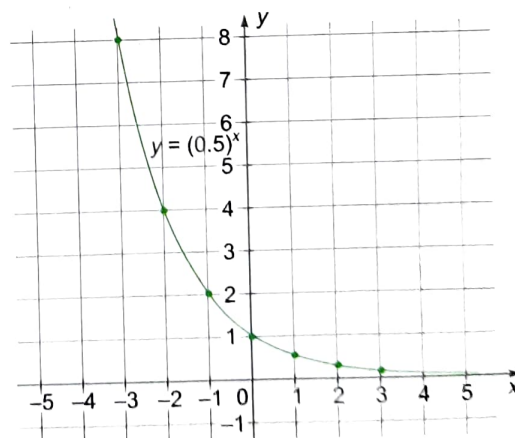
From the above discussion, the graph of $y = f(x) = 2^x$ can be plotted as shown here.

From the graph, we can see that $f(x) = 2^x$ is an increasing function in its domain R . Also, the graph always remains above x -axis, i.e., $2^x > 0$. Thus, range of the function is $(0, \infty)$. Similarly, we can draw the graph of $y = f(x) = 3^x$ passing through the point $(0, 1)$ and having the same nature as the graph of $f(x) = 2^x$. In fact, this pattern of the graph is common for $f(x) = a^x$ for any base which is greater than 1 ($a > 1$). Also, all graphs pass through

the point $(0, 1)$ as $a^0 = 1$ for any value of a . For $a > 1$, when $x_2 > x_1$, we have $a^{x_2} > a^{x_1}$



Now, consider the function $y = f(x) = (0.5)^x$. Here, $f(0) = 1$, and on increasing the value of x from $x = 0$ onwards value of y decreases as $f(1) = 0.5$, $f(2) = 0.25$, When x approaches to infinity, 2^x approaches to zero but never becomes exactly zero. Thus, $f(x)$ is decreasing function. Also, $f(-1) = 2$, $f(-2) = 4$, $f(-3) = 8$ and so on. Thus, value of 2^x increases as the value of x becomes more and more negative. The graph of the function can be plotted as shown here.



From the graph, we can see that $y = f(x) = (0.5)^x$ is an decreasing function in its domain. In fact, this pattern of the graph is common for $f(x) = a^x$ for any base which is less than 1 ($0 < a < 1$). Also, all graphs pass through the point $(0, 1)$ as $a^0 = 1$ for any value of a . For $0 < a < 1$, when $x_2 > x_1$, we have $a^{x_1} > a^{x_2}$.

GRAPH OF EXPONENTIAL FUNCTION FOR DIFFERENT BASES

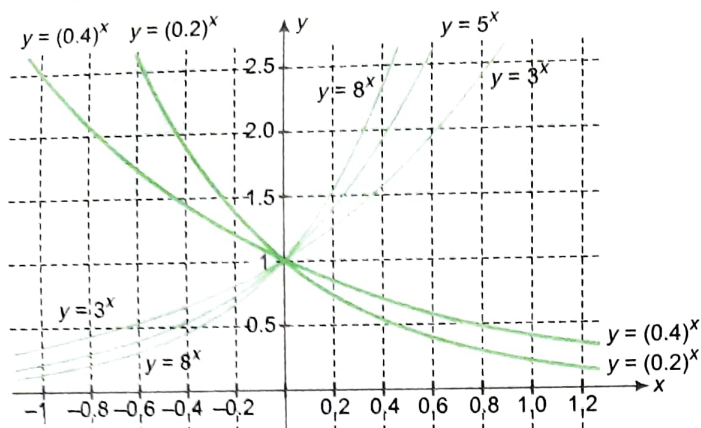


ILLUSTRATION 1.1

Solve for x : $4^x - 3^{x-\frac{1}{2}} = 3^{x+\frac{1}{2}} - 2^{2x-1}$.

Sol. $4^x - 3^{x-\frac{1}{2}} = 3^{x+\frac{1}{2}} - 2^{2x-1}$

or $4^x - \frac{3^x}{\sqrt{3}} = 3^x \sqrt{3} - \frac{4^x}{2}$

or $\frac{3}{2} \times 4^x = 3^x \left(\sqrt{3} + \frac{1}{\sqrt{3}} \right)$

or $\frac{3}{2} \times 4^x = 3^x \times \frac{4}{\sqrt{3}}$

or $\frac{4^{x-1}}{4^{1/2}} = \frac{3^{x-1}}{\sqrt{3}}$

or $4^{x-3/2} = 3^{x-3/2}$

or $\left(\frac{4}{3} \right)^{x-3/2} = 1$

or $x - \frac{3}{2} = 0$

or $x = 3/2$

ILLUSTRATION 1.2

Solve $e^{\sin x} - e^{-\sin x} - 4 = 0$.

Sol. Given that

$$e^{\sin x} - e^{-\sin x} - 4 = 0.$$

Let $e^{\sin x} = y$.

Then, given equation becomes

$$y - \frac{1}{y} - 4 = 0$$

or $y^2 - 4y - 1 = 0$

$\therefore y = e^{\sin x} = 2 + \sqrt{5}, 2 - \sqrt{5}$

Since $e^{\sin x} > 0$, $e^{\sin x} \neq 2 - \sqrt{5}$

Also, maximum value of $e^{\sin x}$ is e when $\sin x = 1$.

So, $e^{\sin x} \neq 2 + \sqrt{5}$

Therefore, given equation has no solution.

ILLUSTRATION 1.3

Solve $|x - 3|^{3x^2 - 10x + 3} = 1$.

Sol. Given that

$$|x - 3|^{3x^2 - 10x + 3} = 1, \text{ we have following cases}$$

If $3x^2 - 10x + 3 = 0$, where $x \neq 3$

$\therefore x = 1/3$

If $|x - 3| = 1$, $\therefore x = 4, 2$

ILLUSTRATION 1.4

Solve $(1/2)^{x^2 - 2x} < 1/4$.

Sol. We have $(1/2)^{x^2 - 2x} < (1/2)^2$

($\because$ base, $1/2 <$

It means $x^2 - 2x > 2$

$$\Rightarrow (x - (1 + \sqrt{3}))(x - (1 - \sqrt{3})) > 0$$

$$\Rightarrow x > 1 + \sqrt{3} \text{ or } x < 1 - \sqrt{3}$$

$$\Rightarrow x \in (-\infty, 1 - \sqrt{3}) \cup (1 + \sqrt{3}, \infty)$$

ILLUSTRATION 1.5

Find the smallest integral value of x satisfying $(x - 2)^{x^2 - 6x + 8} > 1$.

Sol. Clearly, $x > 2$

$$(x - 2)^{x^2 - 6x + 8} > 1$$

$$\Rightarrow (x - 2)^{x^2 - 6x + 8} > (x - 2)^0$$

When $x - 2 > 1$

or $x > 3$

We have

$$x^2 - 6x + 8 > 0$$

$$\Rightarrow (x - 2)(x - 4) > 0$$

$$\Rightarrow x < 2 \text{ or } x > 4$$

From (ii) and (iii), $x > 4$

When $x - 2 < 1$

or $x < 3$

We have

$$x^2 - 6x + 8 < 0$$

$$\therefore (x - 2)(x - 4) < 0$$

$$\therefore 2 < x < 4$$

From (i), (iv) and (v), we have

$$2 < x < 3$$

Thus, $x \in (2, 3) \cup (4, \infty)$.

ILLUSTRATION 1.6

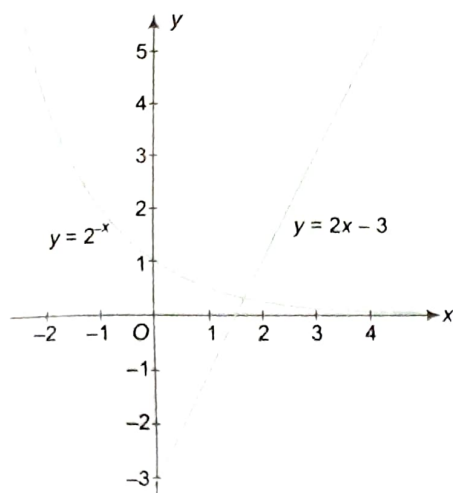
Find the number of solutions of equation $(2x - 3)2^x = 1$.

Sol. We have $(2x - 3)2^x = 1$

or $2x - 3 = 2^{-x}$

To find the number of roots of the above equation, we need to find the number of points of intersection of $y = 2x - 3$ and $y = 2^{-x}$.

The graphs of these functions are as shown in the following figure.



From the figure, graphs intersect at only one point.
Hence, there is only one solution of the given equation.

CONCEPT APPLICATION EXERCISE 1.1

- For $x \leq 2$, solve $x^3 3^{|x-2|} + 3^{x+1} = x^3 \cdot 3^{x-2} + 3^{|x-2|+3}$
- Solve $\left(\frac{1}{2}\right)^{x^6-2x^4} < 2^{x^2}$.
- Solve for x and y : $y^x = x^y$; $x = 2y$.
- Solve $2^{x+2} - 2^{x+3} - 2^{x+4} > 5^{x+1} - 5^{x+2}$
- Solve $\left(\frac{3}{4}\right)^{6x+10-x^2} < \frac{27}{64}$.
- Find the number of solutions of $|x| \cdot 3^{|x|} = 1$.

ANSWERS

- $x = 2$
- $x \in R - \{0, \pm 1\}$
- $x = 4, y = 2$
- $x \in (0, \infty)$
- $-1 < x < 7$
- 2

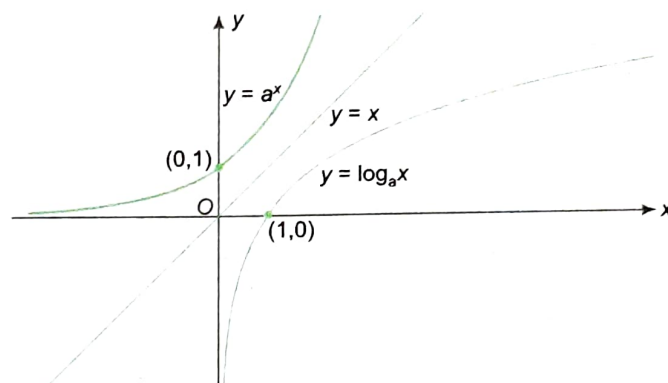
LOGARITHMIC FUNCTION

The **logarithm** of a number to a given base is the exponent to which the base must be raised in order to produce that number. For example, the logarithm of 1000 to base 10 is 3, because 10 to the power of 3 is 1000, i.e., $10^3 = 1000$. We write $\log_{10} 1000 = 3$. Here '10' is called the base of the logarithm. Similarly $\log_2 64$ is the value to which '2' must be raised to get 64. Since $2^6 = 64$, $\log_2 64 = 6$. Thus, if $a^x = y$, then we have $\log_a y = x$. In exponential function $y = a^x$, x can take any real value, so $\log_a y$ can also take any real value. Also, $y = a^x > 0$, so $\log_a y$ is defined only if $y > 0$. Thus, we can define logarithmic function as $f(x) = \log_a x$, $a > 0$, $a \neq 1$, having domain $(0, \infty)$ and range R . Thus, logarithmic function is actually inverse of exponential function. So, domain and range of exponential function are range and domain of logarithmic function, respectively.

GRAPH OF THE LOGARITHMIC FUNCTION

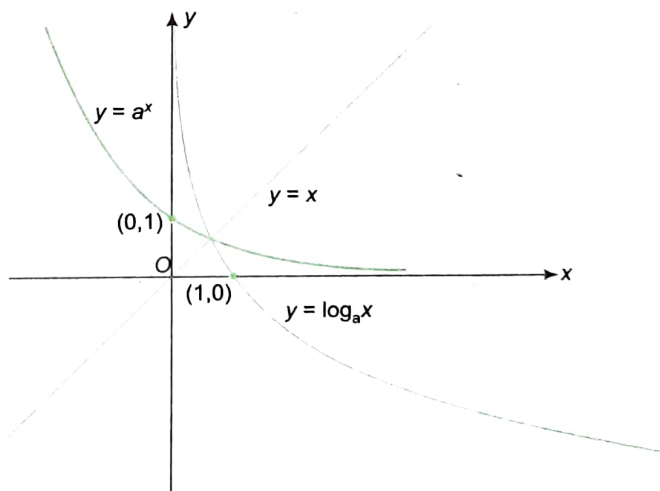
If we consider any point (p, q) on the graph of $y = a^x$, we have $q = a^p$. So, we have $p = \log_a q$, or there is a point (q, p) on the graph of $y = \log_a x$. Thus, for any point (p, q) on the graph of $y = a^x$, we have a point (q, p) on the graph of $y = \log_a x$. Since point (q, p) is the mirror image of the point (p, q) in the line $y = x$, we can draw the graph of $y = \log_a x$ by taking the mirror image of the graph of $y = a^x$ in line $y = x$. Thus, graphs of $y = a^x$ and $y = \log_a x$ are symmetrical about the line $y = x$ as shown in the following figure.

Case I: When $a > 1$



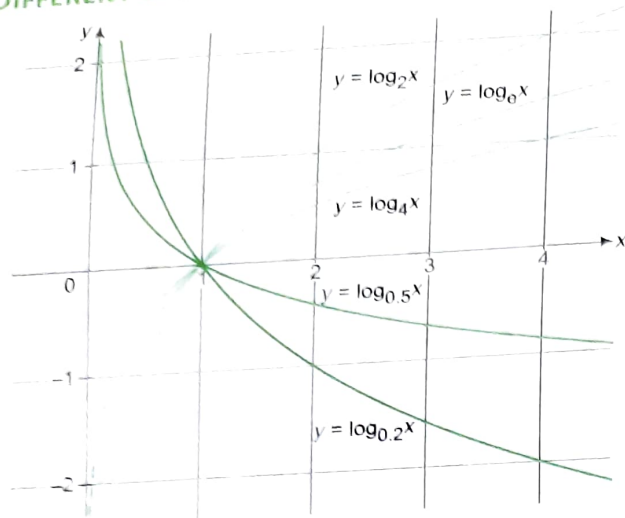
From the graph, we can see that $y = \log_a x$ is an increasing function in its domain. The value of $\log_a x$ approaches to infinity as x approaches to infinity. The value of $\log_a x$ approaches to negative infinity as x approaches to 0. Also, for $x_2 > x_1$, we have $\log_a x_2 > \log_a x_1$.

Case II: When $0 < a < 1$



From the graph, we can see that $y = \log_a x$ is decreasing function in its domain. The value of $\log_a x$ approaches to negative infinity as x approaches to infinity. The value of $\log_a x$ approaches to positive infinity as x approaches to 0. Also, for $x_2 > x_1$, we have $\log_a x_2 < \log_a x_1$.

GRAPH OF THE LOGARITHMIC FUNCTION FOR DIFFERENT BASES



Since $a^0 = 1$, we have $\log_a 1 = 0$. Thus, graph of $y = \log_a x$ for any value of base ' a ' passes through the point $(1, 0)$ on x -axis.

Also, since $a^1 = a$, we have $\log_a a = 1$, for any base ' a '.

Note:

- Common logarithm** is the logarithm with base 10. It is also known as the **decadic logarithm**, named after its base. It is indicated by $\log_{10}(x)$.
- Natural logarithm** is the logarithm to the base e , where e is an irrational constant approximately equal to 2.718281828.

Here, e is defined exactly as $e = \lim_{m \rightarrow \infty} \left(1 + \frac{1}{m}\right)^m$. The natural logarithm is generally written as $\ln(x)$ or $\log_e(x)$.

ILLUSTRATION 1.7

Find the value of $\log_{2\sqrt{3}} 1728$.

Sol. Let $\log_{2\sqrt{3}} 1728 = x$

$$\begin{aligned} &= 1728 = (2\sqrt{3})^x \\ &= (2^4 3^3) = (2\sqrt{3})^x \\ &= (2 \cdot \sqrt{3})^6 = (2\sqrt{3})^x \\ &\Rightarrow x = 6 \end{aligned}$$

ILLUSTRATION 1.8

Prove that $\frac{2}{5} < \log_{10} 3 < \frac{1}{2}$.

Sol. Let $\log_{10} 3 > \frac{2}{5}$

$$\begin{aligned} &\Rightarrow 3 > 10^{\frac{2}{5}} \\ &\Rightarrow 3^5 > 10^2, \text{ which is true} \end{aligned}$$

Now, $\log_{10} 3 < \frac{1}{2}$

$$\begin{aligned} &\Rightarrow 3 < 10^{\frac{1}{2}} \\ &\Rightarrow 3^2 < 10, \text{ which is true} \end{aligned}$$

Hence, $\frac{2}{5} < \log_{10} 3 < \frac{1}{2}$

ILLUSTRATION 1.9

Arrange $\log_2 5$, $\log_{0.5} 5$, $\log_7 5$, $\log_3 5$ in decreasing order.

Sol. $\log_2 5 = \text{exponent of 2 for which we get 5}$

$\log_7 5 = \text{exponent of 7 for which we get 5}$

Clearly, $\log_2 5 > \log_7 5$

With similar reasons, we have $\log_2 5 > \log_3 5 > \log_7 5$.

Also $\log_{0.5} 5 < 0$

$\therefore \log_2 5 > \log_3 5 > \log_7 5 > \log_{0.5} 5$

ILLUSTRATION 1.10

Prove that number $\log_2 7$ is an irrational number.

Sol. Let $\log_2 7$ is a rational number. Then,

$$\log_2 7 = \frac{p}{q}; p, q \in \mathbb{Q} \text{ or } 7 = 2^{p/q} \text{ or } 7^q = 2^p$$

which is not possible for any integral values of p and q .

Hence, $\log_2 7$ is not rational.

ILLUSTRATION 1.11

Which of the following numbers are positive/negative?

- (i) $\log_2 7$ (ii) $\log_{0.2} 3$ (iii) $\log_{1/3} (1/5)$
(iv) $\log_4 3$ (v) $\log_2 (\log_2 9)$

Sol.

- (i) Let $\log_2 7 = x \Rightarrow 7 = 2^x \Rightarrow x > 0$
(ii) Let $\log_{0.2} 3 = x \Rightarrow 3 = 0.2^x \Rightarrow x < 0$
(iii) Let $\log_{1/3} (1/5) = x \Rightarrow 1/5 = (1/3)^x \Rightarrow 5 = 3^x \Rightarrow x > 0$
(iv) Let $\log_4 3 = x \Rightarrow 3 = 4^x \Rightarrow x < 0$
(v) Let $\log_2 (\log_2 9) = x \Rightarrow \log_2 9 = 2^x \Rightarrow 9 = 2^{2^x} \Rightarrow x > 0$

ILLUSTRATION 1.12

Find the value of $\log \tan 1^\circ \log \tan 2^\circ \cdots \log \tan 89^\circ$.

Sol. $\log \tan 1^\circ \log \tan 2^\circ \cdots \log \tan 89^\circ$
 $= \log \tan 1^\circ \log \tan 2^\circ \cdots \log \tan 45^\circ \cdots \log \tan 89^\circ$
 $= \log \tan 1^\circ \log \tan 2^\circ \cdots (\log 1) \cdots \log \tan 89^\circ$
 $= 0 \text{ (as } \log 1 = 0)$

ILLUSTRATION 1.13

If $\log_a 3 = 2$ and $\log_b 8 = 3$, then prove that $\log_a b = \log_3 4$.

Sol. If $\log_a 3 = 2$

$$\Rightarrow 3 = a^2$$

$$\Rightarrow a = \sqrt{3}$$

If $\log_b 8 = 3$

$$\Rightarrow 8 = b^3$$

$$\begin{aligned} \Rightarrow b &= 2 \\ \text{So, } \log_a b &= \log_{\sqrt{3}} 2 = x \text{ (let)} \\ \Rightarrow 2 &= (\sqrt{3})^x \\ \Rightarrow 4 &= 3^x \\ \Rightarrow x &= \log_3 4 \end{aligned}$$

ILLUSTRATION 1.14

If $\log_3 y = x$ and $\log_2 z = x$, find 72^x in terms of y and z .

$$\begin{aligned} \text{Sol. } \log_3 y &= x \\ \Rightarrow y &= 3^x \\ \log_2 z &= x \\ \Rightarrow z &= 2^x \end{aligned}$$

$$\text{Now, } 72^x = (2^3 3^2)^x = 2^{3x} 3^{2x} = (2^x)^3 (3^x)^2 = y^3 z^2$$

ILLUSTRATION 1.15

If $\frac{x(y+z-x)}{\log x} = \frac{y(z+x-y)}{\log y} = \frac{z(x+y-z)}{\log z}$, prove that $x^y y^x = z^y y^z = x^z z^x$.

$$\text{Sol. Let } \frac{x(y+z-x)}{\log_a x} = \frac{y(z+x-y)}{\log_a y} = \frac{z(x+y-z)}{\log_a z} = k$$

$$\Rightarrow \log_a x = \frac{x(y+z-x)}{k}$$

$$\Rightarrow x = a^{\frac{x(y+z-x)}{k}}$$

$$\text{Similarly, } y = a^{\frac{y(x+z-y)}{k}} \text{ and } z = a^{\frac{z(x+y-z)}{k}}$$

$$\begin{aligned} \text{Now } x^y y^x &= a^{\frac{xy(y+z-x)}{k}} a^{\frac{yx(z+x-y)}{k}} \\ &= a^{\frac{xy^2 + x^2 y + x^2 y + xy^2 - x^2 y - xy^2}{k}} = a^{\frac{2xyz}{k}} \end{aligned}$$

$$\text{Similarly, } z^y y^z = x^z z^x = a^{\frac{2xyz}{k}}$$

ILLUSTRATION 1.16

Suppose $x, y, z > 0$ and are not equal to 1 and $\log x + \log y + \log z = 0$. Find the value of $x^{\frac{1}{\log y} + \frac{1}{\log z}} \times y^{\frac{1}{\log z} + \frac{1}{\log x}} \times z^{\frac{1}{\log x} + \frac{1}{\log y}}$ (base 10).

$$\text{Sol. Let } K = x^{\frac{1}{\log y} + \frac{1}{\log z}} \times y^{\frac{1}{\log z} + \frac{1}{\log x}} \times z^{\frac{1}{\log x} + \frac{1}{\log y}}$$

$$\begin{aligned} \therefore \log K &= \log x \left[\frac{1}{\log y} + \frac{1}{\log z} \right] + \log y \left[\frac{1}{\log z} + \frac{1}{\log x} \right] \\ &\quad + \log z \left[\frac{1}{\log x} + \frac{1}{\log y} \right] \quad (i) \end{aligned}$$

Putting $\log x + \log y + \log z = 0$ (given), we get

$$\frac{\log x}{\log y} + \frac{\log z}{\log y} = -1; \frac{\log y}{\log x} + \frac{\log z}{\log x} = -1;$$

$$\frac{\log x}{\log z} + \frac{\log y}{\log z} = -1$$

$\therefore$ R.H.S. of Eq. (i) = -3

$$\Rightarrow \log_{10} K = -3 \text{ or } K = 10^{-3}$$

ILLUSTRATION 1.17

Solve $2(25)^x - 5(10^x) + 2(4^x) \geq 0$.

$$\text{Sol. } 2(25)^x - 5(10^x) + 2(4^x) \geq 0$$

$$\Rightarrow 2(5)^{2x} - 5(5^x)(2^x) + 2(2^{2x}) \geq 0$$

$$\Rightarrow 2\left(\frac{5}{2}\right)^{2x} - 5\left(\frac{5}{2}\right)^x + 2 \geq 0$$

$$\Rightarrow \left[\left(\frac{5}{2}\right)^x - 2 \right] \left[2\left(\frac{5}{2}\right)^x - 1 \right] \geq 0$$

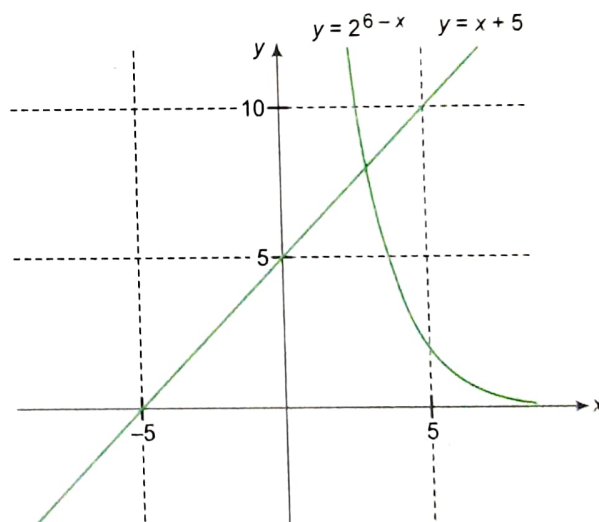
$$\Rightarrow \left(\frac{5}{2}\right)^x \leq \frac{1}{2} \text{ or } \left(\frac{5}{2}\right)^x \geq 2$$

$$\Rightarrow x \leq \log_{2.5} 0.5 \text{ or } x \geq \log_{2.5} 2$$

ILLUSTRATION 1.18

Find the number of solution to equation $\log_2 (x+5) = 6-x$.

$$\text{Sol. Here, } x+5 = 2^{6-x}$$



Now graph of $y = x + 5$ and $y = 2^{6-x}$ intersect only once. Hence, there is only one solution.

ILLUSTRATION 1.19

Find the number of solutions of the following equations:

(i) $x^{-1/2} \log_{0.5} x = 1$

(ii) $x^2 - 4x + 3 - \log_2 x = 0$

Sol.

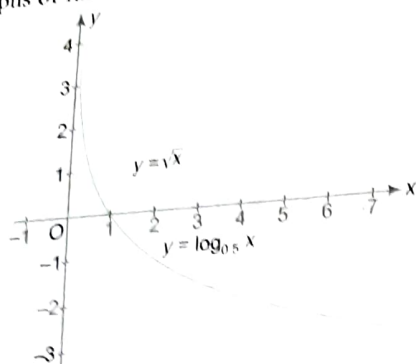
(i) We have, $x^{-1/2} \log_{0.5} x = 1$

$$\Rightarrow \log_{0.5} x = \sqrt{x}$$

To find the number of solutions, we must draw the graphs of $y = \sqrt{x}$ and $y = \log_{0.5} x$ and find the number of points of their intersection.

1.6 Trigonometry

Graphs of functions are as shown in the following figure.



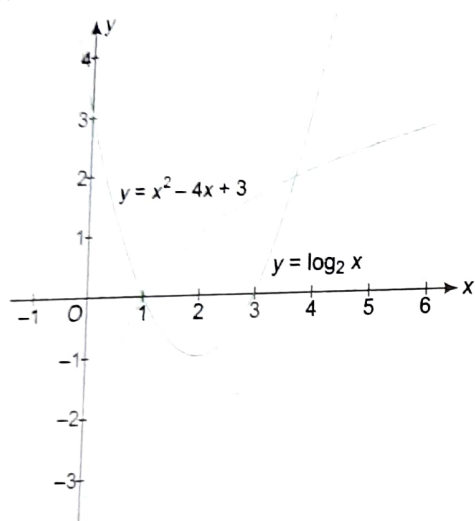
From the figure, we see that graphs intersect at only one point.

Hence, there is only one solution.

(ii) We have $x^2 - 4x + 3 - \log_2 x = 0$

or $x^2 - 4x + 3 = \log_2 x$

Let us draw the graphs of $y = x^2 - 4x + 3$ and $y = \log_2 x$.



From the figure, we see that graphs intersect at two points.

Hence, there are two solutions.

FUNDAMENTAL LAWS OF LOGARITHM

ADDITION AND SUBTRACTION OF LOGARITHM

We have the following laws for addition and subtraction of logarithm:

For $m, n > 0, a > 0, a \neq 1$;

1. $\log_a (mn) = \log_a m + \log_a n$

Proof:

Let $\log_a m = x \Rightarrow a^x = m$

and $\log_a n = y \Rightarrow a^y = n$

$\therefore mn = a^x a^y = a^{x+y}$

$\Rightarrow \log_a (mn) = x + y$

$\Rightarrow \log_a (mn) = \log_a m + \log_a n$

In general, for positive real numbers $x_1, x_2, \dots, x_n$,

$\log_a (x_1 x_2 \dots x_n) = \log_a x_1 + \log_a x_2 + \dots + \log_a x_n$

2. $\log_a \left(\frac{m}{n} \right) = \log_a m - \log_a n$

Proof:

Using (i) and (ii),

$\frac{m}{n} = \frac{a^x}{a^y} = a^{x-y}$

$\Rightarrow \log_a \left(\frac{m}{n} \right) = x - y$

$\Rightarrow \log_a \left(\frac{m}{n} \right) = \log_a m - \log_a n$

ILLUSTRATION 1.20

Find the values of the following:

(i) $\log_{10} 2 + \log_{10} 5$

(ii) $\log_3 (\sqrt{11} - \sqrt{2}) + \log_3 (\sqrt{11} + \sqrt{2})$

(iii) $\log_7 35 - \log_7 5$

Sol.

(i) $\log_{10} 2 + \log_{10} 5 = \log_{10} (2 \times 5) = \log_{10} 10 = 1$

(ii) $\log_3 (\sqrt{11} - \sqrt{2}) + \log_3 (\sqrt{11} + \sqrt{2})$

$= \log_3 (\sqrt{11} - \sqrt{2}) \times (\sqrt{11} + \sqrt{2})$

$= \log_3 (11 - 2) = \log_3 9 = 2 \quad (\text{as } 3^2 = 9)$

(iii) $\log_7 35 - \log_7 5 = \log_7 \frac{35}{5} = \log_7 7 = 1$

ILLUSTRATION 1.21

Find the value of $\log_2 (2\sqrt[3]{9} - 2) + \log_2 (12\sqrt[3]{3} + 4 + 4\sqrt[3]{9})$.

Sol.

$\log_2 (2\sqrt[3]{9} - 2) + \log_2 (12\sqrt[3]{3} + 4 + 4\sqrt[3]{9})$

$= \log_2 ((72)^{1/3} - (8)^{1/3}) + \log_2 ((72)^{2/3} + (8)^{2/3} + (72)^{1/3} (8)^{1/3})$

$= \log_2 ((72)^{1/3} - (8)^{1/3}) ((72)^{2/3} + (8)^{2/3} + (72)^{1/3} (8)^{1/3})$

$= \log_2 (72 - 8)$

$= \log_2 64 = 6$

CONCEPT APPLICATION EXERCISE 1.2

1. Find the value of $3^{2\log_9 3}$.

2. Find the value of $\sqrt{(\log_{0.5} 4)^2}$.

3. If $\log_{\sqrt{8}} b = 3\frac{1}{3}$, then find the value of b .

4. Find the value of $\log_5 \log_2 \log_3 \log_2 512$.

5. If $\log_5 x = a$ and $\log_2 y = a$, find 100^{2a-1} in terms of x and y .

6. Find the value of $\log_{1/3} \sqrt[4]{729 \cdot \sqrt[3]{9^{-1}} \cdot 27^{-4/3}}$.

7. Solve for x : $\log_4 \log_3 \log_2 x = 0$.

8. Prove that $\log_{10} 2$ lies between $\frac{1}{4}$ and $\frac{1}{3}$.

9. Find number of roots of the equation $x^3 - \log_{0.5} x = 0$.

ANSWERS

1. 3
2. 2
3. 32
4. 0
5. $\frac{(xy)^4}{100}$
6. -1
7. $x = 8$
9. one solution

EXPONENTS OF ARGUMENT AND BASE OF LOGARITHM

For $m, n > 0$; $a > 0$, $a \neq 1$; $b \in R$;

$$1. \log_a(m^b) = b \times \log_a m$$

Proof:

$$\text{Let } \log_a(m^b) = p$$

$$\text{Then } (m^b) = a^p$$

$$\Rightarrow m = a^{\frac{p}{b}}$$

$$\Rightarrow \frac{p}{b} = \log_a m$$

$$\Rightarrow p = b \log_a m$$

$$\Rightarrow \log_a(m^b) = b \log_a m$$

$$2. \log_{(a^b)} m = \frac{1}{b} \log_a m$$

Proof:

$$\text{Let } \log_{(a^b)} m = p$$

$$\text{Then } m = (a^b)^p$$

$$\Rightarrow m = a^{bp}$$

$$\Rightarrow \log_a m = bp$$

$$\Rightarrow \log_a m = b \log_{(a^b)} m$$

$$\Rightarrow \log_{(a^b)} m = \frac{1}{b} \log_a m$$

From above two laws, we have

$$\log_{(a^b)} m^c = \frac{c}{b} \log_a m$$

ILLUSTRATION 1.22

What is logarithm of $32\sqrt[5]{4}$ to the base $2\sqrt{2}$?

$$\begin{aligned} \text{Sol. } \log_{2\sqrt{2}} 32\sqrt[5]{4} &= \log_{(2^{3/2})} (2^5 4^{1/5}) \\ &= \log_{(2^{3/2})} (2^{5+\frac{2}{5}}) \\ &= \frac{2}{3} \frac{27}{5} \log_2 2 \\ &= \frac{18}{5} \\ &= 3.6 \end{aligned}$$

ILLUSTRATION 1.23

If $\log_e \left(\frac{a+b}{2} \right) = \frac{1}{2} (\log_e a + \log_e b)$, then find the relation between a and b .

$$\begin{aligned} \text{Sol. } \log_e \left(\frac{a+b}{2} \right) &= \frac{1}{2} (\log_e a + \log_e b) \\ &= \frac{1}{2} (\log_e ab) = (\log_e \sqrt{ab}) \end{aligned}$$

$$\Rightarrow \frac{a+b}{2} = \sqrt{ab}$$

$$\text{or } a+b-2\sqrt{ab}=0$$

$$\text{or } (\sqrt{a}-\sqrt{b})^2=0$$

$$\text{or } a=b$$

ILLUSTRATION 1.24

Which of the following pairs of expressions are defined for the same set of values of x ?

$$(i) f_1(x) = 2\log_{10} x \text{ and } f_2(x) = \log_{10} x^2$$

$$(ii) f_1(x) = \log_x x^2 \text{ and } f_2(x) = 2$$

$$(iii) f_1(x) = \log_{10}(x-2) + \log_{10}(x-3) \text{ and } f_2(x) = \log_{10}(x-2)(x-3)$$

Sol.

$$(i) f_1(x) = 2\log_{10} x \text{ is defined for } x > 0$$

$$f_2(x) = \log_{10} x^2 \text{ is defined for } x^2 > 0 \text{ or } x \in R - \{0\}$$

Therefore $f_1(x)$ and $f_2(x)$ are not defined for same set of values of x .

$$(ii) f_1(x) = \log_x x^2 \text{ is defined for } x > 0, x \neq 1$$

$$\therefore f_1(x) = 2, x > 0, x \neq 1$$

But $f_2(x) = 2$ is defined for all real x .

Therefore $f_1(x)$ and $f_2(x)$ are not defined for same set of values of x .

$$(iii) f_1(x) = \log_{10}(x-2) + \log_{10}(x-3) \text{ is defined if } x-2 > 0 \text{ and } x-3 > 0$$

$$\therefore x > 3$$

$$f_2(x) = \log_{10}(x-2)(x-3) \text{ is defined if } (x-2)(x-3) > 0$$

$$\therefore x < 2 \text{ or } x > 3$$

Therefore $f_1(x)$ and $f_2(x)$ are not defined for same set of values of x .

ILLUSTRATION 1.25

$$\text{Find the value of } 7\log\left(\frac{16}{15}\right) + 5\log\left(\frac{25}{24}\right) + 3\log\left(\frac{81}{80}\right).$$

$$\begin{aligned} \text{Sol. } 7\log\left(\frac{16}{15}\right) + 5\log\left(\frac{25}{24}\right) + 3\log\left(\frac{81}{80}\right) \\ &= \log\left[\left(\frac{16}{15}\right)^7 \left(\frac{25}{24}\right)^5 \left(\frac{81}{80}\right)^3\right] \\ &= \log \frac{2^{28}}{3^7 5^7} \frac{5^{10}}{2^{15} 3^5} \frac{3^{12}}{2^{12} 5^3} \\ &= \log 2 \end{aligned}$$

ILLUSTRATION. 1.26

If sum $\log_2 x + \log_4 x + \log_{16} x + \log_{32} x + \dots = 6$, then find the value of x .

$$\begin{aligned} \text{Sol. } \log_2 x + \log_4 x + \log_{16} x + \log_{32} x + \dots &= 6 \\ \Rightarrow \log_2 x + \log_{2^2} x + \log_{2^4} x + \log_{2^5} x + \dots &= 6 \end{aligned}$$

1.8 Trigonometry

$$\Rightarrow \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right) \log_2 x = 6$$

$$\Rightarrow \frac{1}{1 - \frac{1}{2}} \log_2 x = 6$$

$$\Rightarrow \log_2 x = 3$$

$$\Rightarrow x = 8$$

ILLUSTRATION 1.27

Suppose that a and b are positive real numbers such that $\log_2 a + \log_3 b = 7/2$ and $\log_2 b + \log_3 a = 2/3$. Then find the value of ab .

Sol. $\log_2 a + \log_3 b = \frac{7}{2}$ and $\log_2 b + \log_3 a = \frac{2}{3}$

$$\Rightarrow \frac{1}{3} \log_3 a + \frac{1}{2} \log_3 b = \frac{7}{2}$$

and $\frac{1}{3} \log_3 b + \frac{1}{2} \log_3 a = \frac{2}{3}$

Adding the equations, we get

$$\frac{1}{3} \log_3(ab) + \frac{1}{2} \log_3(ab) = \frac{7}{2} + \frac{2}{3} = \frac{25}{6}$$

or $\frac{5}{6} \log_3(ab) = \frac{25}{6}$

or $\log_3(ab) = 5$

or $ab = 3^5 = 243$

ILLUSTRATION 1.28

Solve for x : $11^{4x-5} \cdot 3^{2x} = 5^{3-x} \cdot 7^{-x}$.

Sol. $11^{4x-5} \cdot 3^{2x} = 5^{3-x} \cdot 7^{-x}$

or $(4x-5)\log 11 + 2x\log 3 = (3-x)\log 5 - x\log 7$

or $x = \frac{\log(11^5 \times 5^3)}{\log(11^4 \times 315)}$

RECIPROCAL OF LOGARITHM

For $m, a > 0, a, m \neq 1$; $\log_a m = \frac{1}{\log_m a}$

Proof:

Let $\log_a m = p$

Then $m = a^p$

$$\Rightarrow a = m^{1/p}$$

$$\Rightarrow \frac{1}{p} = \log_m a$$

$$\Rightarrow \frac{1}{\log_a m} = \log_m a$$

or $\log_a m = \frac{1}{\log_m a}$

ILLUSTRATION 1.29

Which is greater: $x = \log_3 5$ or $y = \log_{17} 25$?

Sol. $\frac{1}{y} = \log_{25} 17 = \frac{1}{2} \log_5 17$ and $\frac{1}{x} = \log_5 3 = \frac{1}{2} \log_5 9$

$\therefore \frac{1}{y} > \frac{1}{x}$ or $x > y$

ILLUSTRATION 1.30

If $n > 1$, then prove that

$$\frac{1}{\log_2 n} + \frac{1}{\log_3 n} + \dots + \frac{1}{\log_{53} n} = \frac{1}{\log_{53!} n}$$

Sol. The given expression is equal to

$$\log_n 2 + \log_n 3 + \dots + \log_n 53 = \log_n (2 \times 3 \times \dots \times 53)$$

$$= \log_n 53! = \frac{1}{\log_{53!} n}$$

ILLUSTRATION 1.31

Let $a = \log_3 20$, $b = \log_4 15$ and $c = \log_5 12$. Then find the value of $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}$.

Sol. We have

$$a+1 = \log_3 20 + \log_3 3 = \log_3 60$$

$$b+1 = \log_4 15 + \log_4 4 = \log_4 60$$

$$c+1 = \log_5 12 + \log_5 5 = \log_5 60$$

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = \frac{1}{\log_3 60} + \frac{1}{\log_4 60} + \frac{1}{\log_5 60}$$

$$= \log_{60} 3 + \log_{60} 4 + \log_{60} 5$$

$$= \log_{60} (3 \times 4 \times 5)$$

$$= \log_{60} 60$$

$$= 1$$

ILLUSTRATION 1.32

If $\log_a(ab) = x$, then evaluate $\log_b(ab)$ in terms of x .

Sol. $\log_a(ab) = x$ or $\log_a a + \log_a b = x$ or $\log_a b = x - 1$

Now, $\log_b(ab) = \log_b a + \log_b b$

$$= \frac{1}{\log_a b} + 1 = \frac{1}{x-1} + 1 = \frac{x}{x-1}$$

ILLUSTRATION 1.33

If $\log_{12} 27 = a$, then find $\log_6 16$ in terms of a .

Sol. Since $a = \log_{12} 27 = \log_{12} (3)^3 = 3 \log_{12} 3$,

$$= \frac{3}{\log_3 12} = \frac{3}{1 + \log_3 4} = \frac{3}{1 + 2 \log_3 2}$$

$$\therefore \log_3 2 = \frac{3-a}{2a}$$

$$\begin{aligned}\text{Then, } \log_6 16 &= \log_6 2^4 = 4 \log_6 2 = \frac{4}{\log_2 6} \\ &= \frac{4}{1 + \log_2 3} = \frac{4}{1 + \frac{2a}{3-a}} = 4 \left(\frac{3-a}{3+a} \right)\end{aligned}$$

CHANGE OF BASE

1. For $m, a, b > 0$ and $a, b \neq 1$; $\log_a m = \frac{\log_b m}{\log_b a}$

Proof:

$$\text{Let } \log_a m = p$$

$$\text{Then } a^p = m$$

$$\Rightarrow \log_b (a^p) = \log_b m \quad [\text{Taking log to the base } b]$$

$$\Rightarrow p \log_b a = \log_b m$$

$$\Rightarrow \log_a m \times \log_b a = \log_b m$$

$$\Rightarrow \log_a m = \frac{\log_b m}{\log_b a}$$

Replacing b by m in the above result, we get

$$\log_a m = \frac{\log_m m}{\log_m a}$$

$$\Rightarrow \log_a m = \frac{1}{\log_m a} \quad [\because \log_m m = 1]$$

2. For $a, b, c > 0$ and $b \neq 1$; $a^{\log_b c} = c^{\log_b a}$

Proof:

$$\text{Let } a^{\log_b c} = p$$

$$\text{Then } \log_b p = \log_a a$$

$$\Rightarrow \frac{\log p}{\log b} = \frac{\log p}{\log a}$$

$$\Rightarrow \frac{\log a}{\log b} = \frac{\log p}{\log c}$$

$$\Rightarrow \log_b a = \log_c p$$

$$\Rightarrow p = c^{\log_b a}$$

$$\Rightarrow a^{\log_b c} = c^{\log_b a}$$

ILLUSTRATION 1.34

If $a^x = b$, $b^y = c$, $c^z = a$, then find the value of xyz .

$$\text{Sol. } a^x = b, b^y = c, c^z = a$$

$$\Rightarrow x = \log_a b, y = \log_b c, z = \log_c a$$

$$\Rightarrow xyz = (\log_a b)(\log_b c)(\log_c a) = \frac{\log b}{\log a} \frac{\log c}{\log b} \frac{\log a}{\log c} = 1$$

ILLUSTRATION 1.35

Find the value of $(\log_3 4)(\log_4 5)(\log_5 6)(\log_6 7)(\log_7 8)(\log_8 9)$.

$$\text{Sol. } \log_3 4 \log_4 5 \log_5 6 \log_6 7 \log_7 8 \log_8 9$$

$$\begin{aligned}&= \frac{\log 4}{\log 3} \times \frac{\log 5}{\log 4} \times \frac{\log 6}{\log 5} \times \frac{\log 7}{\log 6} \times \frac{\log 8}{\log 7} \times \frac{\log 9}{\log 8} \\ &= \frac{\log 9}{\log 3} = \log_3 9 = 2\end{aligned}$$

ILLUSTRATION 1.36

$$\text{Simplify } \frac{1}{1 + \log_a bc} + \frac{1}{1 + \log_b ca} + \frac{1}{1 + \log_c ab}.$$

$$\begin{aligned}\text{Sol. } &\frac{1}{1 + \log_a bc} + \frac{1}{1 + \log_b ca} + \frac{1}{1 + \log_c ab} \\ &= \frac{1}{1 + \frac{\log bc}{\log a}} + \frac{1}{1 + \frac{\log ca}{\log b}} + \frac{1}{1 + \frac{\log ab}{\log c}} \\ &= \frac{\log a}{\log a + \log b + \log c} + \frac{\log b}{\log a + \log b + \log c} \\ &\quad + \frac{\log c}{\log a + \log b + \log c} \\ &= 1\end{aligned}$$

ILLUSTRATION 1.37

If $x = \log_{2a} a$, $y = \log_{3a} 2a$, $z = \log_{4a} 3a$, prove that $1 + xyz = 2yz$.

$$\begin{aligned}\text{Sol. } 1 + xyz &= 1 + (\log_{2a} a)(\log_{3a} 2a)(\log_{4a} 3a) \\ &= 1 + \frac{\log a}{\log 2a} \frac{\log 2a}{\log 3a} \frac{\log 3a}{\log 4a} \\ &= 1 + \frac{\log a}{\log 4a} \\ &= \log_{4a} 4a + \log_{4a} a \\ &= \log_{4a} 4a^2 = 2 \log_{4a} 2a \\ &= 2(\log_{3a} 2a)(\log_{4a} 3a) = 2yz\end{aligned}$$

ILLUSTRATION 1.38

If $\log_b a \log_c a + \log_a b \log_c b + \log_a c \log_b c = 3$ (where a, b, c are different positive real numbers $\neq 1$), then find the value of abc .

$$\begin{aligned}\text{Sol. } &\log_b a \cdot \log_c a + \log_a b \cdot \log_c b + \log_a c \cdot \log_b c = 3 \\ \text{or } &\frac{\log a}{\log b} \frac{\log a}{\log c} + \frac{\log b}{\log a} \frac{\log b}{\log c} + \frac{\log c}{\log a} \frac{\log c}{\log b} = 3 \\ \text{or } &(\log a)^3 + (\log b)^3 + (\log c)^3 = 3(\log a)(\log b)(\log c) \\ \text{or } &\log a + \log b + \log c = 0 \quad (\text{as } a, b, c \text{ are different}) \\ \Rightarrow &\log abc = 0 \quad \text{or } abc = 1\end{aligned}$$

$f(f^{-1}(x))$ WHERE $f(x)$ IS EXPONENTIAL FUNCTION

Here $f^{-1}(x)$ is inverse function of exponential function $f(x)$.

$$\text{If } f(x) = a^x, \text{ then } f^{-1}(x) = \log_a x.$$

$$\therefore f(f^{-1}(x)) = a^{\log_a x}$$

$$\text{Let } y = a^x \quad \dots(1)$$

1.10 Trigonometry

$$\therefore x = \log_a y$$

Putting this value of x in (1), we get

$$\therefore a^{\log_a y} = y$$

$$\text{So, } f(f^{-1}(x)) = a^{\log_a x} = x$$

$$\text{e.g., } 3^{\log_3 8} = 8, \quad 2^{\log_2 5} = 2^{\log_2 5} = 5^3,$$

$$5^{-2 \log_5 3} = 5^{\log_5 3^{-2}} = 3^{-2} = \frac{1}{9} \text{ etc.}$$

ILLUSTRATION 1.39

If $y = 2^{\frac{1}{\log_4 x}}$ then prove that $x = y^2$.

$$\text{Sol. } y = 2^{\frac{1}{\log_4 x}}$$

$$\therefore y = 2^{\log_4 x} \Rightarrow y = 2^{\frac{1}{2} \log_2 x}$$

$$\Rightarrow y^2 = 2^{\log_2 x}$$

$$\Rightarrow y^2 = x \quad (\text{as } y > 0)$$

ILLUSTRATION 1.40

Find the value of $81^{(1/\log_3 3)} + 27^{\log_9 36} + 3^{4/\log_3 9}$.

$$\begin{aligned} \text{Sol. } & 81^{(1/\log_3 3)} + 27^{\log_9 36} + 3^{4/\log_3 9} \\ &= (3^4)^{\log_3 5} + (3^3)^{\log_{3^2} (6^2)} + 3^{4 \log_3 7} \\ &= 3^{\log_3 5^4} + (3^3)^{\log_3 (6)} + 3^{4 \log_3 7} \\ &= 5^4 + 3^{\log_3 6^3} + 3^{2 \log_3 7} \\ &= 5^4 + 6^3 + 3^{\log_3 7^2} \\ &= 625 + 216 + 7^2 \\ &= 890 \end{aligned}$$

ILLUSTRATION 1.41

Prove that $\frac{2^{\log_2 x} - 3^{\log_3 (x^2+1)} - 2x}{7^{4 \log_7 x} - x - 1} > 0; \forall x \in (0, \infty)$.

$$\begin{aligned} \text{Sol. } y &= \frac{2^{\log_2 x} - 3^{\log_3 (x^2+1)} - 2x}{7^{4 \log_7 x} - x - 1} \\ &= \frac{2^{\log_2 x^4} - 3^{\log_3 (x^2+1)} - 2x}{7^{\log_7 x^2} - x - 1} \\ &= \frac{x^4 - (x^2 + 2x + 1)}{x^2 - x - 1} \\ &= x^2 + x + 1 \\ &= \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} > 0; \forall x \in R \end{aligned}$$

ILLUSTRATION 1.42

If $60^a = 3$ and $60^b = 5$, then find the value of $12^{\frac{1-a-b}{2(1-b)}}$.

Sol. We have

$$60^a = 3 \Rightarrow a = \log_{60} 3$$

$$60^b = 5 \Rightarrow b = \log_{60} 5$$

$$\begin{aligned} \text{So, } \frac{1-(a+b)}{2(1-b)} &= \frac{1-(\log_{60} 3 + \log_{60} 5)}{2(\log_{60} 60 - \log_{60} 5)} \\ &= \frac{\log_{60} 60 - \log_{60} 15}{2(\log_{60} 60 - \log_{60} 5)} = \frac{\log_{60} 4}{2 \log_{60} 12} \\ &= \frac{1}{2} \log_{12} 4 = \log_{12} 2 \\ &\Rightarrow 12^{\frac{1-a-b}{2(1-b)}} = 12^{\log_{12} 2} = 2 \end{aligned}$$

CONCEPT APPLICATION EXERCISE 1.3

1. Write each of the following as single logarithm:

$$\text{(a) } 1 + \log_2 5 \quad \text{(b) } 2 - \log_3 7$$

$$\text{(c) } 2 \log_{10} x + 3 \log_{10} y - 5 \log_{10} z$$

2. Prove that $\frac{1}{3} < \log_{20} 3 < \frac{1}{2}$.

3. Prove that $\log_7 \log_7 \sqrt{7 \sqrt{7 \sqrt{7}}} = 1 - 3 \log_7 2$.

4. If $\log_{10} x = y$, then find $\log_{1000} x^2$ in terms of y .

5. If $\log_7 2 = m$, then find $\log_{49} 28$ in terms of m .

6. Find the value of $\log_2 \left(\frac{1}{7^{\log_7 0.125}} \right)$.

7. Find the value of $\left(\frac{4}{\log_2 (2\sqrt{3})} + \frac{2}{\log_3 (2\sqrt{3})} \right)^2$.

8. If x and y are positive real numbers such that $2 \log (2y - 3x) = \log x + \log y$, then find the value of $\frac{x}{y}$.

9. If $a^2 + b^2 = 7ab$, prove that $\log \left(\frac{a+b}{3} \right) = \frac{1}{2} (\log a + \log b)$.

10. If $\log_b n = 2$ and $\log_n 2b = 2$, then find the value of b .

11. If $\log_2 x \times \log_3 x = \log_2 x + \log_3 x$, then find x .

12. If $y^2 = xz$ and $a^x = b^y = c^z$, then prove that $\log_b a = \log_c b$.

13. Prove the following identities:

$$\text{(a) } \frac{\log_a n}{\log_{ab} n} = 1 + \log_a b \quad \text{(b) } \log_{ab} x = \frac{\log_a x \log_b x}{\log_a x + \log_b x}$$

14. Compute $\log_{ab} (\sqrt[3]{a} / \sqrt{b})$ if $\log_{ab} a = 4$.

15. If $a^x = b^y = c^z = d^w$, show that $\log_a (bcd) = x \left(\frac{1}{y} + \frac{1}{z} + \frac{1}{w} \right)$.

16. Find the value of $\left(\frac{1}{49} \right)^{1+\log_7 2} + 5^{-\log_{(1/5)} (7)}$.

ANSWERS

$$1. \text{ (a) } \log_2 10 \quad \text{(b) } \log_3 \frac{9}{7} \quad \text{(c) } \log_{10} \frac{x^2 y^3}{z^5} \quad 4. \frac{2}{3} y$$

$$5. \frac{1+2m}{2} \quad 6. 3 \quad 7. 16 \quad 8. \frac{4}{9} \quad 10. 2^{1/3}$$

$$11. x = 1, 6 \quad 14. \frac{17}{16} \quad 16. \frac{1373}{196}$$

LOGARITHMIC EQUATIONS

While solving logarithmic equations, we tend to simplify the equation. Solving equation after simplification may give some roots which may not define all the terms in the initial equation. Thus, while solving equations involving logarithmic function, we must take care of domain of the equation.

ILLUSTRATION 1.43

Solve $\log_4 8 + \log_4(x+3) - \log_4(x-1) = 2$.

Sol. $\log_4 8 + \log_4(x+3) - \log_4(x-1) = 2$

$$\text{or } \log_4 \frac{8(x+3)}{x-1} = 2$$

$$\text{or } \frac{8(x+3)}{x-1} = 4^2$$

$$\text{or } x+3 = 2x-2$$

$$\Rightarrow x = 5$$

Also for $x = 5$ all terms of the equation are defined.

ILLUSTRATION 1.44

Solve $\log(-x) = 2 \log(x+1)$.

Sol. By definition, $x < 0$ and $x+1 > 0$.

$$\Rightarrow -1 < x < 0$$

$$\text{Now } \log(-x) = 2 \log(x+1)$$

$$\text{or } -x = (x+1)^2$$

$$\text{or } x^2 + 3x + 1 = 0$$

$$\text{or } x = \frac{-3 \pm \sqrt{5}}{2}, \frac{-3 - \sqrt{5}}{2}$$

Hence,

$$x = \frac{-3 + \sqrt{5}}{2} \text{ is the only solution } \left(\because \frac{-3 - \sqrt{5}}{2} < -1 \right)$$

ILLUSTRATION 1.45

Solve $\log_2(3x-2) = \log_{1/2} x$.

Sol. $\log_2(3x-2) = \log_{1/2} x = -\log_2 x = \log_2 x^{-1}$

$$\text{or } 3x-2 = x^{-1} \text{ or } 3x^2 - 2x = 1$$

$$\Rightarrow x = 1 \text{ or } x = -1/3.$$

But $\log_2(3x-2)$ and $\log_{1/2} x$ are meaningful if $x > 2/3$. Hence, $x = 1$.

ILLUSTRATION 1.46

Solve $2^x + 27^{x(x-1)} = 9$.

Sol. Taking log of both sides, we get

$$(x+2) \log 2 + \frac{x}{x-1} \log 27 = \log 9$$

$$\text{or } (x+2) \log 2 + \frac{x}{x-1} 3 \log 3 = 2 \log 3$$

$$\text{or } (x+2) \log 2 + \left(\frac{3x}{x-1} - 2 \right) \log 3 = 0$$

$$\text{or } (x+2) \left[\log 2 + \frac{\log 3}{x-1} \right] = 0$$

$$\Rightarrow x = -2 \text{ or } x-1 = -\frac{\log 3}{\log 2}$$

$$\Rightarrow x = -2, 1 - \frac{\log 3}{\log 2}$$

ILLUSTRATION 1.47

Solve $\log_2(4 \times 3^x - 6) - \log_2(9^x - 6) = 1$.

Sol. $\log_2(4 \times 3^x - 6) - \log_2(9^x - 6) = 1$

$$\text{or } \log_2 \frac{4 \times 3^x - 6}{9^x - 6} = 1$$

$$\text{or } \frac{4 \times 3^x - 6}{9^x - 6} = 2$$

$$\text{or } 4y - 6 = 2y^2 - 12$$

(putting $3^x = y$)

$$\text{or } y^2 - 2y - 3 = 0$$

$$\text{or } y = -1, 3$$

$$\text{or } 3^x = 3$$

$$\text{or } x = 1$$

ILLUSTRATION 1.48

Solve $6(\log_x 2 - \log_4 x) + 7 = 0$.

Sol. $6(\log_x 2 - \log_4 x) + 7 = 0$

$$\text{or } 6 \left(\log_x 2 - \frac{1}{2} \log_2 x \right) + 7 = 0$$

$$\text{or } 6 \left(\frac{1}{y} - \frac{y}{2} \right) + 7 = 0$$

(where $y = \log_2 x$)

$$\text{or } 6 \left(\frac{2-y^2}{2y} \right) + 7 = 0$$

$$\text{or } 3 \left(\frac{2-y^2}{y} \right) + 7 = 0$$

$$\text{or } 6 - 3y^2 + 7y = 0$$

$$\text{or } 3y^2 - 7y - 6 = 0$$

$$\text{or } 3y^2 + 2y - 9y - 6 = 0$$

$$\text{or } (y-3)(3y+2) = 0$$

$$\text{or } y = 3 \text{ or } y = -2/3$$

$$\Rightarrow \log_2 x = 3 \text{ or } -2/3$$

$$\text{or } x = 8 \text{ or } x = 2^{-2/3}$$

ILLUSTRATION 1.49

Solve $4^{\log_2 \log x} = \log x - (\log x)^2 + 1$ (base is e).

Sol. $\log_2 \log x$ is meaningful if $x > 1$.

$$\text{Since } 4^{\log_2 \log x} = 2^{2 \log_2 \log x} = (2^{\log_2 \log x})^2$$

$$= (\log x)^2 \quad (a^{\log_a x} = x, a > 0, a \neq 1)$$

So the given equation reduces to $2(\log x)^2 - \log x - 1 = 0$.

1.12 Trigonometry

Therefore, $\log x = 1$ or $\log x = -1/2$. But for $x > 1$, $\log x > 0$.

Hence, $\log x = 1$, i.e., $x = e$.

ILLUSTRATION 1.50

Solve $4\log_{x/2}(\sqrt{x}) + 2\log_{4x}(x^2) = 3\log_{2x}(x^3)$.

Sol.
$$\frac{4\log_2 \sqrt{x}}{\log_2(x/2)} + \frac{2\log_2(x^2)}{\log_2(4x)} = \frac{3\log_2(x^3)}{\log_2(2x)}$$

$$\Rightarrow \frac{4 \times \frac{1}{2} \log_2(x)}{\log_2 x - 1} + \frac{4\log_2(x)}{2 + \log_2(x)} = \frac{9\log_2(x)}{1 + \log_2(x)}$$

Let $\log_2 x = t$. The given equation reduces to

$$\frac{2t}{t-1} + \frac{4t}{t+2} = \frac{9t}{t+1}$$

$$\Rightarrow t = 0 \text{ or } \frac{2}{t-1} + \frac{4}{t+2} = \frac{9}{t+1}$$

$$\Rightarrow \frac{2t+4+4t-4}{(t-1)(t+2)} = \frac{9}{t+1} \quad \text{or } t = 0$$

$$\Rightarrow t^2 + t - 6 = 0 \quad \text{or } t = 0$$

$$\Rightarrow (t+3)(t-2) = 0 \quad \text{or } t = 0$$

$$\Rightarrow t = 0, 2, \text{ or } -3$$

$$\Rightarrow x = 1, 4, 1/8$$

ILLUSTRATION 1.51

Solve $4^{\log_8 x} - 6x^{\log_8 2} + 2^{\log_3 27} = 0$.

Sol. Given equation is

$$(2^{\log_8 x})^2 - 6(2^{\log_8 x}) + 27^{\log_3 2} = 0$$

Let $2^{\log_8 x} = y$, we get $y^2 - 6y + 8 = 0$

$$\Rightarrow y = 4 \text{ or } 2$$

$$\text{If } 2^{\log_8 x} = 2^2$$

$$\Rightarrow \log_8 x = 2$$

$$\Rightarrow x = 81$$

$$\text{If } 2^{\log_8 x} = 2^1$$

$$\Rightarrow \log_8 x = 1$$

$$\Rightarrow x = 9$$

ILLUSTRATION 1.52

Solve $\frac{1}{4} x^{\log_2 \sqrt{x}} = (2 \cdot x^{\log_2 x})^{\frac{1}{4}}$.

Sol. Taking log on both sides with base 2, we get

$$\log_2 \left(\frac{1}{4} \right) + (\log_2 \sqrt{x})(\log_2 x) = \frac{1}{4} + \frac{1}{4} (\log_2 x)^2$$

$$\text{or } (\log_2 x)^2 = 9 \quad \text{or } \log_2 x = \pm 3$$

$$\therefore x = 8 \quad \text{or} \quad \frac{1}{8}$$

ILLUSTRATION 1.53

Solve $|x-1|^{(\log_{10} x)^2 - \log_{10} x^2} = |x-1|^3$.

Sol. We have

$$|x-1|^{(\log_{10} x)^2 - \log_{10} x^2} = |x-1|^3, x > 0, x \neq 1$$

$$\Rightarrow |x-1| = 1 \text{ or } (\log_{10} x)^2 - \log_{10} x^2 = 3$$

$$\Rightarrow x = 2 \text{ or } (\log_{10} x)^2 - 2\log_{10} x - 3 = 0$$

$$\Rightarrow x = 2 \text{ or } (\log_{10} x - 3)(\log_{10} x + 1) = 0$$

$$\Rightarrow x = 2 \text{ or } \log_{10} x = 3, \log_{10} x = -1$$

$$\Rightarrow x = 2, x = 1000 \text{ or } x = 0.11$$

CONCEPT APPLICATION EXERCISE 1.4

- Solve $\log_2(25^{x+3} - 1) = 2 + \log_2(5^{x+3} + 1)$.
- Solve $\log_4(2 \times 4^{x-2} - 1) + 4 = 2x$.
- Solve: $27^{\log_3 \sqrt{x^2-3x+1}} = \frac{\log_2(x-1)}{|\log_2(x-1)|}$
- Solve $\log_4(x-1) = \log_2(x-3)$.
- Solve $\log_6 9 - \log_9 27 + \log_8 x = \log_{64} x - \log_6 4$.
- Solve $\log_2(2\sqrt{17-2x}) = 1 - \log_{1/2}(x-1)$.
- Solve $3\log_4 x + 2\log_{4x} 4 + 3\log_{16x} 4 = 0$.
- Solve $(\log_3 x)(\log_5 9) - \log_x 25 + \log_3 2 = \log_3 54$.
- Solve $(x^{\log_{10} 3})^2 - (3^{\log_{10} x}) - 2 = 0$.
- Solve $x^{\log_4 x} = 2^{3(\log_4 x + 3)}$.
- Solve $2\log_{2+\sqrt{3}}(\sqrt{x^2+1}+x) + \log_{2-\sqrt{3}}(\sqrt{x^2+1}-x) = 3$
- Prove that the equation $x^{\log_{\sqrt{x}} 2x} = 4$ has no solution.

ANSWERS

- $x = -2$
- $x = 2$
- $x = 3$
- $x = 5$
- $x = \frac{1}{8}$
- $x = 4$
- $x = \frac{1}{2}, \frac{1}{8}$
- $x = \frac{1}{\sqrt{5}}, 25$
- $x = 10^{\log_3 2}$
- $x = 64, \frac{1}{8}$
- $x = \sqrt{3}$

LOGARITHMIC INEQUALITIES

STANDARD LOGARITHMIC INEQUALITIES

- If $\log_a x > \log_a y \Rightarrow \begin{cases} x > y, \text{ if } a > 1 \\ 0 < x < y, \text{ if } 0 < a < 1 \end{cases}$
- If $\log_a x > y \Rightarrow \begin{cases} x > a^y, \text{ if } a > 1 \\ 0 < x < a^y, \text{ if } 0 < a < 1 \end{cases}$
- $\log_a x > 0 \Rightarrow x > 1$ and $a > 1$
or $0 < x < 1$ and $0 < a < 1$

Frequently Used Inequalities

- $(x-a)(x-b) < 0$ ($a < b$) $\Rightarrow a < x < b$
- $(x-a)(x-b) > 0$ ($a < b$) $\Rightarrow x < a$ or $x > b$
- $|x| < a \Rightarrow -a < x < a$
- $|x| > a \Rightarrow x < -a$ or $x > a$

ILLUSTRATION 1.54Solve $\log_2(x-1) > 4$.**Sol.** $\log_2(x-1) > 4$ or $x-1 > 2^4$ or $x > 17$ **ILLUSTRATION 1.55**Solve $\log_3(x-2) \leq 2$.**Sol.** $\log_3(x-2) \leq 2$ or $0 < x-2 \leq 3^2$ or $2 < x \leq 11$ **ILLUSTRATION 1.56**Solve $\log_{0.3}(x^2-x+1) > 0$.**Sol.** $\log_{0.3}(x^2-x+1) > 0$ or $0 < x^2-x+1 < (0.3)^0$ or $0 < x^2-x+1 < 1$ $\Rightarrow x^2-x+1 > 0$ and $x^2-x < 0$ or $x(x-1) < 0$ $\Rightarrow 0 < x < 1$ (as $x^2-x+1 = (x-1/2)^2 + 3/4 > 0$, for all real x)**ILLUSTRATION 1.57**Solve $1 < \log_2(x-2) \leq 2$.**Sol.** $1 < \log_2(x-2) \leq 2$ $\Rightarrow 2^1 < x-2 \leq 2^2$ $\Rightarrow 4 < x \leq 6$ **ILLUSTRATION 1.58**Solve $\log_2|x-1| < 1$.**Sol.** $\log_2|x-1| < 1$ $\Rightarrow 0 < |x-1| < 2^1$ $\Rightarrow -2 < x-1 < 2$ and $x-1 \neq 0$ $\Rightarrow -1 < x < 3$ and $x \neq 1$ $\Rightarrow x \in (-1, 3) - \{1\}$ **ILLUSTRATION 1.59**Solve $\log_{0.2}|x-3| \geq 0$.**Sol.** $\log_{0.2}|x-3| \geq 0$ $\Rightarrow 0 < |x-3| \leq (0.2)^0$ $\Rightarrow 0 < |x-3| \leq 1$ $\Rightarrow -1 \leq x-3 \leq 1$ and $x-3 \neq 0$ $\Rightarrow 2 \leq x \leq 4$ and $x \neq 3$ $\Rightarrow x \in [2, 4] - \{3\}$ **ILLUSTRATION 1.60**Solve $\log_2 \frac{x-1}{x-2} > 0$.**Sol.** $\log_2 \frac{x-1}{x-2} > 0$ or $\frac{x-1}{x-2} > 2^0$ or $\frac{x-1}{x-2} > 1$ or $\frac{x-1}{x-2} - 1 > 0$ or $\frac{x-1-x+2}{x-2} > 0$ or $\frac{1}{x-2} > 0$ or $x > 2$ **ILLUSTRATION 1.61**Solve $\log_{0.5} \frac{3-x}{x+2} < 0$.**Sol.** $\log_{0.5} \frac{3-x}{x+2} < 0$ or $\frac{3-x}{x+2} > (0.5)^0$ or $\frac{3-x}{x+2} > 1$ or $\frac{3-x}{x+2} - 1 > 0$ or $\frac{3-x-x-2}{x+2} > 0$ or $\frac{2x-1}{x+2} < 0$ or $-2 < x < 1/2$ **ILLUSTRATION 1.62**Solve $\log_3(2x^2+6x-5) > 1$.**Sol.** $\log_3(2x^2+6x-5) > 1$ or $2x^2+6x-5 > 3^1$ or $2x^2+6x-8 > 0$ or $x^2+3x-4 > 0$ or $(x-1)(x+4) > 0$ $\Rightarrow x < -4$ or $x > 1$ **ILLUSTRATION 1.63**Solve $\log_{0.04}(x-1) \geq \log_{0.2}(x-1)$.**Sol.** $\log_{0.04}(x-1) \geq \log_{0.2}(x-1)$ or $\log_{(0.2^2)}(x-1) \geq \log_{0.2}(x-1)$ or $\frac{1}{2} \log_{0.2}(x-1) \geq \log_{0.2}(x-1)$ or $\log_{0.2}(x-1) \geq 2 \log_{0.2}(x-1)$ or $\log_{0.2}(x-1) \geq \log_{0.2}(x-1)^2$ or $(x-1) \leq (x-1)^2$ or $(x-1)^2 - (x-1) \geq 0$ or $(x-1)(x-1-1) \geq 0$ or $(x-1)(x-2) \geq 0$ or $x \leq 1$ or $x \geq 2$ Also, $x > 1$;Hence, $x \geq 2$.

ILLUSTRATION 1.64Solve $\log_{x+3}(x^2 - x) < 1$.**Sol.** $\log_{x+3}(x^2 - x) < 1$

$$x(x-1) > 0$$

$$\Rightarrow x > 1 \text{ or } x < 0$$

$$\text{If } x+3 > 1$$

$$\Rightarrow x > -2$$

$$\text{then } x^2 - x < x+3$$

$$\text{or } x^2 - 2x - 3 < 0$$

$$\text{or } (x-3)(x+1) < 0$$

Hence, $x \in (-1, 0) \cup (1, 3)$ [using (i) and (ii)]

$$\text{If } 0 < x+3 < 1$$

$$\therefore -3 < x < -2$$

$$\text{then } x^2 - x > x+3$$

$$\text{or } x^2 - 2x - 3 > 0$$

$$\text{or } (x-3)(x+1) > 0$$

Hence, $x \in (-3, -2)$ [using (i) and (iii)]Thus, $x \in (-1, 0) \cup (1, 3) \cup (-3, -2)$ **ILLUSTRATION 1.65**Solve $2 \log_3 x - 4 \log_x 27 \leq 5$.**Sol.** Let $\log_3 x = y$

$$\Rightarrow x = 3^y$$

Therefore, the given inequality becomes $2 \log_3 x - 12 \log_x 3 \leq 5$

$$\text{or } 2y - \frac{12}{y} \leq 5$$

$$\text{or } 2y^2 - 5y - 12 \leq 0$$

$$\text{or } (2y+3)(y-4) \leq 0$$

$$\Rightarrow y \in \left[-\frac{3}{2}, 4\right]$$

$$\Rightarrow -\frac{3}{2} \leq \log_3 x \leq 4$$

$$\Rightarrow 3^{-3/2} \leq x \leq 81$$

ILLUSTRATION 1.66Solve $\log_{x+\frac{1}{x}} \left(\log_2 \frac{x-1}{x+2} \right) > 0$.**Sol.** $\log_{x+\frac{1}{x}} \left(\log_2 \frac{x-1}{x+2} \right) > 0$

$$\therefore \frac{x-1}{x+2} > 0$$

$$\Rightarrow x < -2 \text{ or } x > 1$$

Also $x + \frac{1}{x} > 0$. Therefore,

$$\frac{x^2+1}{x} > 0$$

$$\Rightarrow x > 0$$

From Eqs. (i) and (ii), $x > 1$.

$$\text{Now } \log_{x+\frac{1}{x}} \left(\log_2 \frac{x-1}{x+2} \right) > 0$$

$$\Rightarrow \log_2 \frac{x-1}{x+2} > 1$$

$$(\because x + \frac{1}{x} > 1 \text{ for } x > 1)$$

$$\text{or } \frac{x-1}{x+2} > 2$$

$$\text{or } \frac{x+5}{x+2} < 0$$

Hence, $x \in (-5, -2)$, which is not possible as $x > 1$.**ILLUSTRATION 1.67**Solve $\log_{\log_2 \left(\frac{x}{2} \right)} (x^2 - 10x + 22) > 0$.**Sol.** $\log_{\log_2 \left(\frac{x}{2} \right)} (x^2 - 10x + 22) > 0$

We must have

$$(i) \quad x^2 - 10x + 22 > 0$$

$$\Rightarrow x \in (-\infty, 5 - \sqrt{3}) \cup (5 + \sqrt{3}, \infty)$$

$$(ii) \quad \frac{x}{2} > 0$$

$$\Rightarrow x > 0$$

Case (i):

$$0 < \log_2 \left(\frac{x}{2} \right) < 1$$

$$\Rightarrow 1 < \frac{x}{2} < 2$$

$$\text{or } 2 < x < 4$$

Therefore, from Eq. (i), $x^2 - 10x + 22 < 1$

$$\text{or } x^2 - 10x + 21 < 0$$

$$\Rightarrow 3 < x < 7$$

From Eqs. (ii), (iii), (iv), and (v), the common solution

$$3 < x < 5 - \sqrt{3}$$

Case (ii):

$$\log_2 \left(\frac{x}{2} \right) > 1$$

$$\text{or } \frac{x}{2} > 2 \text{ or } x > 4$$

Therefore, from Eqs. (i), $x^2 - 10x + 22 > 1$

$$\text{or } x^2 - 10x + 21 > 0$$

$$\text{or } x < 3 \text{ or } x > 7$$

From Eqs. (ii), (iii), (vi), and (vii), the common solution is $x \in (7, \infty)$ Hence, $x \in (3, 5 - \sqrt{3}) \cup (7, \infty)$

ILLUSTRATION 1.68

Solve $\log_{0.1} \left(\log_2 \frac{x^2+1}{|x-1|} \right) < 0$.

Sol. $\log_{0.1} \left(\log_2 \frac{x^2+1}{|x-1|} \right) < 0$

or $\log_2 \left(\frac{x^2+1}{|x-1|} \right) > 1$

or $\frac{x^2+1}{|x-1|} > 2$

Case (i): $x > 1$

$$\frac{x^2+1}{x-1} > 2$$

or $x^2+1 > 2x-2$

or $x^2-2x+3 > 0$, which is always true

$$\Rightarrow x > 1$$

Case (ii): $x < 1$

$$\frac{x^2+1}{1-x} > 2$$

or $x^2+1 > 2-2x$

or $x^2+2x-1 > 0$

$$\Rightarrow x < -1-\sqrt{2} \text{ or } x > 1+\sqrt{2}$$

Hence $x \in (-\infty, -1-\sqrt{2}) \cup (\sqrt{2}-1, 1) \cup (1, \infty)$.

ILLUSTRATION 1.69

Solve $\frac{x-1}{\log_3(9-3^x)-3} \leq 1$.

Sol. We have $\frac{x-1}{\log_3(9-3^x)-3} \leq 1$

 For this, we must have $9-3^x > 0$ or $3^x < 9$ or $x < 2$

The given expression can be expressed as:

$$\frac{(x-1)}{\log_3(9-3^x)-\log_3 27} \leq 1$$

$$\Rightarrow \frac{(x-1)}{\log_3 \left(\frac{9-3^x}{27} \right)} \leq 1$$

$$\Rightarrow (x-1) \cdot \log \left(\frac{9-3^x}{27} \right) \leq 1$$

$$\Rightarrow \log \left(\frac{9-3^x}{27} \right) (3^{x-1}) \leq 1$$

As $x < 2$, $0 < \frac{9-3^x}{27} < 1$

We have $3^{x-1} \geq \frac{9-3^x}{27}$

$$\Rightarrow 9 \times 3^x \geq 9-3^x$$

$$\Rightarrow 10 \times 3^x \geq 9$$

$$\Rightarrow x \geq \log_3 0.9$$

 Therefore, $x \in [\log_3 0.9, 2)$
ILLUSTRATION 1.70

Solve $\left(\frac{1}{2} \right)^{\log_{10} a^2} + 2 > \frac{3}{2^{\log_{10}(-a)}}$.

Sol. $\left(\frac{1}{2} \right)^{\log_{10} a^2} + 2 > \frac{3}{2^{\log_{10}(-a)}}$

 For this to be true, we must have $a < 0$

$$\Rightarrow \frac{1}{2^{2 \log_{10}(-a)}} + 2 > \frac{3}{2^{\log_{10}(-a)}}$$

 Putting $y = \frac{1}{2^{\log_{10}(-a)}}$, we get

$$y^2 + 2 > 3y$$

$$\Rightarrow y^2 - 3y + 2 > 0$$

$$\Rightarrow (y-1)(y-2) > 0$$

$$\Rightarrow y < 1 \text{ or } y > 2$$

$$\Rightarrow 2^{-\log_{10}(-a)} < 1 \text{ or } 2^{-\log_{10}(-a)} > 2$$

$$\Rightarrow -\log_{10}(-a) < 0 \text{ or } -\log_{10}(-a) > 1$$

$$\Rightarrow \log_{10}(-a) > 0 \text{ or } \log_{10}(-a) < -1$$

$$\Rightarrow -a > 1 \text{ or } -a < 10^{-1}$$

$$\Rightarrow a < -1 \text{ or } a > -0.1$$

$$\Rightarrow a \in (-\infty, -1) \cup (-0.1, 0)$$

CONCEPT APPLICATION EXERCISE 1.5

1. Solve $\log_3 |x| > 2$.

2. Solve $\log_2 \frac{x-4}{2x+5} < 1$.

3. Solve $\log_{10}(x^2-2x-2) \leq 0$.

4. Let $f(x) = \sqrt{\log_{10} x^2}$. Find the set of all values of x for which $f(x)$ is real.

5. Solve $2^{\log_2(x-1)} > x+5$.

6. Solve $\log_2 |4-5x| > 2$.

7. Solve $\log_{0.2} \frac{x+2}{x} \leq 1$.

8. Solve $\log_{1/2}(x^2-6x+12) \geq -2$.

9. Solve $(0.5)^{\log_1 \log_{10} \left(x^2 - \frac{4}{5} \right)} > 1$.

 10. Find the values of x for which the function

$$f(x) = \sqrt{\log_{1/2} \frac{x-1}{x+5}}$$
 is defined.

11. Solve $\log_{1-x}(x-2) \geq -1$.
 12. Solve $\log_3(x+2)(x+4) + \log_{1/3}(x+2) < \frac{1}{2} \log_{\sqrt{3}} 7$.
 13. Solve $\log_x(x^2-1) \leq 0$.

ANSWERS

1. $x < -9$ or $x > 9$ 2. $x \in \left(-\infty, -\frac{14}{3}\right) \cup (4, \infty)$
 3. $x \in [-1, 1 - \sqrt{3}) \cup (1 + \sqrt{3}, 3]$ 4. $x \leq -1$ or $x \geq 1$
 5. no solution 6. $x \in (-\infty, 0) \cup \left(\frac{8}{5}, +\infty\right)$
 7. $x \in \left(-\infty, -\frac{5}{2}\right] \cup (0, +\infty)$ 8. $x \in [2, 4]$
 9. $x \in \left(-\frac{3}{\sqrt{5}}, -1\right) \cup \left(1, \frac{3}{\sqrt{5}}\right)$ 10. $x \in (1, \infty)$
 11. no solution 12. $x \in (-2, 3)$ 13. $x \in (1, \sqrt{2})$

FINDING LOGARITHM

To calculate the logarithm of any positive number in decimal form, we always express the given positive number in the decimal form as the product of an integral power of 10 and a number between 1 and 10, i.e., any positive number n in the decimal form is written as

$$n = m \times 10^p$$

where p is an integer and $1 \leq m < 10$. This is called the standard form of n .

CHARACTERISTIC AND MANTISSA OF LOGARITHM

Let n be a positive real number and $m \times 10^p$ be the standard form of n . Then $n = m \times 10^p$, where p is an integer and m is a real number between 1 and 10, i.e., $1 \leq m < 10$. Thus,

$$\begin{aligned}\log_{10} n &= \log_{10} (m \times 10^p) \\ &= \log_{10} m + \log_{10} 10^p \\ &= \log_{10} m + p \log_{10} 10 \\ &= p + \log_{10} m\end{aligned}$$

$$\text{Thus, } \log_{10} 1 \leq \log_{10} m < \log_{10} 10$$

$$\Rightarrow 0 \leq \log_{10} m < 1$$

Thus, the logarithm of a positive real number n consists of two parts:

- The integral part p , called characteristic, which is positive, negative, or zero.
- The decimal part $\log m$, called mantissa, which is a real number between 0 and 1.

$$\text{Thus, } \log n = \text{Characteristic} + \text{Mantissa}$$

Note that it is only the characteristic that changes when the decimal point is moved. An advantage of using the base 10 is thus revealed: if the characteristic is known, the decimal point may easily be placed. If the number is known, the characteristic may be determined by inspection, i.e., by observing the location of the decimal point.

Although an understanding of the relation of the characteristic to the powers of 10 is necessary for a thorough comprehension of logarithms, the characteristic may be determined mechanically by the application of the following rules:

- For a number greater than 1, the characteristic is positive and is one less than the number of digits to the left of the decimal point in the number.
- For a positive number less than 1, the characteristic is negative and has an absolute value one more than the number of zeros between the decimal point and the first nonzero digit of the number.

ILLUSTRATION 1.71

Write the characteristic of each of the following numbers by using their standard forms:

- (i) 1235.5 (ii) 346.41
 (iii) 62.723 (iv) 7.12345
 (v) 0.35792 (vi) 0.034239
 (vii) 0.002385 (viii) 0.0009468

Sol.

Number	Standard form	Characteristic
1235.5	1.2355×10^3	3
346.41	3.4641×10^2	2
62.723	6.2723×10^1	1
7.12345	7.12345×10^0	0
0.35792	3.5792×10^{-1}	-1
0.034239	3.4239×10^{-2}	-2
0.002385	2.385×10^{-3}	-3
0.0009468	9.468×10^{-4}	-4

FINDING MANTISSA USING LOG TABLE

The logarithm table is used to find the mantissa of logarithms of numbers. It contains 90 rows and 20 columns. Every row begins with a two-digit number 10, 11, 12, ..., 98, 99 and every column is headed by a one-digit number 0, 1, 2, 3, ..., 9. On the right of the table, we have a big column which is divided into nine sub-columns headed by the digits 1, 2, 3, ..., 9. This column is called the column of mean differences.

Note that the position of the decimal point in a number is immaterial for finding the mantissa. To find the mantissa of a number, we consider the first four digits from the left most side of the number. If the number in the decimal form is less than one and it has four or more consecutive zeros to the right of the decimal point, then its mantissa is calculated with the help of the number formed by digits beginning with the first nonzero digit. For example, to find the mantissa of 0.000032059, we consider the number 3205. If the given number has only one digit, we replace it by a two-digit number obtained by adjoining zero to the right of the number. Thus, 2 is to be replaced by 20 for finding the mantissa.

The digits used to compute the mantissa of a given number are called its significant digits.

ILLUSTRATION 1.72

Write the significant digits in each of the following numbers to compute the mantissa of their logarithms :

- (i) 3.239 (ii) 8 (iii) 0.9
 (iv) 0.02 (v) 0.0367 (vi) 89
 (vii) 0.0003 (viii) 0.00075

Sol.

Number	Significant digits
3.239	3239
8	80
0.9	90
0.02	20
0.0367	367
89	89
0.0003	30
0.00075	75

NEGATIVE CHARACTERISTICS

When a characteristic is negative, such as -2 , we do not perform the subtraction, because this would involve a negative mantissa. There are several ways of indicating a negative characteristic. Mantissas as presented in the table in the appendix are always positive, and the sign of the characteristic is indicated separately. For example, consider $\log 0.023 = \bar{2}.36173$. Here the bar over 2 indicates that only the characteristic is negative, i.e., the logarithm is $-2 + 0.36173$.

ILLUSTRATION 1.73

Find the mantissa of the logarithm of the number 5395.

Sol. To find the mantissa of $\log 5395$ in the log table, we first look into the row starting with 53. In this row, look at the number in the column headed by 9. The number is 7316.

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

Now, move to the column of mean differences and look under the column headed by 5 in the row corresponding to 53. We see the number 4 there. Add this number 4 to 7316 to get 7320. This is the required mantissa of $\log 5395$.

If we wish to find $\log 5395$, then we compute its characteristic also. Clearly, the characteristic is 3. So, $\log 5395 = 3.7320$.

ILLUSTRATION 1.74

Find the mantissa of the logarithm of the number 0.002359.

Sol. The first four digits beginning with the first nonzero digit on the right of the decimal point form the number 2359. To find the mantissa of $\log (0.002359)$, we first look in the row starting with 23. In this row, look at the number in the column headed by 5. The number is 3711.

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	15	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15

1.18 Trigonometry

Now, move to the column of mean differences and look under the column headed by 9 in the row corresponding to 23. We see the number 17 there.

Add this number to 3711. We get the number 3728. This is the required mantissa of $\log(0.002359)$.

Mantissa of $\log 23.598$, $\log 2.3598$, and 0.023598 is the same (only characteristics are different).

ILLUSTRATION 1.75

Use logarithm tables to find the logarithm of the following numbers:

(i) 25795

(ii) 25.795

Sol.

- (i) The characteristic of the logarithm of 25795 is 4.

To find the mantissa of the logarithm of 25795, we take the first four digits.

The number formed by the first four digits is 2579. Now, we look in the row starting with 25. In this row, look at the number in the column headed by 7. The number is 4099. Now, move to the column of mean differences and look under the column headed by 9 in the row corresponding to 25. We see that the number there is 15.

Add this number to 4099. We get the number 4114. This is the required mantissa. Hence,

$$\log(25795) = 4.4114$$

- (ii) The characteristic of the logarithm of 25.795 is 1, because there are two digits to the left of the decimal point. The mantissa is the same as in the above question. Hence,

$$\log 25.795 = 1.4114.$$

Similarly, $\log 2.5795 = 0.4114$.

$$\text{and } \log(0.25795) = -1 + 0.4114 = \bar{1}.4114$$

Here $-1 + 0.4114$ cannot be written as -1.4114 , as -1.4114 is a negative number of magnitude 1.4114, whereas $-1 + 0.4114$ is equal to -0.5886 . In order to avoid this confusion, we write $\bar{1}$ for -1 ; thus,

$$\log(0.25795) = \bar{1}.4114.$$

ANTILOGARITHM

The positive number n is called the antilogarithm of a number m if $\log n = m$. If n is antilogarithm of m , we write $n = \text{antilog } m$. For example,

$$(i) \log 100 = 2 \quad \Leftrightarrow \quad \text{antilog } 2 = 100$$

$$(ii) \log 431.5 = 2.6350 \quad \Leftrightarrow \quad \text{antilog } (2.6350) = 431.5$$

$$(iii) \log 0.1257 = \bar{1}.993 \quad \Leftrightarrow \quad \text{antilog } (\bar{1}.993) = 0.1257$$

To find the antilog of a given number, we use the antilogarithm tables given as an appendix at the end of the book. To find n , when $\log n$ is given, we use only the mantissa part. The characteristic is used only in determining the number of digits in the integral part or the number of zeros on the right side of the decimal point in the required number.

TO FIND ANTILOG OF A NUMBER

- Step I:** Determine whether the decimal part of the given number is positive or negative. If it is negative, make it positive by adding 1 to the decimal part and subtracting 1 from the integral part. For example, in -2.5983 , the decimal part is -0.5983 which is negative. So, write

$$\begin{aligned} -2.5983 &= -2 - 0.5983 \\ &= -2 - 1 + 1 - 0.5983 \\ &= -3 + 0.4017 \\ &= \bar{3}.4017 \end{aligned}$$

- Step II:** In the antilogarithm table, look into the row containing the first two digits in the decimal part of the given number.

- Step III:** In the row obtained in step II, look at the number in the column headed by the third digit in the decimal part.

- Step IV:** In the row chosen in step III, move in the column of mean differences and look at the number in the column headed by the fourth digit in the decimal part. Add this number to the number obtained in step III.

- Step V:** Obtain the integral part (characteristic) of the given number.

If the characteristic is positive and is equal to n , then insert a decimal point after $(n + 1)$ digits in the number obtained in step IV.

If $n > 4$, then write zeros on the right side to get $(n + 1)$ digits.

If the characteristic is negative and is equal to $-n$, then on the right side of decimal point write $(n + 1)$ consecutive zeros and then write the number obtained in step IV.

ILLUSTRATION 1.76

Find the antilogarithm of each of the following:

- | | | |
|----------------------|--------------------|---------------------|
| (i) 2.7523 | (ii) 3.7523 | (iii) 5.7523 |
| (iv) 0.7523 | (v) $\bar{1}.7523$ | (vi) $\bar{2}.7523$ |
| (vii) $\bar{3}.7523$ | | |

Sol.

- (i) The mantissa of 2.7523 is positive and is equal to 0.7523. Now, in antilog table, look into the row starting with 75 this row, look at the number in the column headed by 3. The number is 5649. Now in the same row move in the column of mean differences and look at the number in the column headed by 2. The number there is 4. Add this number to 5649 to get 5653.

The characteristic is 2. So, the decimal point is put after three digits to get 565.3. Hence, $\text{antilog}(2.7523) = 565.3$.

- (ii) The mantissa of 3.7523 is the same as the mantissa of the number in Step (i), but the characteristic is 3. Hence, $\text{antilog}(3.7523) = 5653.0$.

- (iii) The mantissa of 5.7523 is the same as the mantissa of the number in Step (i), but the characteristic is 5. Hence, $\text{antilog}(5.7523) = 565300.0$.

(iv) Proceeding as above, we have $\text{antilog}(0.7523) = 5.653$.

(v) In this case, the characteristic is $\bar{1}$, i.e., -1 . Hence,
 $\text{antilog}(\bar{1}.7523) = 0.5653$

(vi) In this case, the characteristic is $\bar{2}$, i.e., -2 . So, we write one zero on the right side of the decimal point. Hence,
 $\text{antilog}(\bar{2}.7523) = 0.05653$

(vii) Proceeding as above, $\text{antilog}(\bar{3}.7523) = 0.005623$.

ILLUSTRATION 1.77

Evaluate $\sqrt[3]{72.3}$, if $\log 0.723 = \bar{1}.8591$.

Sol. Let $x = \sqrt[3]{72.3}$. Then

$$\log x = \log(72.3)^{1/3}$$

$$\text{or } \log x = \frac{1}{3} \log 72.3$$

$$\text{or } \log x = \frac{1}{3} \times 1.8591$$

$$\Rightarrow \log x = 0.6197$$

$$\text{or } x = \text{antilog}(0.6197) \\ = 4.166 \quad (\text{using antilog table})$$

ILLUSTRATION 1.78

Using logarithms, find the value of 6.45×981.4 .

Sol. Let $x = 6.45 \times 981.4$. Then,

$$\log x = \log(6.45 \times 981.4)$$

$$= \log 6.45 + \log 981.4$$

$$= 0.8096 + 2.9919$$

$$= 3.8015$$

$$\therefore x = \text{antilog}(3.8015) = 6331 \quad (\text{using antilog table})$$

ILLUSTRATION 1.79

Let $x = (0.15)^{20}$. Find the characteristic and mantissa of the logarithm of x to the base 10. Assume $\log_{10} 2 = 0.301$ and $\log_{10} 3 = 0.477$.

Sol. $\log x = \log(0.15)^{20} = 20 \log\left(\frac{15}{100}\right)$

$$= 20[\log 15 - 2]$$

$$= 20[\log 3 + \log 5 - 2]$$

$$= 20[\log 3 + 1 - \log 2 - 2] \quad \left(\because \log_{10} 5 = \log_{10} \frac{10}{2}\right)$$

$$= 20[-1 + \log 3 - \log 2]$$

$$= 20[-1 + 0.477 - 0.301]$$

$$= -20 \times 0.824$$

$$= -16.48$$

$$= \bar{17}.52$$

Hence, Characteristic = -17 and Mantissa = 0.52

ILLUSTRATION 1.80

If $\log_{10} 2 = 0.30103$, $\log_{10} 3 = 0.47712$, then find the number of digits in $3^{12} \times 2^8$.

Sol. Let $y = 3^{12} \times 2^8$

$$\begin{aligned} \Rightarrow \log_{10} y &= 12 \log_{10} 3 + 8 \log_{10} 2 \\ &= 12 \times 0.47712 + 8 \times 0.30103 \\ &= 5.72544 + 2.40824 \\ &= 8.13368 \end{aligned}$$

$$\therefore \text{Number of digits in } y = 8 + 1 = 9.$$

ILLUSTRATION 1.81

In the 2001 census, the population of India was found to be 8.7×10^7 . If the population increases at the rate of 2.5% every year, what would be the population in 2011?

Sol. Here, $P_0 = 8.7 \times 10^7$, $r = 2.5$, and $n = 10$.

Let P be the population in 2011. Then,

$$\begin{aligned} P &= P_0 \left(1 + \frac{r}{100}\right)^n \\ &= 8.7 \times 10^7 \left(1 + \frac{2.5}{100}\right)^{10} \\ &= 8.7 \times 10^7 (1.025)^{10} \end{aligned}$$

Taking log of both sides, we get

$$\begin{aligned} \log P &= \log [8.7 \times 10^7 (1.025)^{10}] \\ &= \log 8.7 + \log 10^7 + \log (1.025)^{10} \\ &= \log 8.7 + 7 \log 10 + 10 \log (1.025) \\ &= 0.9395 + 7 + 0.1070 \\ &= 8.0465 \end{aligned}$$

$$\Rightarrow P = \text{antilog}(8.0465) = 1.113 \times 10^8 \quad (\text{using antilog table})$$

ILLUSTRATION 1.82

Find the compound interest on ₹ 12000 for 10 years at the rate of 12% per annum compounded annually.

Sol. We know that the amount A at the end of n years at rate of $r\%$ per annum when the interest is compounded annually is given by

$$A = P \left(1 + \frac{r}{100}\right)^n$$

Here, $P = ₹ 12000$, $r = 12$, and $n = 10$.

$$\begin{aligned} \therefore A &= ₹ \left[12000 \left(1 + \frac{12}{100}\right)^{10}\right] \\ &= ₹ \left[12000 \left(1 + \frac{3}{25}\right)^{10}\right] \\ &= ₹ \left[12000 \left(\frac{25+3}{25}\right)^{10}\right] \\ &= ₹ \left[12000 \left(\frac{28}{25}\right)^{10}\right] \end{aligned}$$

$$\text{Now, } A = ₹ 12000 \left(\frac{28}{25}\right)^{10}$$

1.20 Trigonometry

$$\Rightarrow \log A = \log 12000 + 10 (\log 28 - \log 25)$$

$$= 4.0792 + 10 (1.4472 - 1.3979)$$

$$= 4.0792 + 0.493 = 4.5722$$

$$\Rightarrow A = \text{antilog}(4.5722) = 37350.$$

So, the amount after 10 years is ₹ 37350.

Hence, Compound interest = ₹ (37350 - 12000) = ₹ 25350

ILLUSTRATION 1.83

If P is the number of natural numbers whose logarithms to the base 10 have the characteristic p and Q is the number of natural numbers logarithms of whose reciprocals to the base 10 have the characteristic $-q$, then find the value of $\log_{10} P - \log_{10} Q$.

Sol. $10^p \leq P < 10^{p+1}$

$$\Rightarrow P = 10^{p+1} - 10^p$$

$$\Rightarrow P = 9 \times 10^p$$

Similarly, $10^{q-1} < Q \leq 10^q$

$$\Rightarrow Q = 10^q - 10^{q-1} = 10^{q-1}(10 - 1) = 9 \times 10^{q-1}$$

$$\therefore \log_{10} P - \log_{10} Q = \log_{10} (P/Q)$$

$$= \log_{10} 10^{p-q+1}$$

$$= p - q + 1$$

ILLUSTRATION 1.84

Let L denote $\text{antilog}_{32} 0.6$ and M denote the number of positive integers which have the characteristic 4, when the base of log is 5, and N denote the value of $49^{(1-\log_5 2)} + 5^{-\log_5 4}$. Find the value of LM/N .

Sol. $L = \text{antilog}_{32} 0.6 = (32)^{6/10} = 2^{(5 \times 6)/10} = 2^3 = 8$

$$M = \text{Integer from } 625 \text{ to } 3125 = 2500$$

$$N = 49^{(1-\log_5 2)} + 5^{-\log_5 4}$$

$$= 49 \times 7^{-2\log_5 2} + 5^{-\log_5 4}$$

$$= 49 \times \frac{1}{4} + \frac{1}{4} = \frac{50}{4} = \frac{25}{2}$$

$$\therefore \frac{LM}{N} = \frac{8 \times 2500 \times 2}{25} = 1600$$

CONCEPT APPLICATION EXERCISE 1.6

- If $\log_{10} 2 = 0.3010$ and $\log_{10} 3 = 0.477$, then find the number of digits in the following numbers:
(a) 3^{40} (b) $2^{32} \times 5^{25}$ (c) 24^{24}
- If characteristic of three numbers a , b and c are 5, -3 and 2, respectively, then find the maximum number of digits in $N = abc$.
- There are 3 numbers a , b and c such that $\log_{10} a = 5.71$, $\log_{10} b = 6.23$ and $\log_{10} c = 7.89$. Find the number of digits before decimal in $\frac{ab^2}{c}$.
- Rupees 10,000 is invested at 6% interest compounded annually. How long will it take to accumulate Rs 20,000 in the account?

- An initial number of bacteria presented in a culture is 10000. This number doubles every 30 minutes. How long will it take to bacteria to reach the number 100000?
- Charles Richter defined the magnitude of an earthquake to be $M = \log_{10} \frac{I}{S}$, where I is the intensity of the earthquake

(measured by the amplitude of a seismograph reading taken 100 km from the epicentre of the earthquake) and S is the intensity of a "standard earthquake" (whose amplitude is 1 micron = 10^{-1} cm).

Each number increase on the Richter scale indicates an intensity ten times stronger. For example, an earthquake of magnitude 6 is ten times stronger than an earthquake of magnitude 5. An earthquake of magnitude 7 is 100 times stronger than an earthquake of magnitude 5. An earthquake of magnitude 8 is 1000 times stronger than an earthquake of magnitude 5.

The earthquake in city A registered 8.3 on the Richter scale. In the same year, another earthquake was recorded in city B that was four times stronger. What was the magnitude of the earthquake in city B ?

ANSWERS

- | | | | | |
|-------------|----------------|--------|-----------|-------|
| 1. (a) 20 | (b) 28 | (c) 34 | 2. 7 | 3. 11 |
| 4. 12 years | 5. 100 minutes | | 6. 8.9020 | |

Solved Examples

EXAMPLE 1.1

Find the number of solutions of equation $2^x + 3^x + 4^x - 5^x = 0$.

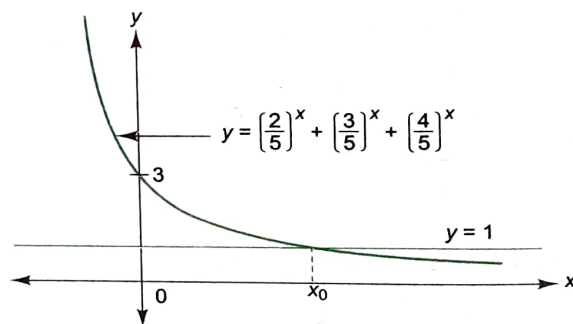
Sol. $2^x + 3^x + 4^x - 5^x = 0$

or $2^x + 3^x + 4^x = 5^x$

or $\left(\frac{2}{5}\right)^x + \left(\frac{3}{5}\right)^x + \left(\frac{4}{5}\right)^x = 1$

Now the number of solutions of the equation is equal to number of times, graphs of

$$y = \left(\frac{2}{5}\right)^x + \left(\frac{3}{5}\right)^x + \left(\frac{4}{5}\right)^x \text{ and } y = 1 \text{ intersect.}$$



From the graph, equation has only one solution, x_0 .

EXAMPLE 1.2

Let a, b, c, d be positive integers such that $\log_a b = 3/2$ and $\log_c d = 5/4$. If $(a - c) = 9$, then find the value of $(b - d)$.

Sol. $b = a^{3/2}$ and $d = c^{5/4}$

Let $a = x^2$ and $c = y^4, x, y \in \mathbb{N}$

$$\Rightarrow b = x^3, d = y^5$$

Given $a - c = 9$, then $x^2 - y^4 = 9$

$$\Rightarrow (x - y^2)(x + y^2) = 9$$

Hence, $x - y^2 = 1$ and $x + y^2 = 9$.

(No other combination in the set of positive integers will be possible.)

$$x = 5 \text{ and } y = 2$$

$$\therefore b - d = x^3 - y^5 = 125 - 32 = 93$$

EXAMPLE 1.3

If $a \geq b > 1$, then find the largest possible value of the expression $\log_a(a/b) + \log_b(b/a)$.

Sol. Let $x = \log_a(a/b) + \log_b(b/a)$
 $= \log_a a - \log_a b + \log_b b - \log_b a$
 $= 2 - (\log_b a + \log_a b)$
 $= -(\sqrt{\log_b a} - \sqrt{\log_a b})^2 \leq 0$

Hence, the maximum value is 0.

EXAMPLE 1.4

If $y = a^{\frac{1}{1 - \log_a x}}$ and $z = a^{\frac{1}{1 - \log_a y}}$, then prove that $x = a^{\frac{1}{1 - \log_a z}}$.

Sol. $\log_a y = \frac{1}{1 - \log_a x}$
 $\Rightarrow 1 - \log_a y = 1 - \frac{1}{1 - \log_a x}$
 $= \frac{-\log_a x}{1 - \log_a x}$

or $\frac{1}{1 - \log_a y} = \frac{1 - \log_a x}{-\log_a x} \quad \dots(i)$

But $z = a^{\frac{1}{1 - \log_a y}}$

$$\Rightarrow \log_a z = \frac{1}{1 - \log_a y} = -\frac{1}{\log_a x} + 1$$

or $\frac{1}{\log_a x} = 1 - \log_a z$

or $\log_a x = \frac{1}{1 - \log_a z}$

or $x = a^{\frac{1}{1 - \log_a z}}$

EXAMPLE 1.5

Solve $\sqrt{\log(-x)} = \log \sqrt{x^2}$ (base is 10).

Sol. Since the equation can be satisfied only for $x < 0$, we have $\sqrt{x^2} = |x| = -x$.

So, given equation reduces to

$$\sqrt{\log(-x)} = \log(-x)$$

or $\log(-x) = [\log(-x)]^2$

or $\log(-x)[1 - \log(-x)] = 0$

if $\log(-x) = 0 \Rightarrow -x = 1 \Rightarrow x = -1$

if $\log_{10}(-x) = 1 \Rightarrow -x = 10 \Rightarrow x = -10$

EXAMPLE 1.6

Solve $3^{(\log_3 x)^2 - \frac{9}{2} \log_3 x + 5} = 3\sqrt{3}$.

Sol. We have $(\log_3 x)^2 - \frac{9}{2} \log_3 x + 5 = \frac{3}{2}$

Putting $\log_3 x = y$, we have $y^2 - \frac{9}{2}y + 5 = \frac{3}{2}$

or $2y^2 - 9y + 7 = 0$,

i.e., $(2y - 7)(y - 1) = 0$

or $y = 7/2, 1$

Therefore, either $\log_3 x = 1$ or $\log_3 x = 7/2$

i.e., either $x = 9$ or $x = 9^{7/2} = 3^7$

EXAMPLE 1.7

Solve for x : $(2x)^{\log_b 2} = (3x)^{\log_b 3}$.

Sol. $(2x)^{\log_b 2} = (3x)^{\log_b 3}$

$$\Rightarrow \log_b 2 [\log 2 + \log x] = \log_b 3 [\log 3 + \log x]$$

or $(\log_b 2)(\log 2) - \log_b 3 \cdot \log 3 = (\log_b 3 - \log_b 2) \log x$

or $\frac{\log 2}{\log b} \cdot \log 2 - \frac{\log 3}{\log b} \cdot \log 3 = \left(\frac{\log 3}{\log b} - \frac{\log 2}{\log b} \right) \log x$

or $\frac{(\log 2)^2 - (\log 3)^2}{\log b} = \left(\frac{\log 3 - \log 2}{\log b} \right) \log x$

or $\log x = -(\log 3 + \log 2) = \log (6)^{-1}$

or $x = 1/6$

EXAMPLE 1.8

Solve the equations for x and y : $(3x)^{\log 3} = (4y)^{\log 4}$, $4^{\log x} = 3^{\log y}$.

Sol. $(3x)^{\log 3} = (4y)^{\log 4}$, $4^{\log x} = 3^{\log y}$

$$\Rightarrow (\log 3)(\log 3x) = (\log 4)(\log 4y)$$

and $(\log x)(\log 4) = (\log y)(\log 3)$

$$\Rightarrow (\log 3)(\log 3 + \log x) = (\log 4)(\log 4 + \log y)$$

and $(\log x)(\log 4) = (\log y)(\log 3)$

$$\Rightarrow (\log 3)(\log 3 + p) = (\log 4)(\log 4 + q)$$

and $p(\log 4) = q(\log 3)$ (where $p = \log x$ and $q = \log y$)

$$\Rightarrow (\log 3) \left(\log 3 + \frac{q \log 3}{\log 4} \right) = \log 4 (\log 4 + q)$$

(eliminating p)

$$\Rightarrow (\log 3)^2 - (\log 4)^2 = \frac{(\log 4)^2 - (\log 3)^2}{\log 4} q$$

$$\Rightarrow q = -\log 4$$

$$\Rightarrow \log y = \log 4^{-1}$$

$$\Rightarrow y = 1/4$$

$$\text{Now, } p(\log 4) = q(\log 3)$$

$$\text{or } p(\log 4) = -(\log 4)(\log 3)$$

$$\text{or } p = -\log 3$$

$$\text{or } \log x = \log 3^{-1}$$

$$\text{or } x = 1/3$$

EXAMPLE 1.9Solve $\log_2 2 + \log_2 x = -3/2$.**Sol.** Given equation is

$$\frac{1}{1 + \log_2 x} + \frac{\log_2 2x}{2} = -\frac{3}{2}$$

$$\text{or } \frac{1}{1 + \log_2 x} + \frac{1 + \log_2 x}{2} = -\frac{3}{2}$$

$$\text{Let } 1 + \log_2 x = y$$

$$\Rightarrow \frac{1}{y} + \frac{y}{2} = -\frac{3}{2}$$

$$\text{or } 2 - y^2 + 3y = 0$$

$$\text{or } y = -1 \text{ or } -2$$

$$\Rightarrow 1 + \log_2 x = -1 \text{ or } -2$$

$$\text{or } \log_2 x = -2 \text{ or } -3$$

$$\Rightarrow x = 2^{-2} \text{ or } 2^{-3}$$

EXAMPLE 1.10Solve $\log_{(2x+3)}(6x^2 + 23x + 21) + \log_{(3x+7)}(4x^2 + 12x + 9) = 4$.**Sol.** The given equation can be written as

$$\log_{(2x+3)}(2x+3)(3x+7) + \log_{(3x+7)}(2x+3)^2 = 4$$

$$\text{or } \log_{(2x+3)}(2x+3) + \log_{(2x+3)}(3x+7) + \frac{2}{\log_{(2x+3)}(3x+7)} = 4$$

Let $\log_{(2x+3)}(3x+7) = t$. Then

$$1 + t + \frac{2}{t} = 4$$

$$\text{or } t^2 - 3t + 2 = 0$$

$$\text{or } (t-1)(t-2) = 0$$

$$\Rightarrow t = 1, t = 2$$

(i) if $t = 1$, then

$$\log_{(2x+3)}(3x+7) = 1$$

$$\text{or } 3x+7 = 2x+3$$

Hence, $x = -4$, which is not possible as $2x+3 > 0$ and $3x+7 > 0$.(ii) if $t = 2$, then

$$\log_{(2x+3)}(3x+7) = 2$$

$$\text{or } 3x+7 = (2x+3)^2$$

$$\text{or } 4x^2 + 9x + 2 = 0$$

$$\text{or } (x+2)(4x+1) = 0$$

$$\Rightarrow x = -2 \text{ and } x = -\frac{1}{4}$$

Since $x = -2$ is not possible, there is only one solution $x = -1/4$.

Exercises

Single Correct Answer Type

- $\log_4 18$ is
 - a rational number
 - an irrational number
 - a prime number
 - none of these
- The number $N = 6 \log_{10} 2 + \log_{10} 31$ lies between two successive integers whose sum is equal to
 - 5
 - 7
 - 9
 - 10
- Given that $\log(2) = 0.3010\dots$, the number of digits in the number 2000^{2000} is
 - 6601
 - 6602
 - 6603
 - 6604
- If $(21.4)^a = (0.00214)^b = 100$, then the value of $\frac{1}{a} - \frac{1}{b}$ is
 - 0
 - 1
 - 2
 - 4
- The value of $\log ab - \log|b| =$
 - $\log a$
 - $\log|a|$
 - $-\log a$
 - none of these
- If a, b, c are consecutive positive integers and $\log(1+ac) = 2K$, then the value of K is
 - $\log b$
 - $\log a$
 - 2
 - 1
- If $\frac{a+\log_4 3}{a+\log_2 3} = \frac{a+\log_8 3}{a+\log_4 3} = b$, then b is equal to
 - $\frac{1}{2}$
 - $\frac{2}{3}$
 - $\frac{1}{3}$
 - $\frac{3}{2}$
- If $p > 1$ and $q > 1$ are such that $\log(p+q) = \log p + \log q$, then the value of $\log(p-1) + \log(q-1)$ is equal to
 - 0
 - 1
 - 2
 - none of these
- The value of $\frac{1+2\log_3 2}{(1+\log_3 2)^2} + (\log_6 2)^2$ is
 - 2
 - 3
 - 4
 - 1
- If $\log_4 5 = a$ and $\log_5 6 = b$, then $\log_3 2$ is equal to
 - $\frac{1}{2a+1}$
 - $\frac{1}{2b+1}$
 - $2ab+1$
 - $\frac{1}{2ab-1}$
- If $\log_{10} 2 = a$, $\log_{10} 3 = b$ then $\log_{0.72}(9.6)$ in terms of a and b is equal to
 - $\frac{2a+3b-1}{5a+b-2}$
 - $\frac{5a+b-1}{3a+2b-2}$
 - $\frac{3a+b-2}{2a+3b-1}$
 - $\frac{2a+5b-2}{3a+b-1}$
- There exists a natural number N which is 50 times its own logarithm to the base 10, then N is divisible by
 - 5
 - 7
 - 9
 - 11
- The value of $\frac{\log_2 24}{\log_{96} 2} - \frac{\log_2 192}{\log_{12} 2}$ is
 - 3
 - 0
 - 2
 - 1
- $\log_{x-1} x \cdot \log_{x-2}(x-1) \cdot \dots \cdot \log_{x-12}(x-11) = 2$, x is equal to
 - 9
 - 16
 - 25
 - none of these
- If $f(x) = \log\left(\frac{1+x}{1-x}\right)$, then
 - $f(x_1) \cdot f(x_2) = f(x_1 + x_2)$
 - $f(x+2) - 2f(x+1) + f(x) = 0$
 - $f(x) + f(x+1) = f(x^2 + x)$
 - $f(x_1) + f(x_2) = f\left(\frac{x_1 + x_2}{1 + x_1 x_2}\right)$
- If $a^4 \cdot b^5 = 1$, then the value of $\log_a(a^5 b^4)$ equals
 - 9/5
 - 4
 - 5
 - 8/5
- The value of $3^{\log_4 5} - 5^{\log_4 3}$ is
 - 0
 - 1
 - 2
 - none of these
- If $2^{x+y} = 6^y$ and $3^{x-1} = 2^{y+1}$, then the value of $(\log 3 - \log 2)/(x-y)$ is
 - 1
 - $\log_2 3 - \log_3 2$
 - $\log(3/2)$
 - none of these
- The value of x satisfying $\sqrt{3}^{-4+2\log_{\sqrt{3}} x} = 1/9$ is
 - 2
 - 3
 - 4
 - none of these
- The value of x satisfying the equation $\sqrt[3]{5^{\log_5 5^{\log_5 5^{\log_5 \left(\frac{x}{2}\right)}}}} = 3$, is
 - 1
 - 3
 - 18
 - 54
- If $\sqrt{\log_2 x} - 0.5 = \log_2 \sqrt{x}$, then x equals
 - odd integer
 - prime number
 - composite number
 - irrational
- If $\log_y x + \log_x y = 2$, $x^2 + y^2 = 12$, then the value of xy is
 - 9
 - 12
 - 15
 - 21
- If $(4)^{\log_9 3} + (9)^{\log_4 4} = (10)^{\log_8 8}$, then x is equal to
 - 2
 - 3
 - 10
 - 30
- If $(x+1)^{\log_{10}(x+1)} = 100(x+1)$, then
 - all the roots are positive real numbers.
 - all the roots lie in the interval $(0, 100)$
 - all the roots lie in the interval $[-1, 99]$
 - none of these

25. If $\log_2 x + \log_x 2 = \frac{10}{3} = \log_2 y + \log_y 2$ and $x \neq y$, then $x + y =$
 (1) 2 (2) 65/8
 (3) 37/6 (4) none of these
26. If $\log_{10} \left[\frac{1}{2^x + x - 1} \right] = x [\log_{10} 5 - 1]$, then $x =$
 (1) 4 (2) 3 (3) 2 (4) 1
27. If $\log_3 \{5 + 4 \log_3 (x - 1)\} = 2$, then x is equal to
 (1) 2 (2) 4 (3) 8 (4) 16
28. If $2x^{\log_4 3} + 3^{\log_4 x} = 27$, then x is equal to
 (1) 2 (2) 4 (3) 8 (4) 16
29. Equation $\log_4 (3 - x) + \log_{0.25} (3 + x) = \log_4 (1 - x) + \log_{0.25} (2x + 1)$ has
 (1) only one prime solution (2) two real solutions
 (3) no real solution (4) none of these
30. The value of b for which the equation $2 \log_{1/25} (bx + 28) = -\log_5 (12 - 4x - x^2)$ has coincident roots is
 (1) $b = -12$ (2) $b = 4$
 (3) $b = 4$ or $b = -12$ (4) $b = -4$ or $b = 12$
31. If the equation $2^x + 4^y = 2^y + 4^x$ is solved for y in terms of x , where $x < 0$, then the sum of the solutions is
 (1) $x \log_2 (1 - 2^x)$ (2) $x + \log_2 (1 - 2^x)$
 (3) $\log_2 (1 - 2^x)$ (4) $x \log_2 (2^x + 1)$
32. The number of solution of $x^{\log_4 (x+3)^2} = 16$ is
 (1) 0 (2) 1 (3) 2 (4) ∞
33. The product of roots of the equation $\frac{\log_8 (8/x^2)}{(\log_8 x)^2} = 3$ is
 (1) 1 (2) 1/2 (3) 1/3 (4) 1/4
34. Let $a > 1$ be a real number. Then the number of roots of equation $a^{2 \log_2 x} = 5 + 4x^{\log_2 a}$ is
 (1) 2 (2) infinite
 (3) 0 (4) 1
35. The number of roots of the equation $\log_{3\sqrt{x}} x + \log_{3x} \sqrt{x} = 0$ is
 (1) 1 (2) 2 (3) 3 (4) 0
36. The set of all x satisfying the equation $x^{\log_3 x^2 + (\log_3 x)^2 - 10} = 1/x^2$ is
 (1) $\{1, 9\}$ (2) $\{1, 9, 1/81\}$
 (3) $\{1, 4, 1/81\}$ (4) $\{9, 1/81\}$
37. Number of real values of x satisfying the equation $\log_2 (x^2 - x) \log_2 \left(\frac{x-1}{x} \right) + (\log_2 x)^2 = 4$, is
 (1) 0 (2) 2 (3) 3 (4) 7
38. If $xy^2 = 4$ and $\log_3 (\log_2 x) + \log_{1/3} (\log_{1/2} y) = 1$, then x equals
 (1) 4 (2) 8 (3) 16 (4) 64
39. If x_1 and x_2 are the roots of the equation $e^2 \cdot x^{\ln x} = x^3$, $x_1 > x_2$, then
 (1) $x_1 = 2x_2$ (2) $x_1 = x_2^2$
 (3) $2x_1 = x_2^2$ (4) $x_1^2 = x_2^3$
40. The number of real values of the parameter k for which $(\log_{16} x)^2 - \log_{16} x + \log_{16} k = 0$ with real coefficients have exactly one solution is
 (1) 2 (2) 1
 (3) 4 (4) none of these
41. $x^{\log_5 x} > 5$ implies
 (1) $x \in (0, \infty)$ (2) $x \in (0, 1/5) \cup (5, \infty)$
 (3) $x \in (1, \infty)$ (4) $x \in (1, 2)$
42. If $S = \{x \in \mathbb{N} : 2 + \log_2 \sqrt{x+1} > 1 - \log_{1/2} \sqrt{4-x^2}\}$, then
 (1) $S = \{1\}$ (2) $S = \mathbb{Z}$
 (3) $S = \mathbb{N}$ (4) none of these
43. If $S = \{x \in \mathbb{R} : (\log_{0.6} 0.216) \log_5 (5 - 2x) \leq 0\}$, then S is equal to
 (1) $[2.5, \infty)$ (2) $[2, 2.5]$
 (3) $(2, 2.5)$ (4) $(0, 2.5)$
44. Solution set of the inequality $\frac{1}{2^x - 1} > \frac{1}{1 - 2^{x-1}}$ is
 (1) $(1, \infty)$ (2) $(0, \log_2 (4/3))$
 (3) $(-1, \infty)$ (4) $(0, \log_2 (4/3)) \cup (1, \infty)$
45. If $\log_2 x + \log_2 y \geq 6$, then the least value of $x + y$ is
 (1) 4 (2) 8 (3) 16 (4) 32
46. Which of the following is not the solution of $\log_x \left(\frac{5}{2} - \frac{1}{x} \right) > \left(\frac{5}{2} - \frac{1}{x} \right)$?
 (1) $\left(\frac{2}{5}, \frac{1}{2} \right)$ (2) $(1, 2)$
 (3) $\left(\frac{2}{5}, 1 \right)$ (4) none of these
47. The solution set of the inequality $\log_{10} (x^2 - 1) \leq \log_{10} (4x - 11)$ is
 (1) $(4, \infty)$ (2) $(4, 5]$
 (3) $(11/4, \infty)$ (4) $(11/4, 5)$
48. Solution set of the inequality $\log_{0.8} \left(\log_6 \frac{x^2 + x}{x + 4} \right) < 0$ is
 (1) $(-4, -3)$ (2) $(-3, 4) \cup (8, \infty)$
 (3) $(-3, \infty)$ (4) $(-4, -3) \cup (8, \infty)$
49. Which of the following is not the solution of $\log_3 (x^2 - 2) < \log_3 \left(\frac{3}{2} |x| - 1 \right)$ is
 (1) $(\sqrt{2}, 2)$ (2) $(-2, -\sqrt{2})$
 (3) $(-\sqrt{2}, 2)$ (4) none of these

50. The true solution set of inequality $\log_{(x+1)}(x^2 - 4) > 1$ is equal to

- (1) $(2, \infty)$ (2) $\left(2, \frac{1+\sqrt{21}}{2}\right)$
 (3) $\left(\frac{1-\sqrt{21}}{2}, \frac{1+\sqrt{21}}{2}\right)$ (4) $\left(\frac{1+\sqrt{21}}{2}, \infty\right)$

Multiple Correct Answers Type

1. For $a > 0, \neq 1$, the roots of the equation $\log_{ax} a + \log_x a^2 + \log_{a^3x} a^3 = 0$ are given by

- (1) $a^{-4/3}$ (2) $a^{-3/4}$ (3) a (4) $a^{-1/2}$

2. The real solutions of the equation $2^{x+2} \cdot 5^{6-x} = 10^{x^2}$ is/are

- (1) 1 (2) 2
 (3) $-\log_{10}(250)$ (4) $\log_{10} 4 - 3$

3. If $\frac{\log x}{b-c} = \frac{\log y}{c-a} = \frac{\log z}{a-b}$, then which of the following is/are true?

- (1) $xyz = 1$ (2) $x^a y^b z^c = 1$
 (3) $x^{b-c} y^{c-a} z^{a-b} = 1$ (4) $xyz = x^a y^b z^c$

4. If $\log_k x \cdot \log_5 k = \log_x 5$, $k \neq 1$, $k > 0$, then x is equal to

- (1) k (2) $1/5$
 (3) 5 (4) none of these

5. If $p, q \in \mathbb{N}$ satisfy the equation $x^{\sqrt{x}} = (\sqrt{x})^x$, then

- (1) $p + q = 5$
 (2) $|p - q| = 4$
 (3) $pq = 4$
 (4) if $\log_p p$ is defined, then $\log_p q$ is not and vice versa

6. Which of the following, when simplified, reduces to unity?

- (1) $\log_{10} 5 \cdot \log_{10} 20 + (\log_{10} 2)^2$
 (2) $\frac{2 \log 2 + \log 3}{\log 48 - \log 4}$
 (3) $-\log_5 \log_3 \sqrt[5]{9}$
 (4) $\frac{1}{6} \log_{\sqrt{3}/2} \left(\frac{64}{27}\right)$

7. If $\log_a x = b$ for permissible values of a and x , then identify the statement(s) which can be correct.

- (1) If a and b are two irrational numbers, then x can be rational.
 (2) If a is rational and b is irrational, then x can be rational.
 (3) If a is irrational and b is rational, then x can be rational.
 (4) If a and b are rational, then x can be rational.

8. The equation $\log_{x+1}(x - 0.5) = \log_{x-0.5}(x + 1)$ has

- (1) two real solutions (2) no prime solution
 (3) one integral solution (4) no irrational solution

9. The equation $\sqrt{1 + \log_x \sqrt{27}} \log_3 x + 1 = 0$ has

- (1) no integral solution (2) one irrational solution
 (3) two real solutions (4) no prime solution

10. If $\log_{1/2}(4 - x) \geq \log_{1/2} 2 - \log_{1/2}(x - 1)$, then x belongs to

- (1) $(1, 2]$ (2) $[3, 4)$
 (3) $(1, 3]$ (4) $[1, 4)$

11. If the equation $x^{\log_a x} = \frac{x^{k-2}}{a^k}$, $a \neq 0$, has exactly one

solution for x , then the value of k is/are

- (1) $6 + 4\sqrt{2}$ (2) $2 + 6\sqrt{3}$
 (3) $6 - 4\sqrt{2}$ (4) $2 - 6\sqrt{3}$

12. The values of x satisfying the equation

$$|x - 1|^{\log_3 x^2 - 2 \log_3 9} = (x - 1)^7 \text{ is/are}$$

- (1) $\frac{1}{\sqrt{3}}$ (2) 1 (3) 2 (4) 81

13. If $x = 9$ is one of the solutions of $\log_e(x^2 + 15a^2) - \log_e(a - 2) = \log_e\left(\frac{8ax}{a - 2}\right)$, then

- (1) $a = \frac{3}{5}$ (2) $a = 3$ (3) $x = 15$ (4) $x = 2$

14. In which of the following, $m > n$ ($m, n \in \mathbb{R}$)?

- (1) $m = (\log_2 5)^2$ and $n = \log_2 20$
 (2) $m = \log_{10} 2$ and $n = \log_{10} \sqrt[3]{10}$
 (3) $m = \log_{10} 5 \cdot \log_{10} 20$ and $n = 1$
 (4) $m = \log_{1/2} \left(\frac{1}{3}\right)$ and $n = \log_{1/3} \left(\frac{1}{2}\right)$

15. If $\log_{10} 5 = a$ and $\log_{10} 3 = b$, then

- (1) $\log_{30} 8 = \frac{3(1-a)}{b+1}$ (2) $\log_{40} 15 = \frac{a+b}{3-2a}$
 (3) $\log_{243} 32 = \frac{1-a}{b}$ (4) none of these

16. The value of $\frac{6 a^{\log_e b} (\log_a b) (\log_b a)}{e^{\log_e a \cdot \log_e b}}$ is

- (1) independent of a (2) independent of b
 (3) dependent on a (4) dependent on b

17. The inequality $\sqrt{x^{\log_2 \sqrt{x}}} \geq 2$ is satisfied by

- (1) only one value of x (2) $x \in \left(0, \frac{1}{4}\right]$
 (3) $x \in [4, \infty)$ (4) $x \in (1, 2)$

Linked Comprehension Type

For Problems 1 and 2

Consider the system of equations

$$\log_3 (\log_2 x) + \log_{1/3} (\log_{1/2} y) = 1 \text{ and } xy^2 = 9.$$

1. The value of x lies in the interval

- (1) $(200, 300)$ (2) $(400, 500)$
 (3) $(700, 800)$ (4) none of these

2. The value of
- $\sqrt{13}$
- lies in the interval

(1) (5, 7) (2) (7, 10) (3) (11, 15) (4) (25, 30)

For Problems 3 and 4Consider equations $x^{\log_e x} = 2$ and $y^{\log_e y} = 16$.

3. The value of
- x
- is

(1) $2^{\sqrt{2}}$ (2) $2^{\sqrt{4}}$ (3) $2^{\sqrt{64}}$ (4) $2^{\sqrt{256}}$

4. The value of
- y
- is

(1) $2^{\sqrt{2}}$ (2) $2^{\sqrt{4}}$ (3) $2^{\sqrt{128}}$ (4) $2^{\sqrt{16}}$ **For Problems 5 and 6**Consider $E = 2^{\left(\sqrt{\log_3 4 + \log_3 4} - \sqrt{\log_3 \sqrt{b} + \log_3 \sqrt{a}}\right) \sqrt{\log_3 b}}$

5. The value of
- E
- if
- $b \geq a > 1$
- is

(1) 1 (2) 2 (3) $2^{\log_e b}$ (4) $2^{\log_e a}$

6. The value of
- E
- if
- $1 < b < a$
- is

(1) 1 (2) 2 (3) $2^{\log_e b}$ (4) $2^{\log_e a}$ **Matrix Match Type**

1.

List I	List II
a. The smallest integer greater than $\frac{1}{\log_3 \pi} + \frac{1}{\log_4 \pi}$ is	p. 10
b. Let $3^a = 4$, $4^b = 5$, $5^c = 6$, $6^d = 7$, $7^e = 8$, and $8^f = 9$. Then the value of the product $(abcdef)$ is	q. 3
c. Characteristic of the logarithm of 2008 to the base 2 is	r. 1
d. If $\log_2(\log_2(\log_3 x)) = \log_2(\log_3(\log_2 y)) = 0$, then the value of $(x - y)$ is	s. 2

2.

List I	List II
a. $2^{\log_{1/5} 15}$ is	p. rational
b. $\sqrt[3]{5^{1/\log_5 5} + \frac{1}{\sqrt{(-\log_{10} 0.1)}}}$ is	q. irrational
c. $\log_3 5 \cdot \log_{25} 27$ is	r. composite
d. Product of roots of equation $x^{\log_{10} x} = 100x$ is	s. prime

3.

List I	List II
a. The value of $\log_2 \log_2 \log_4 256 + \log_{\sqrt{2}} 4$ is	p. 1
b. If $\log_3 (5x - 2) - 2 \log_3 \sqrt{3x + 1} = 1 - \log_3 4$, then $x =$	q. 6
c. Product of roots of the equation $7^{\log_7 (x^2 - 4x + 5)} = (x - 1)$ is	r. 3
d. Number of integers satisfying $\log_2 \sqrt{x} - 2(\log_{1/4} x)^2 + 1 > 0$ are	s. 5

Codes

	a	b	c	d
(1)	s	p	q	r
(2)	r	s	q	p
(3)	q	r	p	s
(4)	r	p	q	s

Numerical Value Type

- If $\log_a b = 2$, $\log_b c = 2$, and $\log_3 c = 3 + \log_3 a$, then the value of $c/(ab)$ is _____.
- The value of $(\log_{10} 2)^3 + \log_{10} 8 \cdot \log_{10} 5 + (\log_{10} 5)^3$ is _____.
- If $\log_4 A = \log_6 B = \log_9 (A + B)$, then $[4(B/A)]$ (where $[]$ represents the greatest integer function) equals _____.
- Integral value of x which satisfies the equation $\log_6 54 \log_x 16 = \log_{\sqrt{2}} x - \log_{36} (4/9)$ is _____.
- If $a = \log_{245} 175$ and $b = \log_{1715} 875$, then the value of $\frac{1-ab}{a-b}$ is _____.
- The difference of roots of the equation $(\log_{27} x^3)^2 = \log_{27} x$ is _____.
- Sum of all integral values of x satisfying the inequality $(3^{(5/2) \log_3 (12-3x)}) - (3^{\log_2 x}) > 32$ is _____.
- The least integer greater than $\log_2 15 \cdot \log_{1/6} 2 \cdot \log_3 1/6$ is _____.
- The reciprocal of $\frac{2}{\log_4 (2000)^6} + \frac{3}{\log_5 (2000)^6}$ is _____.
- Sum of integers satisfying $\sqrt{\log_2 x - 1} - 1/2 \log_2 (x^3) + 2 > 1$ is _____.
- Number of integers satisfying the inequality $\log_{1/2} |x - 3| > -1$ is _____.
- Number of integers ≤ 10 satisfying the inequality $2 \log_{1/2} (x - 1) \leq \frac{1}{3} - \frac{1}{\log_{x^2 - x} 8}$ is _____.
- The value of $\log(\sqrt{3+2\sqrt{2}} + \sqrt{3-2\sqrt{2}})^{2^9}$ is _____.
- The value of $5^{\log_{15} (1/2)} + \log_{\sqrt{2}} \frac{4}{\sqrt{7} + \sqrt{3}} + \log_{1/2} \frac{1}{10 + 2\sqrt{21}}$ is _____.
- The value of $N = \frac{\log_5 250}{\log_{50} 5} - \frac{\log_5 10}{\log_{1250} 5}$ is _____.
- The number of positive integers satisfying $x + \log_{10} (2^{x+1}) = x \log_{10} 5 + \log_{10} 6$ is _____.
- If x, y, z are positive real numbers such that $\log_2 x^z = 1$, $\log_5 z = 6$, and $\log_{xy} z = 2/3$, then the value of $(1/2z)$ is _____.
- If $a = \log_{12} 18$, $b = \log_{24} 54$, then the value of $ab + 5(a - b)$ is _____.

Archives

JEE MAIN

Single Correct Answer Type

1. The equation $e^{\sin x} - e^{-\sin x} - 4 = 0$ has:

- (1) infinite number of real roots
- (2) no real roots
- (3) exactly one real root
- (4) exactly four real roots

(AIEEE 2012)

Multiple Correct Answers Type

1. If $3^x = 4^{x-1}$, then $x =$

- (1) $\frac{2 \log_3 2}{2 \log_3 2 - 1}$
- (2) $\frac{2}{2 - \log_2 3}$
- (3) $\frac{1}{1 - \log_4 3}$
- (4) $\frac{2 \log_2 3}{2 \log_2 3 - 1}$

(JEE Advanced 2013)

Numerical Value Type

1. The value of

$$6 + \log_{3/2} \left(\frac{1}{3\sqrt{2}} \sqrt{4 - \frac{1}{3\sqrt{2}}} \sqrt{4 - \frac{1}{3\sqrt{2}}} \sqrt{4 - \frac{1}{3\sqrt{2}}} \dots \right)$$

is _____.

(IIT-JEE 2012)

2. The value of $((\log_2 9)^2)^{\frac{1}{\log_2(\log_2 9)}} \times (\sqrt{7})^{\frac{1}{\log_2 7}}$ is _____.

(JEE Advanced 2018)

Answers Key

EXERCISES

Single Correct Answer Type

- | | | | | |
|---------|---------|---------|---------|---------|
| 1. (2) | 2. (2) | 3. (3) | 4. (3) | 5. (2) |
| 6. (1) | 7. (3) | 8. (1) | 9. (4) | 10. (4) |
| 11. (2) | 12. (1) | 13. (1) | 14. (2) | 15. (4) |
| 16. (1) | 17. (1) | 18. (3) | 19. (4) | 20. (4) |
| 21. (2) | 22. (1) | 23. (3) | 24. (3) | 25. (4) |
| 26. (4) | 27. (2) | 28. (4) | 29. (4) | 30. (3) |
| 31. (2) | 32. (1) | 33. (4) | 34. (4) | 35. (2) |
| 36. (2) | 37. (2) | 38. (4) | 39. (2) | 40. (1) |
| 41. (2) | 42. (1) | 43. (2) | 44. (4) | 45. (3) |
| 46. (3) | 47. (2) | 48. (4) | 49. (3) | 50. (4) |

Multiple Correct Answers Type

- | | |
|--------------------|----------------|
| 1. (1) (4) | 2. (2) (3) (4) |
| 3. (1) (2) (3) (4) | 4. (2) (3) |
| 5. (1) (3) (4) | 6. (1) (2) (3) |
| 7. (1) (2) (3) (4) | 8. (2) (3) (4) |
| 9. (1) (4) | 10. (1) (2) |
| 11. (1) (3) | 12. (3) (4) |
| 13. (2) (3) | 14. (1) (4) |
| 15. (1) (2) (3) | 16. (1) (2) |
| 17. (2) (3) | |

Linked Comprehension Type

- | | | | | |
|--------|--------|--------|--------|--------|
| 1. (3) | 2. (2) | 3. (2) | 4. (4) | 5. (2) |
| 6. (3) | | | | |

Matrix Match Type

1. $a \rightarrow q; b \rightarrow s; c \rightarrow p; d \rightarrow r$
2. $a \rightarrow q; b \rightarrow p, s; c \rightarrow p; d \rightarrow p, r$
3. (1)

Numerical Value Type

- | | | | | |
|---------|---------|---------|---------|---------|
| 1. (3) | 2. (1) | 3. (6) | 4. (4) | 5. (5) |
| 6. (8) | 7. (3) | 8. (3) | 9. (6) | 10. (5) |
| 11. (2) | 12. (9) | 13. (6) | 14. (6) | 15. (2) |
| 16. (1) | 17. (5) | 18. (1) | | |

ARCHIVES

JEE MAIN

Single Correct Answer Type

1. (2)

JEE ADVANCED

Single Correct Answer Type

1. (3)

Multiple Correct Answers Type

1. (1), (2), (3)

Numerical Value Type

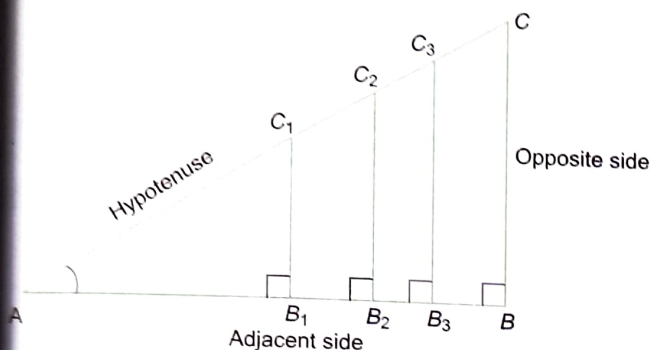
1. (4)
2. (8)

2

Trigonometric Functions

TRIGONOMETRIC RATIOS/ FUNCTIONS OF ACUTE ANGLES

Consider a right-angled triangle ABC with right angle at B .



Here, sides BC and AB are called opposite side and adjacent side (or base) w.r.t. angle A , respectively.

Now, we have similar triangles AB_1C_1 , AB_2C_2 , AB_3C_3 , AB_4C_4 and ABC .

$$\therefore \text{Therefore, } \frac{B_1C_1}{AC_1} = \frac{B_2C_2}{AC_2} = \frac{B_3C_3}{AC_3} = \frac{BC}{AC} = \lambda_1 \text{ (constant)}$$

$$\text{Also, } \frac{AB_1}{AC_1} = \frac{AB_2}{AC_2} = \frac{AB_3}{AC_3} = \frac{AB}{AC} = \lambda_2 \text{ (constant)}$$

$$\text{and } \frac{B_1C_1}{AB_1} = \frac{B_2C_2}{AB_2} = \frac{B_3C_3}{AB_3} = \frac{BC}{AB} = \lambda_3 \text{ (constant)}$$

All the above ratios depend on angle A . If we change angle A , then the values of λ_1 , λ_2 and λ_3 will change.

Thus, ratio of the lengths of two sides of the right-angled triangle ABC is completely determined by angle A alone and is independent of the size of the triangle. There are six possible ratios that can be formed from the three sides of right-angled triangle ABC . Each of them has been given a name as follows:

- (i) $\sin A = \frac{\text{opposite side}}{\text{hypotenuse}}$
- (ii) $\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}}$
- (iii) $\tan A = \frac{\text{opposite side}}{\text{adjacent side}}$
- (iv) $\cot A = \frac{\text{adjacent side}}{\text{opposite side}}$
- (v) $\sec A = \frac{\text{hypotenuse}}{\text{adjacent side}}$
- (vi) $\text{cosec } A = \frac{\text{hypotenuse}}{\text{opposite side}}$

The abbreviations stand for sine, cosine, tangent, cotangent, secant and cosecant of A , respectively. These functions of angle A are called trigonometric functions or trigonometric ratios.

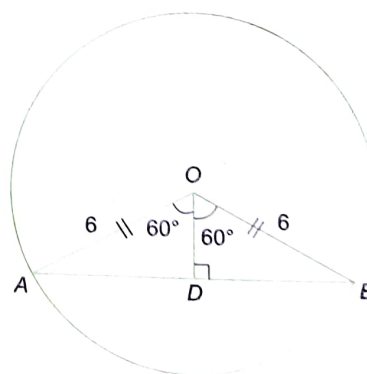
TRIGONOMETRIC RATIOS OF STANDARD ANGLES

Angle (θ) → T-Ratio ↓	0°	30°	45°	60°	90°
$\sin \theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan \theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined
$\text{cosec } \theta$	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$\sec \theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined
$\cot \theta$	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

ILLUSTRATION 2.1

Find the length of the chord which subtends an angle of 120° at the centre of the circle of radius 6 cm.

Sol. Let the chord be AB and O be the centre of the circle.



In the figure, D is the mid-point of AB and OD is perpendicular to AB .

$$OA = OB = 6 \text{ cm, } \angle AOB = 120^\circ.$$

Clearly, triangles AOD and BOD are congruent.

$$\therefore \angle AOD = \angle BOD = \frac{1}{2} \angle AOB = 60^\circ$$

$$\text{In } \triangle AOD, \sin 60^\circ = \frac{AD}{AO} = \frac{AD}{6}$$

2.2 Trigonometry

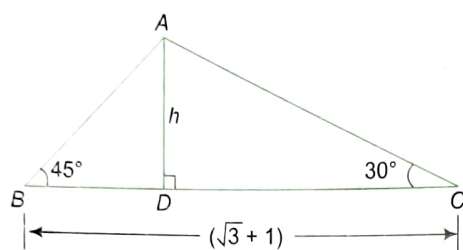
$$\therefore AD = 6 \sin 60^\circ = 6 \times \frac{\sqrt{3}}{2} = 3\sqrt{3}$$

$$\therefore AB = 2 \times AD = 6\sqrt{3} \text{ cm}$$

ILLUSTRATION 2.2

If two angles of a triangle are 30° and 45° and the included side is $(\sqrt{3} + 1)$ cm, then find the area of the triangle.

Sol. Triangle with given specifications is drawn in the following figure.



To find the area of the triangle ABC , we need to find the length of altitude AD .

Let AD measure h .

Now, in triangle ADC ,

$$\tan 30^\circ = \frac{h}{CD}$$

$$\Rightarrow CD = h \cot 30^\circ = \sqrt{3}h$$

In triangle ADB ,

$$\tan 45^\circ = \frac{h}{BD}$$

$$\Rightarrow BD = h \cot 45^\circ = h$$

Now, $BC = BD + DC$

$$\therefore h + \sqrt{3}h = \sqrt{3} + 1$$

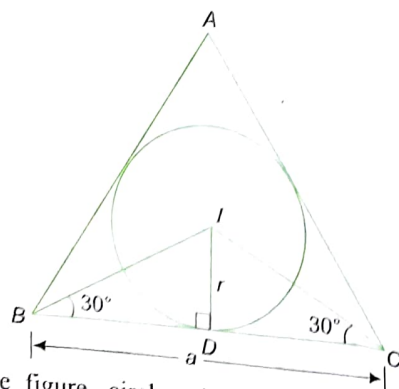
$$\therefore h = 1$$

$$\therefore \text{Area of triangle } ABC = \frac{1}{2} \times AD \times BC = \frac{\sqrt{3} + 1}{2} \text{ cm}^2$$

ILLUSTRATION 2.3

A circle is inscribed in an equilateral triangle of side a . Find the area of any square inscribed in this circle.

Sol.



In the above figure, circle with centre I is inscribed in an equilateral triangle ABC .

BC and BA are two tangents from B to circle. must be the angle bisector of $\angle B$.

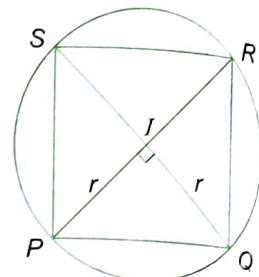
$$\text{But } \angle B = 60^\circ$$

$$\therefore \angle IBD = 30^\circ$$

$$\therefore \tan 30^\circ = \frac{r}{a/2}$$

$$\therefore r = \frac{a}{2\sqrt{3}}$$

Now, inside the circle, square $PQRS$ is inscribed.



In the above figure, IPQ is isosceles-right angled.

$$\therefore PQ^2 = IP^2 + IQ^2 = r^2 + r^2 = 2r^2$$

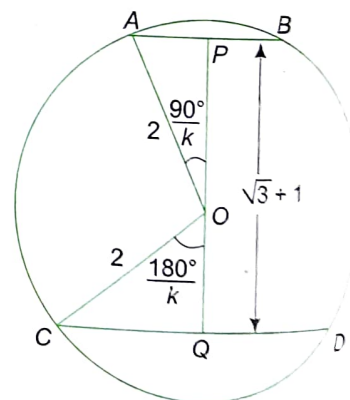
$$\Rightarrow PQ = \sqrt{2} \times \frac{a}{2\sqrt{3}} = \frac{a}{\sqrt{6}}$$

$$\therefore \text{Area of square} = \left(\frac{a}{\sqrt{6}} \right)^2 = \frac{a^2}{6} \text{ sq. unit}$$

ILLUSTRATION 2.4

Two parallel chords of a circle of radius 2 units apart. If these chords subtend, at the centre, $\frac{90^\circ}{k}$ and $\frac{180^\circ}{k}$, where $k > 0$, then find the value

Sol.



In the figure, two parallel chords AB and CD are at units.

Chord AB and CD subtend angles $\frac{90^\circ}{k}$ and $\frac{180^\circ}{k}$ at O , respectively.

$$\therefore \angle AOP = \frac{90^\circ}{k} \text{ and } \angle COQ = \frac{180^\circ}{k}$$

$$\text{In triangle } AOP, OP = 2 \cos \frac{90^\circ}{k}$$

$$\text{In triangle } COQ, OQ = 2 \cos \frac{180^\circ}{k}$$

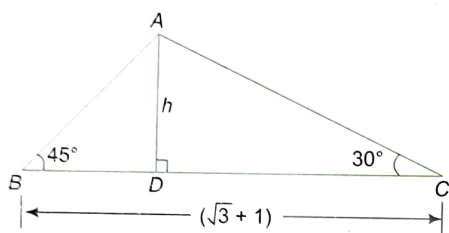
$$\therefore AD = 6 \sin 60^\circ = 6 \times \frac{\sqrt{3}}{2} = 3\sqrt{3}$$

$$\therefore AB = 2 \times AD = 6\sqrt{3} \text{ cm}$$

ILLUSTRATION 2.2

If two angles of a triangle are 30° and 45° and the included side is $(\sqrt{3} + 1)$ cm, then find the area of the triangle.

Sol. Triangle with given specifications is drawn in the following figure.



To find the area of the triangle ABC , we need to find the length of altitude AD .

Let AD measure h .

Now, in triangle ADC ,

$$\tan 30^\circ = \frac{h}{CD}$$

$$\Rightarrow CD = h \cot 30^\circ = \sqrt{3}h$$

In triangle ADB ,

$$\tan 45^\circ = \frac{h}{BD}$$

$$\Rightarrow BD = h \cot 45^\circ = h$$

Now, $BC = BD + DC$

$$\therefore h + \sqrt{3}h = \sqrt{3} + 1$$

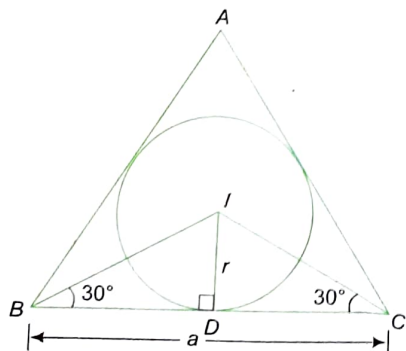
$$\therefore h = 1$$

$$\therefore \text{Area of triangle } ABC = \frac{1}{2} \times AD \times BC = \frac{\sqrt{3} + 1}{2} \text{ cm}^2$$

ILLUSTRATION 2.3

A circle is inscribed in an equilateral triangle of side a . Find the area of any square inscribed in this circle.

Sol.



In the above figure, circle with centre I is inscribed in an equilateral triangle ABC .

BC and BA are two tangents from B to circle, therefore must be the angle bisector of $\angle B$.

$$\text{But } \angle B = 60^\circ$$

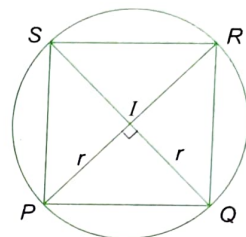
($\because \triangle ABC$ is an equilateral)

$$\therefore \angle IBD = 30^\circ$$

$$\therefore \tan 30^\circ = \frac{r}{a/2}$$

$$\therefore r = \frac{a}{2\sqrt{3}}$$

Now, inside the circle, square $PQRS$ is inscribed.



In the above figure, IPQ is isosceles-right angled triangle.

$$\therefore PQ^2 = IP^2 + IQ^2 = r^2 + r^2 = 2r^2$$

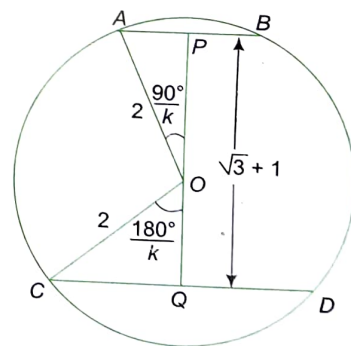
$$\Rightarrow PQ = \sqrt{2} \times \frac{a}{2\sqrt{3}} = \frac{a}{\sqrt{6}}$$

$$\therefore \text{Area of square} = \left(\frac{a}{\sqrt{6}}\right)^2 = \frac{a^2}{6} \text{ sq. unit}$$

ILLUSTRATION 2.4

Two parallel chords of a circle of radius 2 units are $(\sqrt{3} + 1)$ units apart. If these chords subtend, at the centre, angles $\frac{90^\circ}{k}$ and $\frac{180^\circ}{k}$, where $k > 0$, then find the value of k .

Sol.



In the figure, two parallel chords AB and CD are at distance $\sqrt{3} + 1$ units.

Chord AB and CD subtend angles $\frac{90^\circ}{k}$ and $\frac{180^\circ}{k}$ at the centre O , respectively.

$$\therefore \angle AOP = \frac{90^\circ}{k} \text{ and } \angle COQ = \frac{180^\circ}{k}$$

$$\text{In triangle } AOP, OP = 2 \cos \frac{90^\circ}{k}$$

$$\text{In triangle } COQ, OQ = 2 \cos \frac{180^\circ}{k}$$

Now, $PQ = OP + OQ = 2 \cos \frac{90^\circ}{k} + 2 \cos \frac{180^\circ}{k}$

$$2 \cos \frac{90^\circ}{k} + 2 \cos \frac{180^\circ}{k} = \sqrt{3} + 1 \quad (\text{Given})$$

$$\cos \frac{90^\circ}{k} + \cos \frac{180^\circ}{k} = \frac{\sqrt{3}}{2} + \frac{1}{2}$$

Clearly, value of k is 3.

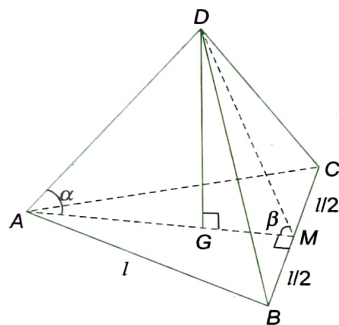
ILLUSTRATION 2.5

Find the height of the regular pyramid with each edge measuring l cm.

Also,

- if α is the angle between any edge and face not containing that edge, then prove that $\cos \alpha = \frac{1}{\sqrt{3}}$
- if β is the angle between the two faces, then prove that $\cos \beta = \frac{1}{3}$

Sol. In the figure we have regular pyramid.



All the edges have length l cm.

So, each face of the pyramid is an equilateral triangle.

From vertex D drop perpendicular to meet opposite face ABC at point G , which is the centre of the triangle ABC .

Here, G is also centroid of triangle ABC as it is an equilateral triangle.

Clearly, $AG = \frac{2}{3} AM$

In triangle AMB , $AM = AB \sin 60^\circ = \frac{\sqrt{3}}{2} l$ cm

$$AG = \frac{2}{3} \times AM = \frac{2}{3} \times \frac{\sqrt{3}}{2} l = \frac{1}{\sqrt{3}} l \text{ cm}$$

And $GM = \frac{1}{3} \times \frac{\sqrt{3}}{2} l = \frac{1}{2\sqrt{3}} l$ cm

In triangle AGD , using Pythagoras, we get

$$AD^2 = AG^2 + GD^2$$

$$l^2 = \frac{1}{3} l^2 + GD^2$$

$$GD = \frac{\sqrt{2}}{3} l \text{ cm}$$

For angle between any edge and face not containing that edge, consider edge AD and face ABC .

Angle between them is α , which is angle between AD and AM .

In triangle AGD , $\cos \alpha = \frac{AG}{AD} = \frac{\frac{1}{\sqrt{3}} l}{l} = \frac{1}{\sqrt{3}}$

Angle between two faces ABC and BCD is the angle between AM and DM , which is β .

In triangle DGM , $\cos \beta = \frac{GM}{DM} = \frac{1}{3}$

ILLUSTRATION 2.6

By geometrical interpretation, prove that

(i) $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \sin \beta \cos \alpha$

(ii) $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$

Sol. The following figure is self-explanatory.

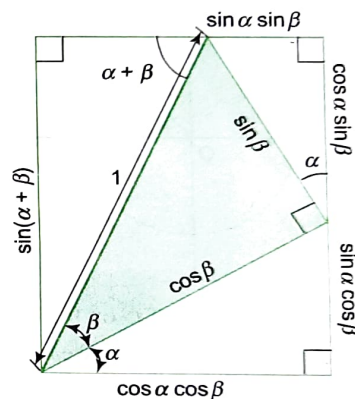


ILLUSTRATION 2.7

By geometrical interpretation, prove that

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

Sol. The following figure is self-explanatory.

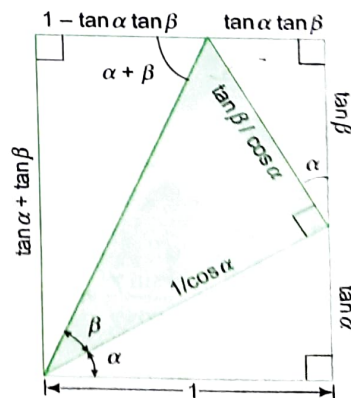


ILLUSTRATION 2.8

Find the minimum value of $2 \cos \theta + \frac{1}{\sin \theta} + \sqrt{2} \tan \theta$, where θ is acute angle.

Sol. Now, using A.M. $\geq$ G.M., we have

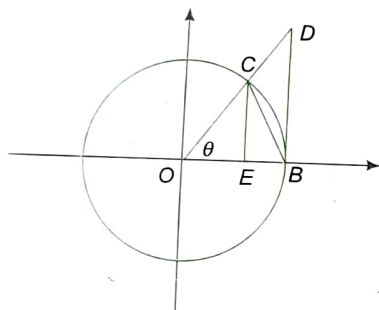
$$\frac{2 \cos \theta + \frac{1}{\sin \theta} + \sqrt{2} \tan \theta}{3} \geq \left(2 \cos \theta \cdot \frac{1}{\sin \theta} \cdot \sqrt{2} \tan \theta \right)^{\frac{1}{3}}$$

$$\Rightarrow 2 \cos \theta + \frac{1}{\sin \theta} + \sqrt{2} \tan \theta \geq 3\sqrt{2}$$

ILLUSTRATION 2.9

For acute angle θ , prove that $\sin \theta < \theta < \tan \theta$.

Sol. Consider the circle with unit radius as shown in the following figure.



From the figure, for any position of C on the circle
Area of $\triangle OBC <$ Area of sector $OBC <$ Area of $\triangle OBD$

$$\therefore \frac{1}{2}(CE)(OB) < \frac{1}{2}\theta < \frac{1}{2}(OB)(BD)$$

$$\Rightarrow \frac{1}{2}(\sin \theta)(1) < \frac{1}{2}\theta < \frac{1}{2}(1)(\tan \theta)$$

$$\Rightarrow \sin \theta < \theta < \tan \theta$$

ILLUSTRATION 2.10

If $0 < \alpha < \beta < \gamma < \pi/2$, then prove that

$$\tan \alpha < \frac{\sin \alpha + \sin \beta + \sin \gamma}{\cos \alpha + \cos \beta + \cos \gamma} < \tan \gamma$$

Sol.

$$\therefore 0 < \alpha < \beta < \gamma < \frac{\pi}{2}$$

$$\therefore \sin \alpha < \sin \beta < \sin \gamma$$

$$\text{and } \cos \alpha > \cos \beta > \cos \gamma$$

$$\text{So } \frac{\sin \alpha + \sin \beta + \sin \gamma}{\cos \alpha + \cos \beta + \cos \gamma} < \frac{3 \sin \gamma}{3 \cos \gamma} \text{ and } > \frac{3 \sin \alpha}{3 \cos \alpha}$$

$$\text{Thus, } \tan \alpha < \frac{\sin \alpha + \sin \beta + \sin \gamma}{\cos \alpha + \cos \beta + \cos \gamma} < \tan \gamma$$

CONCEPT APPLICATION EXERCISE 2.1

- Two sides of a parallelogram are 12 cm and 8 cm. If one of the interior angles is 135° , then find the area of the parallelogram.
- In triangle ABC , $AB = 6$, $AC = 3\sqrt{6}$, $\angle B = 60^\circ$ and $\angle C = 45^\circ$. Find length of side BC .
- The circumference of a circle circumscribing an equilateral triangle is 24π units. Find the area of the circle inscribed in the equilateral triangle.
- In an equilateral triangle, three identical coins of radius 1 unit each, are kept so that they touch each other and also the sides of a triangle. Find the area of the triangle.
- A polygon of nine sides, each of length 2, is inscribed in a circle. Prove that the radius of the circle is $\operatorname{cosec} 20^\circ$.
- Two circles of radii 4 cm and 1 cm touch each other externally and θ is the angle contained by their direct common tangents. Find $\sin \frac{\theta}{2} + \cos \frac{\theta}{2}$.
- If angle C of triangle ABC is 90° , then prove that $\tan A + \tan B = \frac{c^2}{ab}$ (where, a, b, c are sides opposite to angles A, B, C , respectively).
- If $\cos^2 \alpha - \sin^2 \alpha = \tan^2 \beta$, then prove that $\tan^2 \alpha = \cos^2 \beta - \sin^2 \beta$.

ANSWERS

- $48\sqrt{2}$ sq. cm
- $3 + 3\sqrt{3}$ units
- 36π sq. unit
- $(4\sqrt{3} + 6)$ sq. unit
- $7/5$

TRIGONOMETRIC IDENTITIES

We know that an identity is an equation which is always true. These can be trivially true, like " $x = x$ " or usefully true, such as the Pythagorean Theorem for right triangles. In trigonometry also, we have many identities. Trigonometric identities are equalities that involve trigonometric functions and are true for every value of the occurring variables where both sides of the equality are defined. The most commonly used trigonometric identities are:

- $\sin^2 \theta + \cos^2 \theta = 1$
- $\sec^2 \theta - \tan^2 \theta = 1$
- $\operatorname{cosec}^2 \theta - \cot^2 \theta = 1$

ILLUSTRATION 2.11

Show that $2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x) + 1 = 0$.

Sol.

$$\begin{aligned} & 2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x) + 1 \\ &= 2[(\sin^2 x)^3 + (\cos^2 x)^3] - 3(\sin^4 x + \cos^4 x) + 1 \\ &= 2[(\sin^2 x + \cos^2 x)^3 - 3\sin^2 x \cos^2 x (\sin^2 x + \cos^2 x)] - 3[(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x] + 1 \\ &= 2[1 - 3\sin^2 x \cos^2 x] - 3[1 - 2\sin^2 x \cos^2 x] + 1 = 0 \end{aligned}$$

ILLUSTRATION 2.12

Prove that $\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = \sec\theta + \tan\theta$.

Sol. L.H.S. = $\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = \sqrt{\frac{1+\sin\theta}{1-\sin\theta} \cdot \frac{1+\sin\theta}{1+\sin\theta}}$

$$= \sqrt{\frac{(1+\sin\theta)^2}{1-\sin^2\theta}} = \sqrt{\frac{(1+\sin\theta)^2}{\cos^2\theta}}$$

$$= \frac{1+\sin\theta}{\cos\theta} = \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}$$

$$= \sec\theta + \tan\theta = \text{R.H.S.}$$

ILLUSTRATION 2.13

Prove that $\frac{1}{\sec A - \tan A} - \frac{1}{\cos A} = \frac{1}{\cos A} - \frac{1}{\sec A + \tan A}$.

Sol. Given, $\frac{1}{\sec A - \tan A} - \frac{1}{\cos A} = \frac{1}{\cos A} - \frac{1}{\sec A + \tan A}$

or $\frac{1}{\sec A - \tan A} + \frac{1}{\sec A + \tan A} = \frac{1}{\cos A} + \frac{1}{\cos A}$

Here, R.H.S. = $\frac{2}{\cos A}$

Now L.H.S. = $\frac{1}{\sec A - \tan A} + \frac{1}{\sec A + \tan A}$

$$= \frac{\sec A + \tan A + \sec A - \tan A}{(\sec A - \tan A)(\sec A + \tan A)}$$

$$= \frac{2}{\cos A}$$

Thus, L.H.S. = R.H.S.

ILLUSTRATION 2.14

If $3\sin\theta + 5\cos\theta = 5$, then show that $5\sin\theta - 3\cos\theta = \pm 3$.

Sol. Given,

$$3\sin\theta + 5\cos\theta = 5 \quad (i)$$

Let $5\sin\theta - 3\cos\theta = x \quad (ii)$

Squaring and adding, we get

$$(9\sin^2\theta + 25\cos^2\theta + 30\sin\theta\cos\theta) + (25\sin^2\theta + 9\cos^2\theta - 30\sin\theta\cos\theta) = 25 + x^2$$

or $9(\sin^2\theta + \cos^2\theta) + 25(\sin^2\theta + \cos^2\theta) = 25 + x^2$

or $34 = 25 + x^2$ or $x^2 = 9$

or $x = \pm 3$

ILLUSTRATION 2.15

If $(\sec A + \tan A)(\sec B + \tan B)(\sec C + \tan C) = (\sec A - \tan A)(\sec B - \tan B)(\sec C - \tan C)$, prove that the value of each side is ± 1 .

Sol. Let

$$(\sec A + \tan A)(\sec B + \tan B)(\sec C + \tan C) = x \quad (i)$$

$$\therefore (\sec A - \tan A)(\sec B - \tan B)(\sec C - \tan C) = x \quad (ii)$$

Multiplying Eqs. (i) and (ii), we get

$$(\sec^2 A - \tan^2 A)(\sec^2 B - \tan^2 B)(\sec^2 C - \tan^2 C) = x^2$$

or $x^2 = 1$

$$\therefore x = \pm 1$$

Hence, each side is equal to ± 1 .

ILLUSTRATION 2.16

If $\tan\theta + \sec\theta = 1.5$, find $\sin\theta$, $\tan\theta$, and $\sec\theta$.

Sol. Given, $\sec\theta + \tan\theta = \frac{3}{2} \quad (i)$

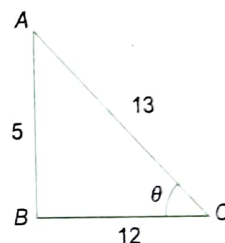
Now, $\sec\theta - \tan\theta = \frac{1}{\sec\theta + \tan\theta} = \frac{2}{3} \quad (ii)$

Adding Eqs. (i) and (ii), we get $2\sec\theta = \frac{3}{2} + \frac{2}{3} = \frac{13}{6}$

$$\therefore \sec\theta = \frac{13}{12}$$

$$\therefore \tan\theta = \frac{5}{12}$$

and $\sin\theta = \frac{5}{13}$

**ILLUSTRATION 2.17**

If $\operatorname{cosec}\theta - \sin\theta = m$ and $\sec\theta - \cos\theta = n$, eliminate θ .

Sol. Given,

$$\operatorname{cosec}\theta - \sin\theta = m$$

or $\frac{1}{\sin\theta} - \sin\theta = m$

or $\frac{1 - \sin^2\theta}{\sin\theta} = m$ or $\frac{\cos^2\theta}{\sin\theta} = m \quad (i)$

Also $\sec\theta - \cos\theta = n$

or $\frac{1}{\cos\theta} - \cos\theta = n$

or $\frac{1 - \cos^2\theta}{\cos\theta} = n$ or $\frac{\sin^2\theta}{\cos\theta} = n \quad (ii)$

From Eq. (i), $\sin\theta = \frac{\cos^2\theta}{m} \quad (iii)$

Putting the value of $\sin\theta$ in Eq. (ii), we get

$$\frac{\cos^4\theta}{m^2 \cos\theta} = n \quad \text{or} \quad \cos^3\theta = m^2 n$$

$$\therefore \cos\theta = (m^2 n)^{1/3} \quad \text{or} \quad \cos^2\theta = (m^2 n)^{2/3} \quad (iv)$$

From Eq. (iii), $\sin\theta = \frac{\cos^2\theta}{m} = \frac{(m^2 n)^{2/3}}{m}$

$$= \frac{m^{4/3} n^{2/3}}{m} = m^{1/3} n^{2/3} = (mn^2)^{1/3}$$

$$\therefore \sin^2 \theta = (mn^2)^{2/3} \quad (v)$$

Adding Eqs. (iv) and (v), we get

$$(m^2 n)^{2/3} + (mn^2)^{2/3} = \cos^2 \theta + \sin^2 \theta$$

$$\text{or } (m^2 n)^{2/3} + (mn^2)^{2/3} = 1$$

ILLUSTRATION 2.18

If $\frac{\cos^4 A}{\cos^2 B} + \frac{\sin^4 A}{\sin^2 B} = 1$, then prove that

$$(i) \sin^4 A + \sin^4 B = 2 \sin^2 A \sin^2 B$$

$$(ii) \frac{\cos^4 B}{\cos^2 A} + \frac{\sin^4 B}{\sin^2 A} = 1$$

Sol. Given,

$$\frac{\cos^4 A}{\cos^2 B} + \frac{\sin^4 A}{\sin^2 B} = 1 = (\cos^2 A + \sin^2 A)$$

$$\text{or } \frac{\cos^4 A}{\cos^2 B} - \cos^2 A = \sin^2 A - \frac{\sin^4 A}{\sin^2 B}$$

$$\text{or } \frac{\cos^2 A (\cos^2 A - \cos^2 B)}{\cos^2 B} = \sin^2 A \frac{(\sin^2 B - \sin^2 A)}{\sin^2 B}$$

$$\text{or } \frac{\cos^2 A}{\cos^2 B} (\cos^2 A - \cos^2 B) = \frac{\sin^2 A}{\sin^2 B} [(1 - \cos^2 B) - (1 - \cos^2 A)]$$

$$\text{or } \frac{\cos^2 A}{\cos^2 B} (\cos^2 A - \cos^2 B) = \frac{\sin^2 A}{\sin^2 B} (\cos^2 A - \cos^2 B)$$

$$\text{or } (\cos^2 A - \cos^2 B) \left(\frac{\cos^2 A}{\cos^2 B} - \frac{\sin^2 A}{\sin^2 B} \right) = 0$$

When $\cos^2 A - \cos^2 B = 0$, we have

$$\cos^2 A = \cos^2 B \quad (i)$$

When $\frac{\cos^2 A}{\cos^2 B} - \frac{\sin^2 A}{\sin^2 B} = 0$, we have

$$\cos^2 A \sin^2 B = \sin^2 A \cos^2 B$$

$$\text{or } \cos^2 A (1 - \cos^2 B) = (1 - \cos^2 A) \cos^2 B$$

$$\text{or } \cos^2 A - \cos^2 A \cos^2 B = \cos^2 B - \cos^2 A \cos^2 B$$

$$\text{or } \cos^2 A = \cos^2 B \quad (ii)$$

Thus, in both the cases, $\cos^2 A = \cos^2 B$. Therefore,

$$\therefore 1 - \sin^2 A = 1 - \sin^2 B \quad \text{or } \sin^2 A = \sin^2 B \quad (iii)$$

$$(i) \text{ L.H.S.} = \sin^4 A + \sin^4 B \\ = (\sin^2 A - \sin^2 B)^2 + 2 \sin^2 A \sin^2 B \\ = 2 \sin^2 A \sin^2 B = \text{R.H.S.} \quad [\because \sin^2 A = \sin^2 B]$$

$$(ii) \text{ L.H.S.} = \frac{\cos^4 B}{\cos^2 A} + \frac{\sin^4 B}{\sin^2 A} = \frac{\cos^4 B}{\cos^2 B} + \frac{\sin^4 B}{\sin^2 B} \\ = \cos^2 B + \sin^2 B = 1 = \text{R.H.S.}$$

ILLUSTRATION 2.19

If $x = \sec \theta - \tan \theta$ and $y = \operatorname{cosec} \theta + \cot \theta$, then prove that $xy + 1 = y - x$.

$$\begin{aligned} \text{Sol. } xy + 1 &= \left(\frac{1 - \sin \theta}{\cos \theta} \right) \left(\frac{1 + \cos \theta}{\sin \theta} \right) + 1 = \frac{1 - \sin \theta + \cos \theta}{\sin \theta \cos \theta} \\ &= \frac{(\sin^2 \theta + \cos^2 \theta)}{\sin \theta \cos \theta} - \frac{(\sin \theta - \cos \theta)}{\sin \theta \cos \theta} \\ &= (\tan \theta + \cot \theta) - (\sec \theta - \operatorname{cosec} \theta) \\ &= (\operatorname{cosec} \theta + \cot \theta) - (\sec \theta - \tan \theta) \\ &= y - x \end{aligned}$$

ILLUSTRATION 2.20

For acute angle θ , prove the following:

$$(i) \sec^2 \theta \operatorname{cosec}^2 \theta \geq 4$$

$$(ii) \sec^2 \theta + \operatorname{cosec}^2 \theta \geq 4$$

Sol. Using A.M. $\geq$ G.M.

$$\begin{aligned} \frac{\sin^2 \theta + \cos^2 \theta}{2} &\geq (\sin^2 \theta \cos^2 \theta)^{1/2} \\ \Rightarrow \frac{1}{2} &\geq (\sin^2 \theta \cos^2 \theta)^{1/2} \Rightarrow \sin^2 \theta \cos^2 \theta \leq \frac{1}{4} \\ \Rightarrow \sec^2 \theta \operatorname{cosec}^2 \theta &\geq 4 \Rightarrow \frac{\sin^2 \theta + \cos^2 \theta}{\sin^2 \theta \cos^2 \theta} \geq 4 \\ \Rightarrow \sec^2 \theta + \operatorname{cosec}^2 \theta &\geq 4 \end{aligned}$$

CONCEPT APPLICATION EXERCISE 2.2

1. Prove that $\frac{\sin x - \cos x + 1}{\sin x + \cos x - 1} = \sec x + \tan x$.
2. If $15 \sin^4 \alpha + 10 \cos^4 \alpha = 6$ then find the value of $8 \operatorname{cosec}^6 \alpha + 27 \sec^6 \alpha$.
3. If $\sec \theta + \tan \theta = p$, then find the value of $\tan \theta$.
4. If $(1 + \sin A)(1 + \sin B)(1 + \sin C) = (1 - \sin A)(1 - \sin B)(1 - \sin C)$, then prove that $(1 + \sin A)(1 + \sin B)(1 + \sin C) = \pm \cos A \cdot \cos B \cdot \cos C$.
5. If $(\sec \theta + \tan \theta)(\sec \phi + \tan \phi)(\sec \psi + \tan \psi) = \tan \theta \tan \phi \tan \psi$, then prove that $(\sec \theta - \tan \theta)(\sec \phi - \tan \phi)(\sec \psi - \tan \psi) = \cot \theta \cdot \cot \phi \cdot \cot \psi$.
6. If $\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1$, $\frac{x}{a} \sin \theta - \frac{y}{b} \cos \theta = 1$, then eliminate θ .
7. If $a + b \tan \theta = \sec \theta$ and $b - a \tan \theta = 3 \sec \theta$, then find the value of $a^2 + b^2$.
8. If $a \sin^2 x + b \cos^2 x = c$, $b \sin^2 y + a \cos^2 y = d$, and $a \tan x = b \tan y$, then prove that $\frac{a^2}{b^2} = \frac{(d-a)(c-a)}{(b-c)(b-d)}$.

ANSWERS

2. 65 3. $\frac{p^2 - 1}{2p}$ 6. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$ 7. 10

DEGREE MEASUREMENT

If a rotation from the initial side to the terminal side is $(1/360)$ th of a revolution, the angle is said to have a measure of one degree, written as 1° . A degree is divided into 60 minutes, and a minute is divided into 60 seconds. One sixtieth of a degree is called a minute, written as $1'$; one sixtieth of minute is called a second, written as $1''$. Thus, $1^\circ = 60'$ and $1' = 60''$.

RADIAN MEASUREMENT

Radian describes the plane angle subtended by a circular arc as the length of the arc divided by the radius of the arc. One radian is the angle subtended at the center of a circle by an arc that is equal in length to the radius of the circle. More generally, the magnitude in radians of such a subtended angle is equal to the ratio of the arc length to the radius of the circle; that is, $\theta = s/r$, where θ is the subtended angle in radians, s is the arc length, and r is the radius. Conversely, the length of the enclosed arc is equal to the radius multiplied by the magnitude of the angle in radians; that is, $s = r\theta$. As the ratio of two lengths, the radian is a "pure number" that needs no unit symbol, and in mathematical writing the symbol "rad" is almost always omitted. In the absence of any symbol, radians are assumed. When degrees are meant, the symbol " $^\circ$ " is used.

Although the radian is a unit of measure, it is a dimensionless quantity.

The radian measurement of an angle is a real number associated with each angle measured in degrees. This makes the analysis of trigonometric function simple and elegant as we plot the graph of functions on the Cartesian plane where arguments of a function on the x-axis are real numbers.

CONVERSION BETWEEN RADIAN AND DEGREES

It follows that the magnitude in radians of one complete revolution (360 degree) is the length of the entire circumference divided by the radius, or $2\pi r/r$ or 2π . Thus, 2π radian is equal to 360 degree, meaning that one radian is equal to $180/\pi$ degree. As stated, one radian is equal to $180/\pi$ degree. Thus, to convert radians into degrees, multiply by $180/\pi$.

$$\text{Angle in degrees} = \left(\text{Angle in radians} \times \frac{180}{\pi} \right)^\circ$$

For example,

$$1 \text{ rad} = \left(1 \times \frac{180}{\pi} \right)^\circ \approx 57.2958^\circ$$

Conversely, to convert degrees into radians, multiply by $\pi/180$.

$$\text{Angle in radians} = \text{Angle in degrees} \times \frac{\pi}{180}$$

For example,

$$1^\circ = 1 \times \frac{\pi}{180} \approx 0.0175 \text{ rad}$$

Conversion of Some Common Angles

$$30^\circ = \frac{\pi}{6} \text{ rad} \quad 45^\circ = \frac{\pi}{4} \text{ rad}$$

$$60^\circ = \frac{\pi}{3} \text{ rad} \quad 90^\circ = \frac{\pi}{2} \text{ rad}$$

RELATION BETWEEN RADIAN AND REAL NUMBERS

Consider a unit circle with center O . Let A be any point on the circle. Consider OA as the initial side of an angle. Then the length of an arc of the circle gives the radian measure of the angle which the arc subtends at the center of the circle.

Consider line PAQ which is tangent to the circle at A . Let point A represents the real number zero, AP represents a positive real number, and AQ represents a negative real number (see figure). If we rope line AP in the counterclockwise direction along the circle, and AQ in the clockwise direction, then every real number corresponds to a radian measure and conversely. Thus, radian measures and real numbers can be considered as one and the same.

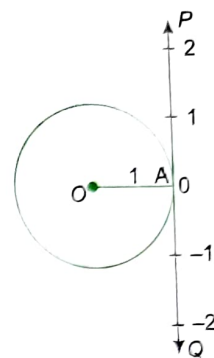


ILLUSTRATION 2.21

Express $45^\circ 20' 10''$ in radian measure ($\pi = 3.1415$).

$$\text{Sol. } 10'' = \frac{10}{60} \text{ min} = \left(\frac{10}{60 \times 60} \right)^\circ = \left(\frac{1}{360} \right)^\circ$$

$$20' = \left(\frac{20}{60} \right)^\circ = \left(\frac{1}{3} \right)^\circ$$

$$\begin{aligned} \therefore 45^\circ 20' 10'' &= \left(45 + \frac{1}{360} + \frac{1}{3} \right)^\circ \\ &= \left(\frac{16200 + 1 + 120}{360} \right)^\circ = \left(\frac{16321}{360} \right)^\circ \end{aligned}$$

$$\begin{aligned} \text{Now } \left(\frac{16321}{360} \right)^\circ &= \frac{16321}{360} \times \frac{\pi}{180} \text{ rad} \\ &= \frac{16321}{360} \times \frac{3.1416}{180} = \frac{51274.054}{64800} \\ &= 0.79 \text{ rad} \end{aligned}$$

ILLUSTRATION 2.22

Express 1.2 rad in degree measure.

$$\begin{aligned} \text{Sol. } (1.2)^R &= 1.2 \times \left(\frac{180}{\pi} \right)^\circ \\ &= \left(1.2 \times \frac{180 \times 7}{22} \right)^\circ \\ &= 68.7272^\circ \\ &= 68^\circ (0.7272 \times 60)' \\ &= 68^\circ (43.63)' \\ &= 68^\circ 43' (0.63 \times 60)'' \\ &= 68^\circ 43' 37.8'' \end{aligned}$$

ILLUSTRATION 2.23

Find the length of an arc of a circle of radius 5 cm subtending a central angle measuring 15° .

Sol. Let s be the length of the arc subtending an angle θ^R at the center of a circle of radius r . Then, $\theta = s/r$.

$$\text{Here, } r = 5 \text{ cm and } \theta = 15^\circ = \left(15 \times \frac{\pi}{180}\right)^R = \left(\frac{\pi}{12}\right)^R$$

$$\therefore \theta = \frac{s}{r} \text{ or } \frac{\pi}{12} = \frac{s}{5} \text{ or } s = \frac{5\pi}{12} \text{ cm}$$

ILLUSTRATION 2.24

Find in degrees the angle subtended at the center of a circle of diameter 50 cm by an arc of length 11 cm.

Sol. Here, $r = 25$ cm and $s = 11$ cm. Therefore,

$$\theta = \left(\frac{s}{r}\right)^R$$

$$\begin{aligned} \text{or } \theta &= \left(\frac{11}{25}\right)^R = \left(\frac{11}{25} \times \frac{180}{\pi}\right)^\circ = \left(\frac{11}{25} \times \frac{180}{22} \times 7\right)^\circ \\ &= \left(\frac{126}{5}\right)^\circ = \left(25\frac{1}{5}\right)^\circ = 25^\circ \left(\frac{1}{5} \times 60\right)' = 25^\circ 12' \end{aligned}$$

ILLUSTRATION 2.25

If arcs of same length in two circles subtend angles of 60° and 75° at their centers, find the ratios of their radii.

Sol. Let r_1 and r_2 be the radii of the given circles and let their arcs be of same length s subtending angles of 60° and 75° at their centers. Now,

$$60^\circ = \left(60 \times \frac{\pi}{180}\right)^R = \left(\frac{\pi}{3}\right)^R$$

$$\text{and } 75^\circ = \left(75 \times \frac{\pi}{180}\right)^R = \left(\frac{5\pi}{12}\right)^R$$

$$\therefore \frac{\pi}{3} = \frac{s}{r_1} \text{ and } \frac{5\pi}{12} = \frac{s}{r_2}$$

$$\Rightarrow \frac{\pi}{3} r_1 = s \text{ and } \frac{5\pi}{12} r_2 = s$$

$$\Rightarrow \frac{\pi}{3} r_1 = \frac{5\pi}{12} r_2 \Rightarrow 4r_1 = 5r_2 \Rightarrow \frac{r_1}{r_2} = \frac{5}{4}$$

Hence, $r_1 : r_2 = 5 : 4$.

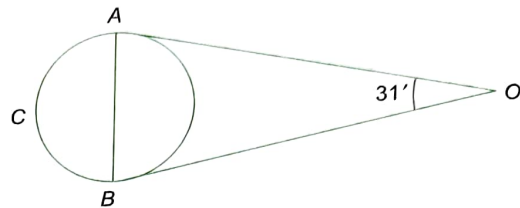
ILLUSTRATION 2.26

Assuming the distance of the earth from the moon to be 3,84,400 km and the angle subtended by the moon at the eye of a person on the earth to be $31'$, find the diameter of the moon.

Sol. Let AB be the diameter of the moon and O be the observer.

$$\text{Given, } \angle AOB = 31' = \frac{31}{60} \times \frac{\pi}{180} \text{ rad}$$

Since the angle subtended by the moon is very small, its diameter will be approximately equal to the small arc of a circle whose center is the eye of the observer and the radius is the distance of the earth from the moon. Also the moon subtends an angle of $31'$ at the center of this circle. Thus,



$$\theta = \frac{l}{r}$$

$$\therefore \frac{31}{60} \times \frac{\pi}{180} = \frac{AB}{384400}$$

$$\text{or } AB = \frac{31}{60} \times \frac{22}{7 \times 180} \times 384400 = 3467.74 \text{ km}$$

ILLUSTRATION 2.27

Find the angle between the minute hand and the hour hand of a clock when the time is 7:20 AM.

Sol. We know that the hour hand completes one rotation in 12 h, while the minute hand completes one rotation in 60 min.

Therefore, the angle traced by the hour hand in 12 h is 360° .

Angle traced by the hour hand in 7 h 20 min, i.e., $\frac{22}{3}$ h

$$= \left(\frac{360}{12} \times \frac{22}{3}\right)^\circ = 220^\circ$$

Also, angle traced by the minute hand in 60 min = 360°

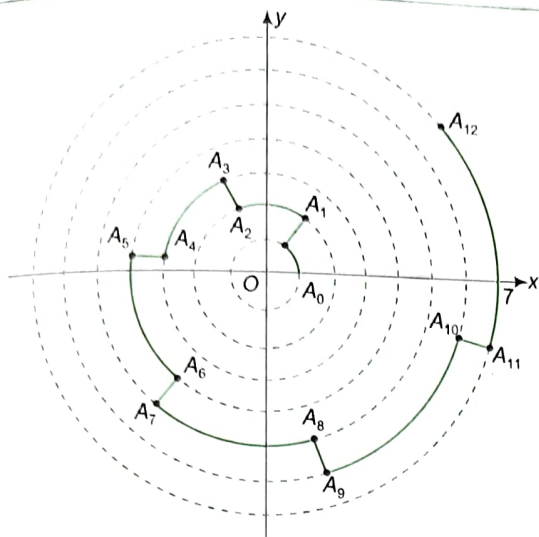
Angle traced by the minute hand in 20 min = $\left(\frac{360}{60} \times 20\right)^\circ = 120^\circ$

Hence, required angle between the two hands is $220^\circ - 120^\circ = 100^\circ$.

ILLUSTRATION 2.28

For each natural number k , let C_k denote a circle with radius k centimeters and center at origin O . On the circle C_k a particle moves k centimeters in the counterclockwise direction. After completing its motion on C_k , the particle moves to C_{k+1} in the radial direction. The motion of the particle continues in this manner. The particle starts at $(1, 0)$. If the particle crosses the positive direction of the x -axis for the first time on the circle C_n , then find the value of n .

Sol. The motion of the particle on the circles is shown with bold lines in the figure starting at $A_0(1, 0)$. Note that on every circle the particle travels just 1 rad. The particle crosses the positive direction of the x -axis first time on C_n , where n is the least positive integer such that $n \geq 2\pi \Rightarrow n = 7$.



For example, if OA and OB are the initial and final positions of the revolving ray, then the angle formed will be $\angle AOB$.

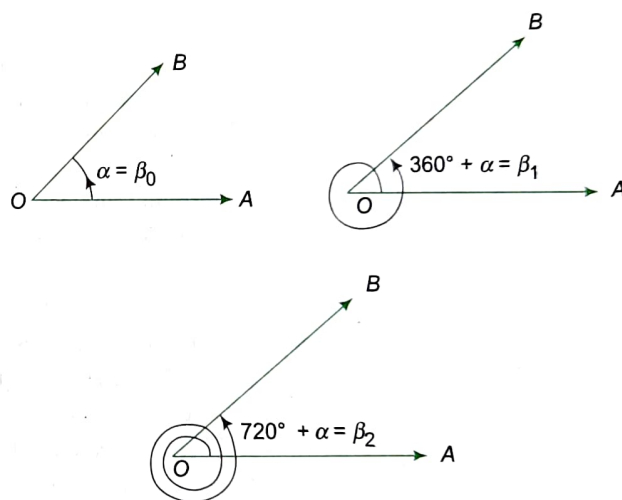
Angles Exceeding 360°

In geometry, we confine ourselves to angles from 0° to 360° . But there may be problems in which rotation involves more than one revolution, for example, the rotation of a flywheel. In trigonometry, we generalize the concept of angle to angles greater than 360° . This angle can be formed in the following way:

The revolving line (radius vector) starts from the initial position $\overrightarrow{OA}$ and makes n complete revolutions in counterclockwise direction and also a further angle α in the same direction. We then have a certain angle β_n given by $\beta_n = 360^\circ \times n + \alpha$, where $0^\circ < \alpha < 360^\circ$ and n is a positive integer or zero.

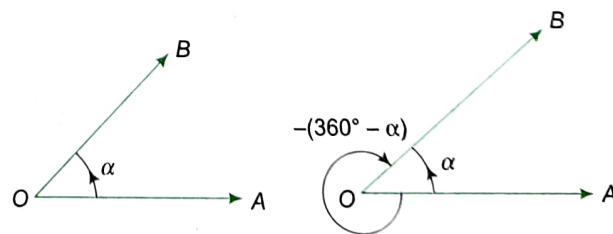
Thus, there are infinitely many β_n angles with initial side $\overrightarrow{OA}$ and final side $\overrightarrow{OB}$.

For example, $\beta_0 = \alpha$, $\beta_1 = 360^\circ + \alpha$, $\beta_2 = 720^\circ + \alpha$, etc.



Sign of Angles

Angles formed by the counterclockwise rotation of the radius vector are taken as positive, whereas angles formed by the clockwise rotation of the radius vector are taken as negative.



Quadrant of Angle

It is often useful to superimpose a Cartesian coordinate system on an angle so that the vertex is at the origin and the initial side is along the positive x -axis. The angle is then said to be in standard position.

If the terminal side of an angle θ is on one of the coordinate axes, then θ is called a quadrantal angle. If θ is not a quadrantal angle, then it is said to be in a certain quadrant if its terminal side lies in that quadrant.

CONCEPT APPLICATION EXERCISE 2.3

1. A horse is tied to a post by a rope. If the horse moves along a circular path always keeping the rope tight and describes 88 m when it has traced out 72° at the center, find the length of the rope.
2. If the angular diameter of the moon is $30'$, how far from the eye a coin of diameter 2.2 cm can be kept to hide the moon?
3. Find in degrees and radians the angle between the hour hand and the minute hand of a clock at half past three.
4. There is an equilateral triangle with each side 4 unit and a circle with the center on one of the vertex of that triangle. The arc of that circle divides the triangle into two parts of equal area. How long is the radius of the circle?

ANSWERS

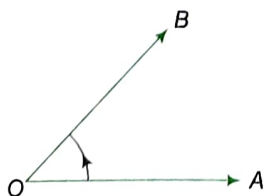
1. 70 m
2. 252 cm
3. 75° or $\frac{5\pi}{12}$ rad
4. $\sqrt{\frac{12\sqrt{3}}{\pi}}$

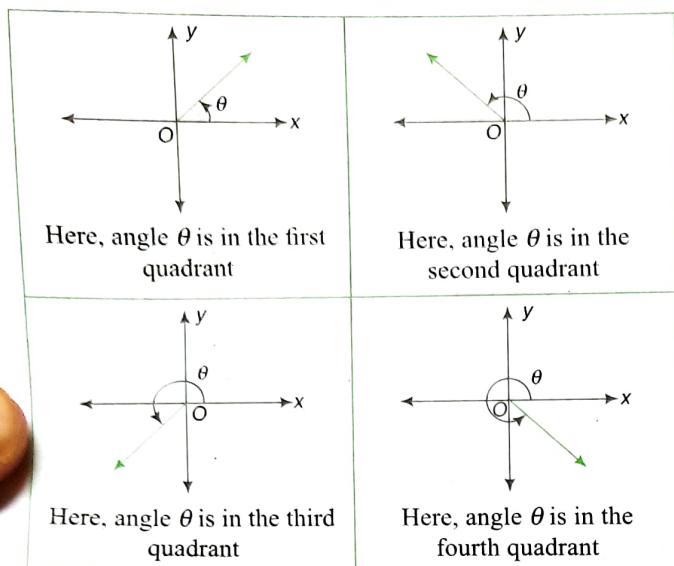
TRIGONOMETRIC FUNCTIONS OF AN ANGLE

ANGLES IN TRIGONOMETRY

In trigonometry, the idea of angle is more general; it may be positive or negative and has any magnitude.

In trigonometry, as in case of geometry, the measure of angle is the amount of rotation from the direction of one ray of the angle to the other. The initial and final positions of the revolving ray are, respectively, called the initial side (arm) and the terminal side (arm), and the revolving line is called the generating line or the radius vector.





Coterminal Angles

Angles in standard position that have the same terminal sides are called coterminal angles. Given any angle α , there is an unlimited number of angles coterminal with α (some positive and some negative).

Do not confuse equal angles with coterminal angles. Equal angles are angles with the same measure, so 30° and -330° are not equal angles, but are coterminal angles. (In fact, how could a positive number ever be equal to a negative number?) Equal angles are always coterminal angles, but coterminal angles are not necessarily equal. For example, if we say that $\angle ABC$ is equal to $\angle CBA$, we mean that both name the same angle; however, if we say that $\angle ABC$ is coterminal with $\angle CBA$, we mean that they share the same terminal side, but are not necessarily equal.

From the above discussion it is clear that coterminal angles differ by $(n \times 360^\circ)$ or $2\pi n$, where $n \in I$.

ILLUSTRATION 2.29

State if the given angles are coterminal.

(i) $\alpha = 185^\circ, \beta = -545^\circ$ (ii) $\alpha = \frac{17\pi}{36}, \beta = \frac{161\pi}{36}$

Sol.

(i) $\alpha = 185^\circ, \beta = -545^\circ$

Here, $\alpha - \beta = 185^\circ - (-545^\circ) = 730^\circ$

Hence, the given angles are not coterminal.

(ii) $\alpha = \frac{17\pi}{36}, \beta = \frac{161\pi}{36}$

Here, $\beta - \alpha = \frac{161\pi}{36} - \frac{17\pi}{36} = \frac{144\pi}{36} = 4\pi$

Hence, the given angles are coterminal.

Reference Angle

Let θ be an angle in standard position. Its reference angle is the acute angle θ' formed by the terminal side of θ and the x -axis.

Figure shows reference angles with angle θ lying in different quadrants.

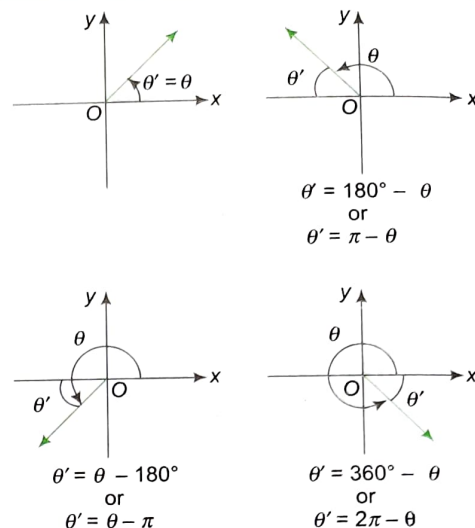


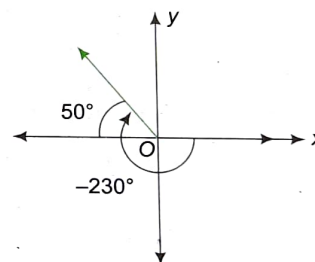
ILLUSTRATION 2.30

Find the reference angles corresponding to each of the following angles. It may help if you sketch θ in standard position.

(i) $\theta = -230^\circ$ (ii) $\frac{31\pi}{9}$ (iii) $\theta = 640^\circ$

Sol.

(i) $\theta = -230^\circ$



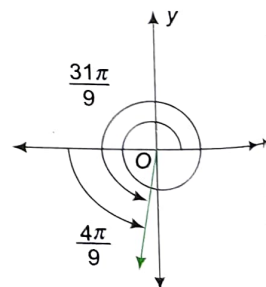
Hence, the reference angle is 50° .

(ii) $\theta = \frac{31\pi}{9}$

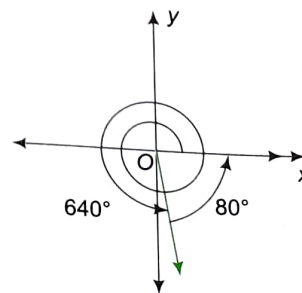
$\theta = \frac{31\pi}{9} = 3\pi + \frac{4\pi}{9}$

Hence, the reference angle

is $\frac{4\pi}{9}$.



(iii) $\theta = 640^\circ$

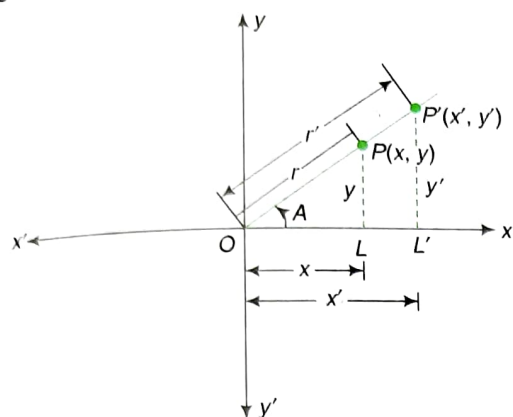


Here, $\theta = 2 \times 360^\circ - 80^\circ$

Hence, the reference angle is 80° .

TRIGONOMETRIC FUNCTIONS OF ANY ANGLE

Consider an acute angle A . Let us place the coordinate axis on this angle such that vertex of the angle is at origin and initial side of an angle is on the positive x -axis as shown in the given figure.



Obviously, for acute angle, terminal side lies in first quadrant.

Now, we choose any point P on the terminal side of angle A .

Let the coordinates of P be (x, y) and its distance from the origin be r . Then we define trigonometric ratios as follows:

$$(i) \sin A = \frac{y}{r} = \frac{y'}{r'}$$

$$(ii) \cos A = \frac{x}{r} = \frac{x'}{r'}$$

$$(iii) \tan A = \frac{y}{x} = \frac{y'}{x'}$$

$$(iv) \operatorname{cosec} A = \frac{r}{y} = \frac{r'}{y'}$$

$$(v) \sec A = \frac{r}{x} = \frac{r'}{x'}$$

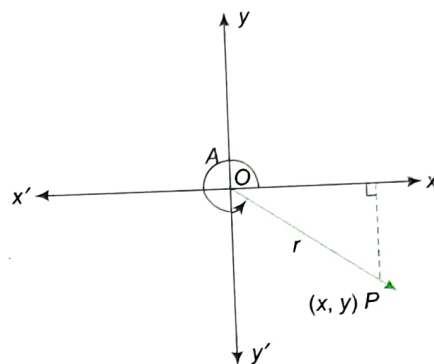
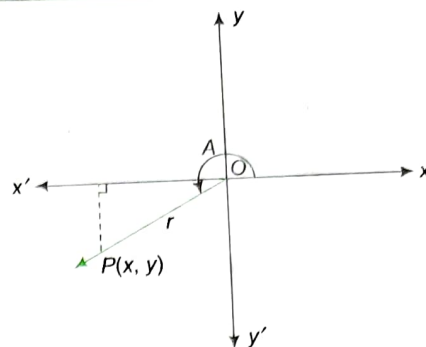
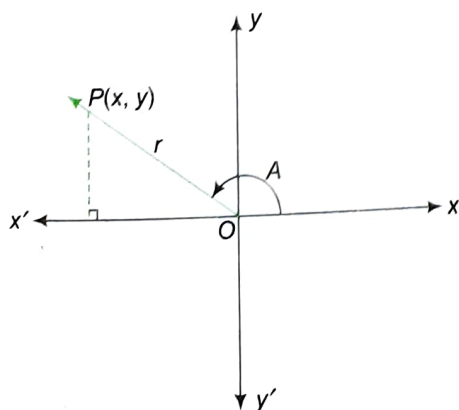
$$(vi) \cot A = \frac{x}{y} = \frac{x'}{y'}$$

In the above definition, there is nothing new. The only change is that the angle A is viewed with reference to the coordinate axis.

All above quantities are functions of the angle A alone. They do not depend on the choice of the point P on the terminal side. If we choose a different point P' (x', y') on the terminal side of angle A at a distance r' from the origin, we observe that above trigonometric ratios are same as sides of triangle $OP'L'$ are involved. This is because triangles OPL and $OP'L'$ are similar.

Since in first quadrant x and y are positive, all the above trigonometric ratios are positive.

Now, let us extend the above idea to define trigonometric ratios for any angle A for which terminal side lies in any of the four quadrants.



In all above cases again, we have

$$(i) \sin A = \frac{y}{r}$$

$$(ii) \cos A = \frac{x}{r}$$

$$(iii) \tan A = \frac{y}{x}$$

$$(iv) \operatorname{cosec} A = \frac{r}{y}$$

$$(v) \sec A = \frac{r}{x}$$

$$(vi) \cot A = \frac{x}{y}$$

Also, in all the cases, coordinates of point P on the terminal side are $(r \cos A, r \sin A)$. In each of the above cases, A is positive as terminal side is attained by anticlockwise rotation. A will be negative if position of the terminal ray is attained by clockwise rotation.

Any trigonometrical function of an angle A° is equal to the same trigonometrical function of an angle $(360 \times n + A)^\circ$, where n is any integer since all these angles will have the same terminal ray, e.g., $\sin 60^\circ = \sin 420^\circ = \sin (-300^\circ)$.

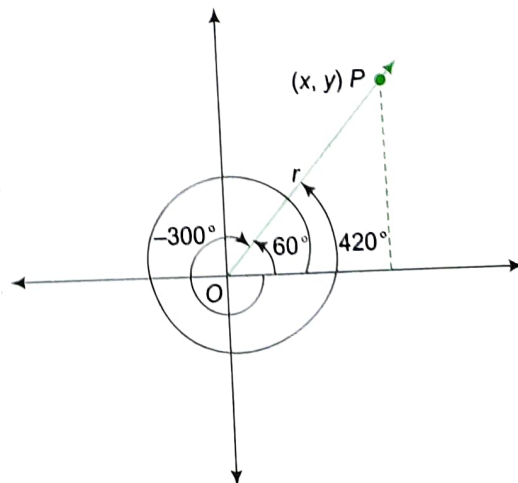


ILLUSTRATION 2.31

Suppose the point with coordinates $(-12, 5)$ is on the terminal side of angle θ . Find the values of the six trigonometric functions of θ .

Sol. Here, $r = \sqrt{(-12)^2 + 5^2} = 13$

Since terminal ray lies in the second quadrant, we have

$$\sin \theta = \frac{5}{13} \quad \operatorname{cosec} \theta = \frac{13}{5}$$

$$\cos \theta = -\frac{12}{13} \quad \sec \theta = -\frac{13}{12}$$

$$\tan \theta = -\frac{5}{12} \quad \cot \theta = -\frac{12}{5}$$

ILLUSTRATION 2.32

Evaluate the sine, cosine, and tangent of each of the following angles:

$$300^\circ, -405^\circ, \frac{7\pi}{6}, \frac{11\pi}{4}$$

Sol.

Angle, θ	Quadrant of θ	Reference angle (θ')	$\sin \theta$	$\cos \theta$	$\tan \theta$
$300^\circ = 360^\circ - 60^\circ$	Fourth	60°	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$
$-405^\circ = -360^\circ - 45^\circ$	Fourth	45°	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1
$\frac{7\pi}{6} = \pi + \frac{\pi}{6}$	Third	$\frac{\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{11\pi}{4} = 3\pi - \frac{\pi}{4}$	Second	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1

CONCEPT APPLICATION EXERCISE 2.4

- Let $(-3, -4)$ be a point on the terminal side of θ . Find the sine, cosine, and tangent of θ .
- Find the reference angle θ' for the following angles in standard position:
 - $\theta = 300^\circ$
 - $\theta = 2.3$
 - $\theta = -135^\circ$
- Evaluate each of the following trigonometric functions:
 - $\cos \frac{4\pi}{3}$
 - $\tan(-210^\circ)$
 - $\operatorname{cosec} \frac{11\pi}{4}$
- State if the given pairs of angles are coterminal.
 - $-185^\circ, 535^\circ$
 - $1000^\circ, 270^\circ$
 - $\frac{15\pi}{4}, -\frac{17\pi}{4}$

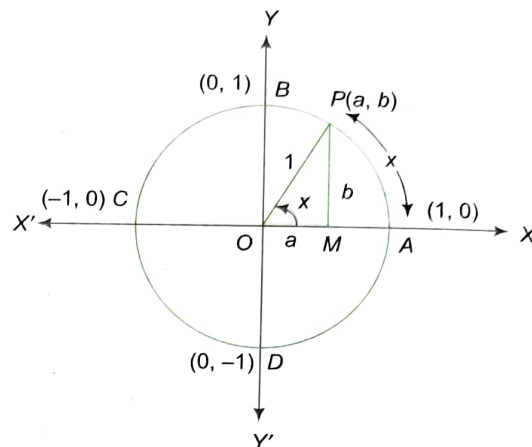
ANSWERS

- $\sin \theta = -\frac{4}{5}, \cos \theta = -\frac{3}{5}, \tan \theta = \frac{4}{3}$
- 60°
 - 0.8416
 - 45°
- $-\frac{1}{2}$
 - $-\frac{1}{\sqrt{3}}$
 - $\sqrt{2}$
- coterminal
 - not coterminal
 - coterminal

BEHAVIOUR OF TRIGONOMETRIC FUNCTIONS

TRIGONOMETRIC FUNCTIONS DEFINED AS CIRCULAR FUNCTIONS

The behaviour of trigonometric functions is best understood if we define them on unit circle with centre at origin of the coordinate axis.



Let $P(a, b)$ be any point on the circle with angle $AOP = x$ radians, i.e., length of arc $AP = x$. Clearly $\cos x = a$ and $\sin x = b$

In right triangle OMP ,

$$OM^2 + MP^2 = OP^2$$

$$\text{or } a^2 + b^2 = 1$$

Thus, for every point on the unit circle, we have

$$a^2 + b^2 = 1$$

$$\text{or } \cos^2 x + \sin^2 x = 1$$

Since one complete revolution subtends an angle of 2π radians at the centre of the circle, $\angle AOB = \frac{\pi}{2}$, $\angle AOC = \pi$ and $\angle AOD = \frac{3\pi}{2}$.

All angles which are integral multiples of $\frac{\pi}{2}$ are quadrantal angles.

The coordinates of the points A, B, C and D are, respectively, $(1, 0), (0, 1), (-1, 0)$ and $(0, -1)$.

Therefore, for quadrantal angles, we have

$$\cos 0^\circ = 1, \quad \sin 0^\circ = 0,$$

$$\cos \frac{\pi}{2} = 0, \quad \sin \frac{\pi}{2} = 1$$

$$\cos \pi = -1, \quad \sin \pi = 0$$

$$\cos \frac{3\pi}{2} = 0, \quad \sin \frac{3\pi}{2} = -1$$

$$\cos 2\pi = 1, \quad \sin 2\pi = 0$$

Now, if we take one complete revolution from the point P , we again come back to same point P .

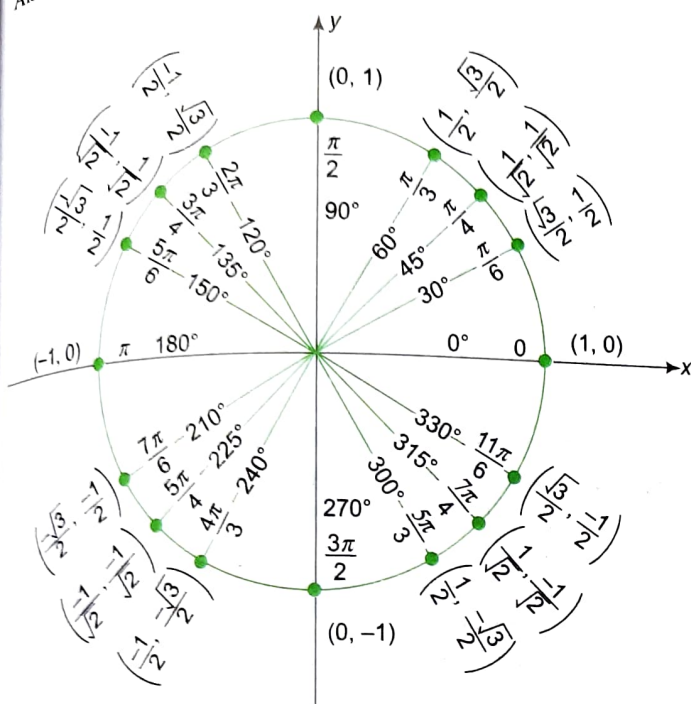
Thus, we also observe that if x increases (or decreases) by an integral multiple of 2π , the values of sine and cosine functions do not change.

Thus, $\sin(2n\pi + x) = \sin x, n \in \mathbb{Z}, \cos(2n\pi + x) = \cos x, n \in \mathbb{Z}$. Further, $\sin x = 0$, if $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \dots$, i.e., $\sin x = 0$ when x is an integral multiple of π .

And, $\cos x = 0$, if $x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \dots$ i.e., $\cos x$ vanishes when x is an odd multiple of $\pm \frac{\pi}{2}$.

Thus, $\sin x = 0$ implies $x = n\pi$, where n is any integer.

Also, $\cos x = 0$ implies $x = (2n+1)\frac{\pi}{2}$, where n is any integer.



Domain of Trigonometric Functions

We observe that point $P(a, b)$ can be obtained by any amount of rotation of ray OP , clockwise or anticlockwise.

Therefore, $\sin x$ and $\cos x$ functions are defined for all real values of x . Domain of these functions is R .

Functions $\tan x$ and $\sec x$ are defined for all values x except values of x where $\cos x = 0$.

So, domain of both the functions $\tan x$ and $\sec x$ is

$$R - \left\{ (2n+1)\frac{\pi}{2}, n \in Z \right\}.$$

Functions $\cot x$ and $\operatorname{cosec} x$ are defined for all values x except values of x where $\sin x = 0$.

So, domain of both the functions $\cot x$ and $\operatorname{cosec} x$ is $R - \{n\pi, n \in Z\}$.

Variation in the Values of Trigonometric Functions in Different Quadrants

We observe that in the first quadrant, as x increases from 0 to $\pi/2$, $\sin x$ increases from 0 to 1 and $\cos x$ decreases from 1 to 0.

So, in first quadrant, $\operatorname{cosec} x$ decreases from infinity (when x tends to zero) to 1 and $\sec x$ increases from 1 to infinity (when x tends to $\pi/2$). Also, $\tan x$ increases from 0 to infinity (when x tends to $\pi/2$) and $\cot x$ decreases from infinity (when x tends to zero) to 0.

In the second quadrant, as x increases from $\pi/2$ to π , $\sin x$ decreases from 1 to 0 and $\cos x$ decreases from 0 to -1 . From these, we can find variation of other trigonometric functions in this quadrant.

Variations of all trigonometric functions in all the quadrants are tabulated as follows:

($\uparrow$ means increasing and $\downarrow$ means decreasing)

	1 st quadrant	2 nd quadrant	3 rd quadrant	4 th quadrant
$\sin x$	$\uparrow$ from 0 to 1	$\downarrow$ from 1 to 0	$\downarrow$ from 0 to -1	$\uparrow$ from -1 to 0
$\cos x$	$\downarrow$ from 1 to 0	$\downarrow$ from 0 to -1	$\uparrow$ from -1 to 0	$\uparrow$ from 0 to 1
$\tan x$	$\uparrow$ from 0 to ∞	$\uparrow$ from $-\infty$ to 0	$\uparrow$ from 0 to ∞	$\uparrow$ from $-\infty$ to 0
$\cot x$	$\downarrow$ from ∞ to 0	$\downarrow$ from 0 to $-\infty$	$\downarrow$ from ∞ to 0	$\downarrow$ from 0 to $-\infty$
$\sec x$	$\uparrow$ from 1 to ∞	$\uparrow$ from $-\infty$ to -1	$\downarrow$ from -1 to $-\infty$	$\downarrow$ from ∞ to 1
$\operatorname{cosec} x$	$\downarrow$ from ∞ to 1	$\uparrow$ from 1 to ∞	$\uparrow$ from $-\infty$ to -1	$\downarrow$ from -1 to $-\infty$

Range of Trigonometric Functions

Since for every point $P(a, b)$ on the unit circle, $-1 \leq a \leq 1$ and $-1 \leq b \leq 1$, we have $-1 \leq \cos x \leq 1$ and $-1 \leq \sin x \leq 1$ for all x .

Thus, range of each of $\sin x$ and $\cos x$ is $[-1, 1]$.

From the above table, we establish that the range of each of $\tan x$ and $\cot x$ is set R (all real numbers).

Also, the range of each of $\sec x$ and $\operatorname{cosec} x$ is $R - (-1, 1)$ or $(-\infty, -1] \cup [1, \infty)$.

Also, we have

$$|\sin x|, |\cos x|, \sin^2 x, \cos^2 x \in [0, 1].$$

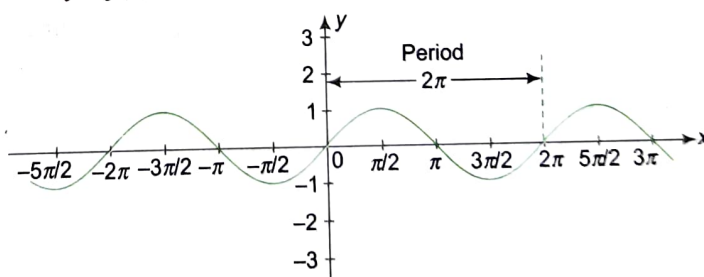
$$|\sec x|, |\operatorname{cosec} x|, \sec^2 x, \operatorname{cosec}^2 x \in [1, \infty)$$

$$|\tan x|, |\cot x|, \tan^2 x, \cot^2 x \in [0, \infty)$$

GRAPH OF TRIGONOMETRIC FUNCTIONS

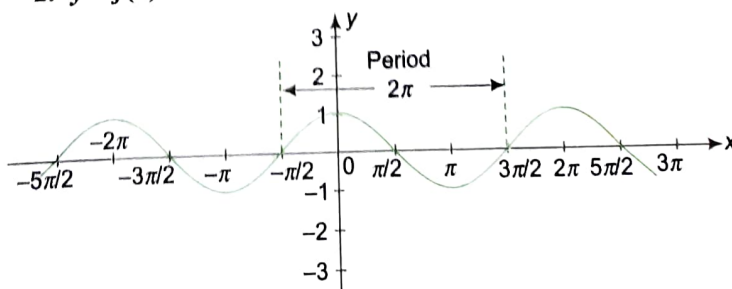
From the above discussion, we can draw the graphs of trigonometric functions as follows:

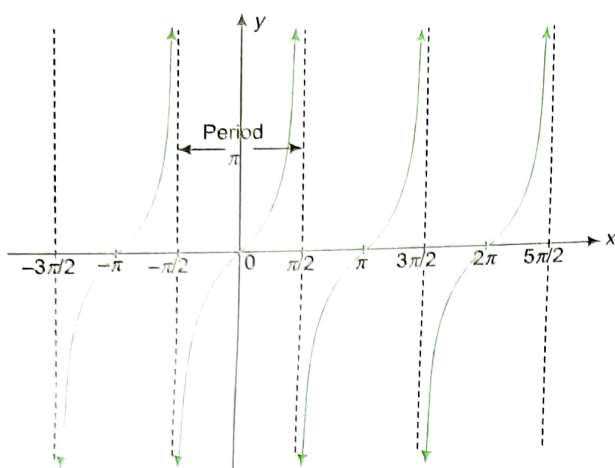
$$1. y = f(x) = \sin x$$



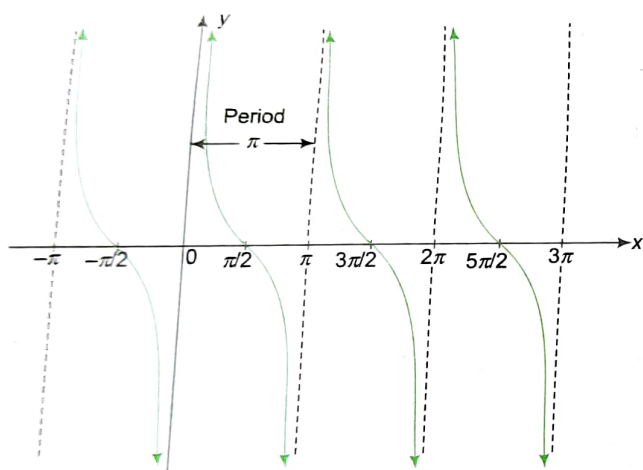
We observe that $\sin x$ completes one full cycle of its possible values (from -1 to 1) in the interval of length 2π . So, period of $\sin x$ is 2π . The graph drawn in the interval $[0, 2\pi]$ repeats to the right and the left.

$$2. y = f(x) = \cos x$$

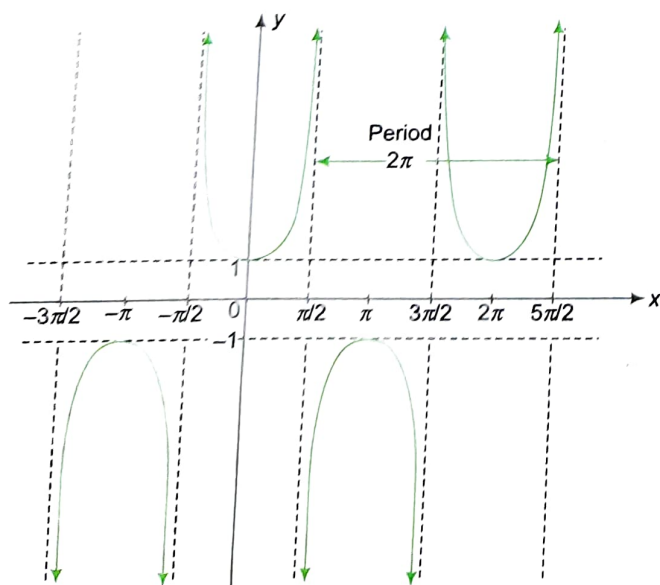
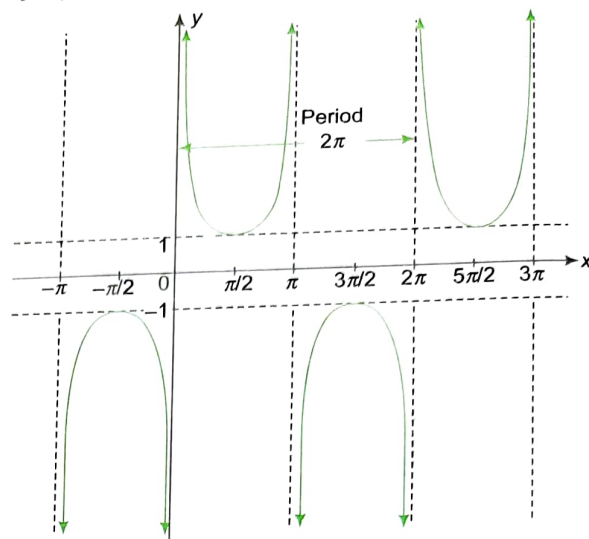


3. $y = f(x) = \tan x$ 

We observe that function $\tan x$ increases in each of the intervals ... $(-3\pi/2, -\pi/2)$, $(-\pi/2, \pi/2)$, $(\pi/2, 3\pi/2)$

4. $y = f(x) = \cot x$ 

We observe that function $\cot x$ decreases in each of the intervals ... $(-\pi, 0)$, $(0, \pi)$, $(\pi, 2\pi)$

5. $y = f(x) = \sec x$ 6. $y = f(x) = \operatorname{cosec} x$ 

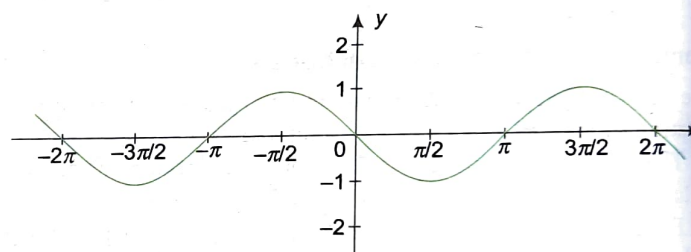
Transformation of Graph of Trigonometric Functions

1. The graph of $y = -f(x)$ from the graph of $y = f(x)$.

Clearly, $-f(x) > 0$ when $f(x) < 0$ and $-f(x) < 0$ when $f(x) > 0$.

So, graph of $y = -f(x)$ is obtained by flipping the graph of $y = f(x)$ upside down.

e.g., graph of $y = -\sin x$ is as shown in the following figure.



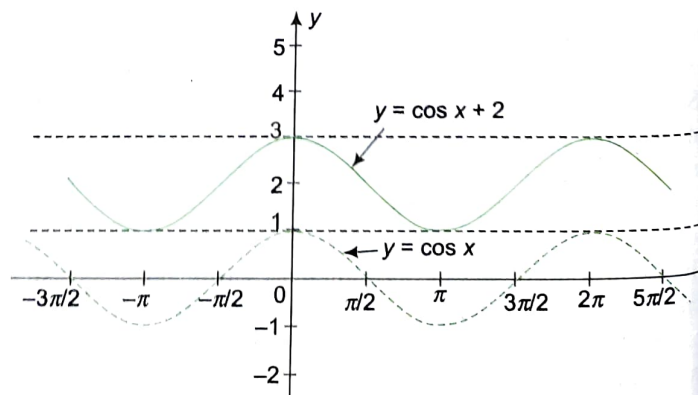
Clearly, period of function does not change in such transformation.

2. The graph of $y = f(x) + a$ from the graph of $y = f(x)$.

Graph of $y = f(x) + a$ is obtained by shifting the graph of $y = f(x)$, by a units upward if $a > 0$ and a units downward if $a < 0$.

The shape of the transformed graph remains the same. So period does not change but range changes.

e.g., graph of $y = \cos x + 2$ is as shown in the following figure:



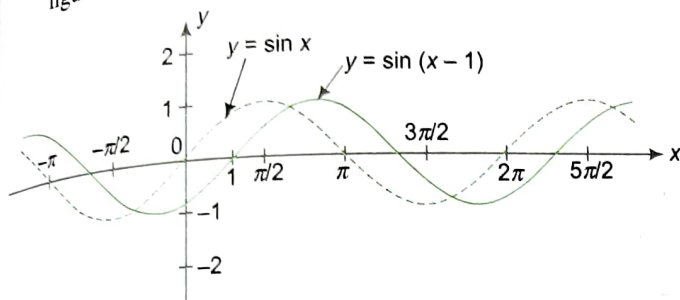
We observe that graph of $y = \cos x + 2$ lies between lines $y = 1$ and $y = 3$. So, range of function is $[1, 3]$.

3. The graph of $y = f(x + a)$ from the graph of $y = f(x)$.

Graph of $y = f(x + a)$ is obtained by shifting the graph of $y = f(x)$, by a units to the right if $a < 0$ and a units to the left if $a > 0$.

The shape of the transformed graph remains the same. So, period does not change. However, domain may change.

e.g., graph of $y = \sin(x - 1)$ is as shown in the following figure:

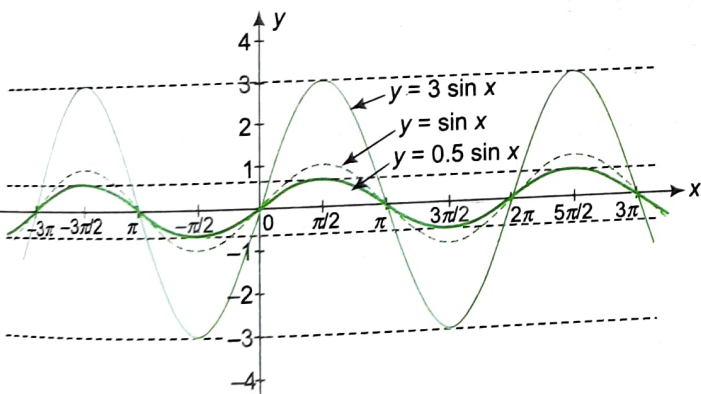


4. The graph of $y = af(x)$ from the graph of $y = f(x)$.

Graph of $y = af(x)$ is obtained by

- stretching vertically the graph of $y = f(x)$ if $a > 1$
- compressing vertically the graph of $y = f(x)$ if $0 < a < 1$
- stretching and flipping vertically the graph of $y = f(x)$ if $a < -1$
- compressing and flipping vertically the graph of $y = f(x)$ if $-1 < a < 0$

e.g., graphs of $y = 3 \sin x$, $y = 0.5 \sin x$ are as shown in the following figure.



We observe that period does not change but range changes. i.e., range of $y = 3 \sin x$ is $[-3, 3]$ and that of $y = 0.5 \sin x$ is $[-0.5, 0.5]$.

5. The graph of $y = f(ax)$ from the graph of $y = f(x)$.

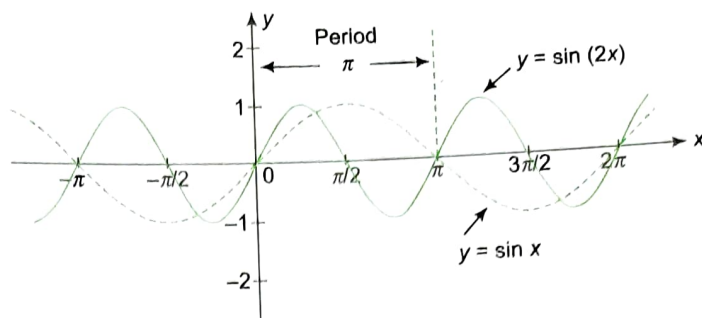
Consider function $y = \sin(2x)$.

If $x \in [0, \pi]$, then $2x \in [0, 2\pi]$.

So, $y = \sin(2x)$ completes its one cycle (of all its possible values) in the period of length π .

We can say that period of $y = \sin(2x)$ is π . So, to draw the graph of $y = \sin(2x)$, compress the graph of $y = \sin x$ horizontally such that period is π . Clearly, range does not change.

Graph of $y = \sin 2x$ is as shown in the following figure:



In general, if period of $y = f(x)$ is T then that of $y = f(ax)$ is $T/|a|$.

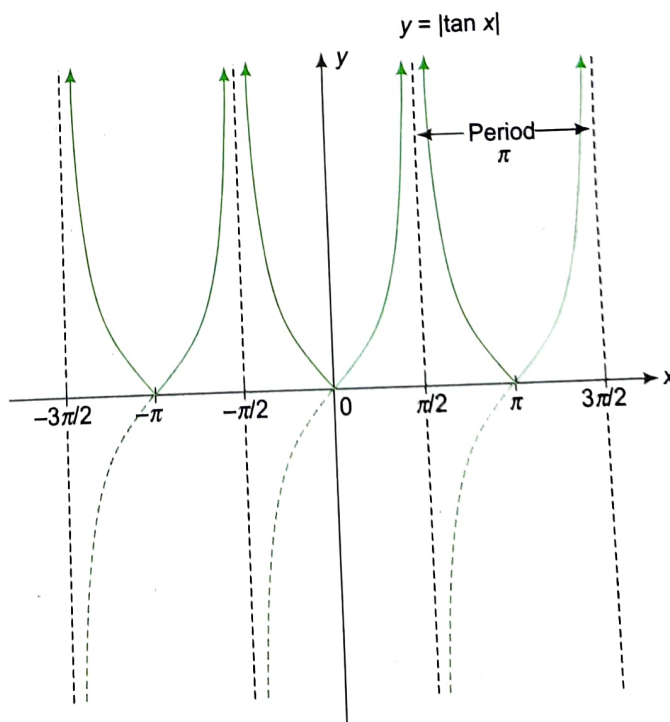
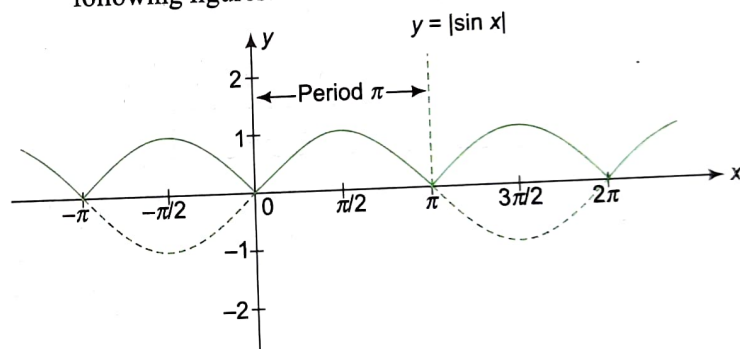
i.e., period of $y = \cos(x/3)$ is 6π , so graph of function is obtained by stretching the graph of $y = \cos x$ horizontally.

6. The graph of $y = |f(x)|$ from the graph of $y = f(x)$.

$$|f(x)| = \begin{cases} f(x), & f(x) \geq 0 \\ -f(x), & f(x) < 0 \end{cases}$$

So, graph of $y = |f(x)|$ coincides with the graph of $y = f(x)$, when $f(x) \geq 0$. For $f(x) < 0$, graph of $y = |f(x)|$ can be obtained by flipping the graph of $y = f(x)$ vertically upward.

e.g., graph of $y = |\sin x|$ and $y = |\tan x|$ are as shown in the following figures:



Note:

Period of $|f(x)|$, where $f(x)$ is trigonometric function is π .

ILLUSTRATION 2.33

Which of the following is possible?

- (a) $\sin \theta = \frac{5}{3}$ (b) $\tan \theta = 1002$
 (c) $\cos \theta = \frac{1+p^2}{1-p^2}$, ($p \neq 0, \pm 1$) (d) $\sec \theta = \frac{1}{2}$

Sol.

(b) $\sin \theta = \frac{5}{3}$ is not possible as $-1 \leq \sin \theta \leq 1$.

$\cos \theta = \frac{1+p^2}{1-p^2}$ is not possible, as in $\frac{1+p^2}{1-p^2}$, the numerator is always greater than the denominator for any value of p

other than $p = 0$. Hence, $\frac{1+p^2}{1-p^2}$ does not lie in $[-1, 1]$.

$\sec \theta = \frac{1}{2}$ is not possible as $\sec \theta \in (-\infty, -1] \cup [1, \infty)$.

$\tan \theta = 1002$ is possible as $\tan \theta$ can take any real value.

ILLUSTRATION 2.34

Find the values of p so that the equation $2 \cos^2 x - (p+3) \cos x + 2(p-1) = 0$ has a real solution.

Sol. We have $2 \cos^2 x - (p+3) \cos x + 2(p-1) = 0$

$$\Rightarrow \cos x = \frac{(p+3) \pm \sqrt{(p+3)^2 - 8(2p-2)}}{4}$$

$$= \frac{(p+3) \pm (p-5)}{4}$$

$$= \frac{p-1}{2}, 2$$

$$\Rightarrow \cos x = \frac{p-1}{2} \quad (\text{as } \cos x \neq 2)$$

$$\text{Since } -1 \leq \cos x \leq 1$$

$$\Rightarrow -2 \leq p-1 \leq 2$$

$$\Rightarrow -1 \leq p \leq 3$$

ILLUSTRATION 2.35

Find the values of a for which $a^2 - 6 \sin x - 5a \leq 0, \forall x \in \mathbb{R}$.

Sol. $a^2 - 6 \sin x - 5a \leq 0, \forall x \in \mathbb{R}$

$$\Rightarrow \frac{a^2 - 5a}{6} \leq \sin x, \forall x \in \mathbb{R}$$

$$\Rightarrow \frac{a^2 - 5a}{6} \leq -1$$

$$\Rightarrow a^2 - 5a + 6 \leq 0$$

$$\Rightarrow (a-3)(a-2) \leq 0$$

$$\Rightarrow a \in [2, 3]$$

ILLUSTRATION 2.36

Which of the following is the greatest?

- (a) $\tan 1$ (b) $\tan 4$ (c) $\tan 7$ (d) $\tan 10$

Sol.

$$(a) \tan 4 = \tan(\pi + (4 - \pi)) = \tan(4 - \pi) = \tan(0.86)$$

$$\tan 7 = \tan(2\pi + (7 - 2\pi)) = \tan(7 - 2\pi) = \tan(0.72)$$

$$\tan 10 = \tan(3\pi + (10 - 3\pi)) = \tan(10 - 3\pi) = \tan(0.58)$$

$$\text{Now, } 1 > 0.86 > 0.72 > 0.58$$

$$\Rightarrow \tan 1 > \tan(0.86) > \tan(0.72) > \tan(0.58)$$

[as angles 1, 0.86, 0.72, 0.58 lie in the first quadrant and tangent function increases in all the quadrant]

Hence, among the given values, $\tan 1$ is the greatest.

ILLUSTRATION 2.37

Which of the following is the least?

- (a) $\sin 3$ (b) $\sin 2$ (c) $\sin 1$ (d) $\sin 7$

Sol.

$$(d) \sin 3 = \sin[\pi - (\pi - 3)] = \sin(\pi - 3) = \sin(0.14)$$

$$\sin 2 = \sin[\pi - (\pi - 2)] = \sin(\pi - 2) = \sin(1.14)$$

$$\sin 7 = \sin[2\pi + (7 - 2\pi)] = \sin(7 - 2\pi) = \sin(0.72)$$

$$\text{Now, } 1.14 > 1 > 0.72 > 0.14$$

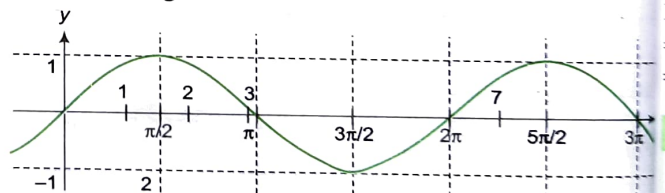
$$\Rightarrow \sin(1.14) > \sin 1 > \sin(0.72) > \sin(0.14)$$

[as 1.14, 1, 0.72, 0.14 lie in the first quadrant and sine function increase in the first quadrant]

Hence, among the given values, $\sin 3$ is the least.

Alternate solution:

We draw the graph of $y = \sin x$ and compare the height of graph for different angles.



From the graph, clearly $\sin 3$ is the least as height of graph is least for angle '3' radians.

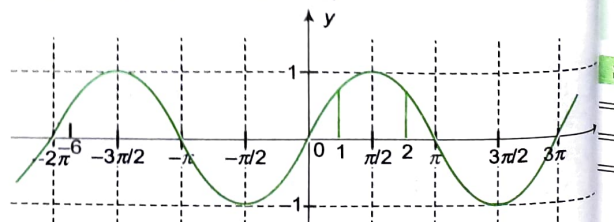
ILLUSTRATION 2.38

Which of the following is the greatest?

- (a) $\operatorname{cosec} 1$ (b) $\operatorname{cosec} 2$ (c) $\operatorname{cosec} 4$ (d) $\operatorname{cosec}(-6)$

Sol.

- (d) Consider $\sin 1$, $\sin 2$, and $-\sin 6$ ($\sin 4$ is negative; hence $\operatorname{cosec} 4$ cannot be maximum).



From the graph, $\sin(-6)$ is least and positive hence, $\operatorname{cosec}(-6)$ is maximum.

ILLUSTRATION 2.39

If $A = 4 \sin \theta + \cos^2 \theta$, then which of the following is not true?

- (a) Maximum value of A is 5.
- (b) Minimum value of A is -4 .
- (c) Maximum value of A occurs when $\sin \theta = 1/2$.
- (d) Minimum value of A occurs when $\sin \theta = 1$.

Sol. (a), (c), (d)

$$f(\theta) = 4 \sin \theta + \cos^2 \theta = 4 \sin \theta + 1 - \sin^2 \theta$$

$$= 5 - (4 \sin \theta + \sin^2 \theta) = 5 - (\sin \theta - 2)^2$$

Now maximum value of $f(\theta)$ occurs when $(\sin \theta - 2)^2$ is minimum.

Minimum value of $(\sin \theta - 2)^2$ occurs when $\sin \theta = 1$, then maximum value of $f(\theta)$ is $5 - (1 - 2)^2 = 4$.

Also minimum value of $f(\theta)$ occurs when $(\sin \theta - 2)^2$ is maximum.

Maximum value of $(\sin \theta - 2)^2$ occurs when $\sin \theta = -1$, then minimum value of $f(\theta)$ is $5 - (-1 - 2)^2 = -4$.

ILLUSTRATION 2.40

Find the values of x for which $3 \cos \theta = x^2 - 8x + 19$ holds good.

Sol. $3 \cos \theta = x^2 - 8x + 19$

$$\Rightarrow 3 \cos \theta = (x - 4)^2 + 3$$

$$\text{Now, L.H.S.} = 3 \cos \theta \leq 3$$

or L.H.S. has greatest value 3

$$\text{But R.H.S., } (x - 4)^2 + 3 \geq 3$$

or R.H.S. has least value 3

$$\text{Hence, L.H.S.} = \text{R.H.S. when } 3 \cos \theta = (x - 4)^2 + 3 = 3$$

$$\Rightarrow \cos \theta = 1 \text{ and } x - 4 = 0$$

$$\Rightarrow \theta = 2n\pi \text{ and } x = 4, \text{ where } n \in \mathbb{Z}.$$

ILLUSTRATION 2.41

Show that the equation $\sin \theta = x + \frac{1}{x}$ is not possible if x is real.

Sol. Given, $\sin \theta = x + \frac{1}{x}$

$$\therefore \sin^2 \theta = x^2 + \frac{1}{x^2} + 2 = \left(x + \frac{1}{x}\right)^2 + 4 \geq 4$$

which is not possible since $\sin^2 \theta \leq 1$.

ILLUSTRATION 2.42

If $\sin^2 \theta_1 + \sin^2 \theta_2 + \sin^2 \theta_3 = 0$, then which of the following is not the possible value of $\cos \theta_1 + \cos \theta_2 + \cos \theta_3$?

- (a) 3
- (b) -3
- (c) -1
- (d) -2

Sol. (d) $\sin^2 \theta_1 + \sin^2 \theta_2 + \sin^2 \theta_3 = 0$

$$\Rightarrow \sin^2 \theta_1 = \sin^2 \theta_2 = \sin^2 \theta_3 = 0$$

$$\Rightarrow \cos^2 \theta_1, \cos^2 \theta_2, \cos^2 \theta_3 = 1$$

$$\Rightarrow \cos \theta_1, \cos \theta_2, \cos \theta_3 = \pm 1$$

- $\cos \theta_1 + \cos \theta_2 + \cos \theta_3$ can be -3 (when all are -1)
 - or 3 (when all are $+1$)
 - or -1 (when any two are -1 and one is $+1$)
 - or 1 (when any two are $+1$ and one is -1)
- but -2 is not a possible value.

ILLUSTRATION 2.43

For real values of θ , which of the following is/are always positive?

- (a) $\cos(\cos \theta)$
- (b) $\cos(\sin \theta)$
- (c) $\sin(\cos \theta)$
- (d) $\sin(\sin \theta)$

Sol. a, b. $\cos \theta, \sin \theta \in [-1, 1]$ or value lies in 1st or 4th quadrant for which $\cos(\cos \theta)$ and $\cos(\sin \theta)$ are always greater than 0. $\sin(\cos \theta) < 0$, when $\cos \theta \in [-1, 0]$ and $\sin(\sin \theta) > 0$ only when $\sin \theta \in (0, 1]$

ILLUSTRATION 2.44

$$\text{Find the range of } f(x) = \frac{1}{4 \cos x - 3}.$$

Sol. $-1 \leq \cos x \leq 1$

$$\text{or } -4 \leq 4 \cos x \leq 4$$

$$\text{or } -7 \leq 4 \cos x - 3 \leq 1$$

$$\text{or } -7 \leq 4 \cos x - 3 < 0 \text{ or } 0 < 4 \cos x - 3 \leq 1$$

$$(\because 4 \cos x - 3 \neq 0)$$

$$\text{or } -\frac{1}{7} \geq \frac{1}{4 \cos x - 3} > -\infty \text{ or } \infty > \frac{1}{4 \cos x - 3} \geq 1$$

$$\text{or } \frac{1}{4 \cos x - 3} \in \left(-\infty, -\frac{1}{7}\right] \cup [1, \infty)$$

ILLUSTRATION 2.45

Find the range of $f(x) = \cos^2 x + \sec^2 x$.

Sol. We have

$$f(x) = \cos^2 x + \sec^2 x$$

$$= (\cos x - \sec x)^2 + 2 \cos x \sec x$$

$$= 2 + (\cos x - \sec x)^2 \geq 2$$

ILLUSTRATION 2.46

$$\text{Find the range of } f(x) = \frac{1}{5 \sin x - 6}.$$

Sol. $-1 \leq \sin x \leq 1$

$$\text{or } -5 \leq 5 \sin x \leq 5$$

$$\text{or } -11 \leq 5 \sin x - 6 \leq -1$$

$$\text{or } -1 \leq \frac{1}{5 \sin x - 6} \leq -1/11$$

$$\text{or } \frac{1}{5 \sin x - 6} \in [-1, -1/11]$$

ILLUSTRATION 2.47Find the range of $f(x) = \sin^2 x - 3 \sin x + 2$.

$$\begin{aligned}\text{Sol. } f(x) &= \sin^2 x - 3 \sin x + 2 \\ &= (\sin x - 3/2)^2 + 2 - 9/4 \\ &= (\sin x - 3/2)^2 - 1/4\end{aligned}$$

$$\text{Now } -1 \leq \sin x \leq 1$$

$$\text{or } -5/2 \leq \sin x - 3/2 \leq -1/2$$

$$\text{or } 1/4 \leq (\sin x - 3/2)^2 \leq 25/4$$

$$\text{or } 0 \leq (\sin x - 3/2)^2 - 1/4 \leq 6$$

ILLUSTRATION 2.48Find the range of $f(x) = \sqrt{\sin^2 x - 6 \sin x + 9} + 3$.

$$\begin{aligned}\text{Sol. } f(x) &= \sqrt{\sin^2 x - 6 \sin x + 9} + 3 \\ &= \sqrt{(\sin x - 3)^2} + 3 \\ &= |\sin x - 3| + 3\end{aligned}$$

$$\text{Now } -1 \leq \sin x \leq 1$$

$$\text{or } -4 \leq \sin x - 3 \leq -2$$

$$\text{or } 2 \leq |\sin x - 3| \leq 4$$

$$\text{or } 5 \leq |\sin x - 3| + 3 \leq 7$$

ILLUSTRATION 2.49If (x, y) satisfies the equation

$$1 + 4x - x^2 = \sqrt{9 \sec^2 y + 4 \operatorname{cosec}^2 y},$$

then find the value of x and $\tan^2 y$.**Sol.** We have

$$1 + 4x - x^2 = \sqrt{9 \sec^2 y + 4 \operatorname{cosec}^2 y}$$

$$\text{Now, } 1 + 4x - x^2 = 5 - (x - 2)^2 \leq 5$$

$$9 \sec^2 y + 4 \operatorname{cosec}^2 y = 13 + 9 \tan^2 y + 4 \cot^2 y$$

$$\geq (3 \tan y - 2 \cot y)^2 + 25 \geq 25$$

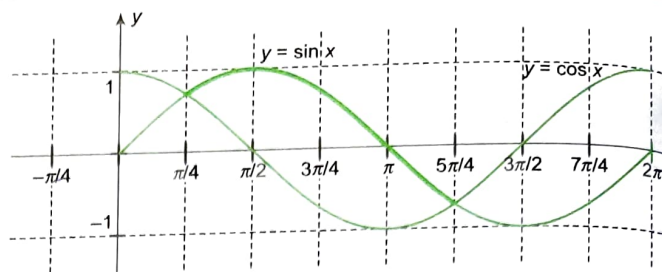
$$\sqrt{9 \sec^2 y + 4 \operatorname{cosec}^2 y} \geq 5$$

Hence, $1 + 4x - x^2 = \sqrt{9 \sec^2 y + 4 \operatorname{cosec}^2 y}$ is possible only when

$$1 + 4x - x^2 = \sqrt{9 \sec^2 y + 4 \operatorname{cosec}^2 y} = 5$$

$$\Rightarrow x = 2 \text{ and } 9 \tan^2 y + 4 \cot^2 y = 12$$

$$\Rightarrow x = 2 \text{ and } \tan^2 y = \frac{2}{3}$$

ILLUSTRATION 2.50Find the value of x for which $f(x) = \sqrt{\sin x - \cos x}$ is defined, $x \in [0, 2\pi]$.**Sol.** $f(x) = \sqrt{\sin x - \cos x}$ is defined if $\sin x \geq \cos x$.From the graph, $\sin x \geq \cos x$, for $x \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.**ILLUSTRATION 2.51**

Which of the following is/are correct?

$$(a) (\tan x)^{\ln(\cos x)} < (\cot x)^{\ln(\cos x)} \quad \forall x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$$

$$(b) (\sin x)^{\ln(\sec x)} > (\cos x)^{\ln(\sec x)} \quad \forall x \in \left(0, \frac{\pi}{4}\right)$$

$$(c) \left(\sec \frac{\pi}{3}\right)^{\ln(\tan x)} > \left(\sec \frac{\pi}{3}\right)^{\ln(\cos x)} \quad \forall x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$$

$$(d) \left(\frac{1}{2}\right)^{\ln(\sin x)} > \left(\frac{3}{4}\right)^{\ln(\sin x)} \quad \forall x \in \left(0, \frac{\pi}{2}\right)$$

Sol.

$$(a) \text{ For } \frac{\pi}{4} < x < \frac{\pi}{2}, \tan x > \cot x$$

$$\text{but } \ln(\cos x) < 0$$

$$\therefore (\tan x)^{\ln(\cos x)} < (\cot x)^{\ln(\cos x)}$$

Hence, the statement is correct.

$$(b) \text{ For } x \in \left(0, \frac{\pi}{4}\right), \cos x > \sin x$$

$$\text{but } \ln(\sec x) > 0 \text{ (as } \sec x > 1)$$

$$\therefore (\sin x)^{\ln(\sec x)} < (\cos x)^{\ln(\sec x)}$$

Hence, the statement is not correct

$$(c) \text{ For } x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right), \tan x > 1$$

$$\therefore \ln(\tan x) > 0$$

$$\text{but } \ln(\cos x) < 0$$

$$\therefore \left(\sec \frac{\pi}{3}\right)^{\ln(\tan x)} > \left(\sec \frac{\pi}{3}\right)^{\ln(\cos x)}$$

Hence, the statement is correct.

$$(d) \text{ For } x \in \left(0, \frac{\pi}{2}\right)$$

$$\ln(\sin x) < 0$$

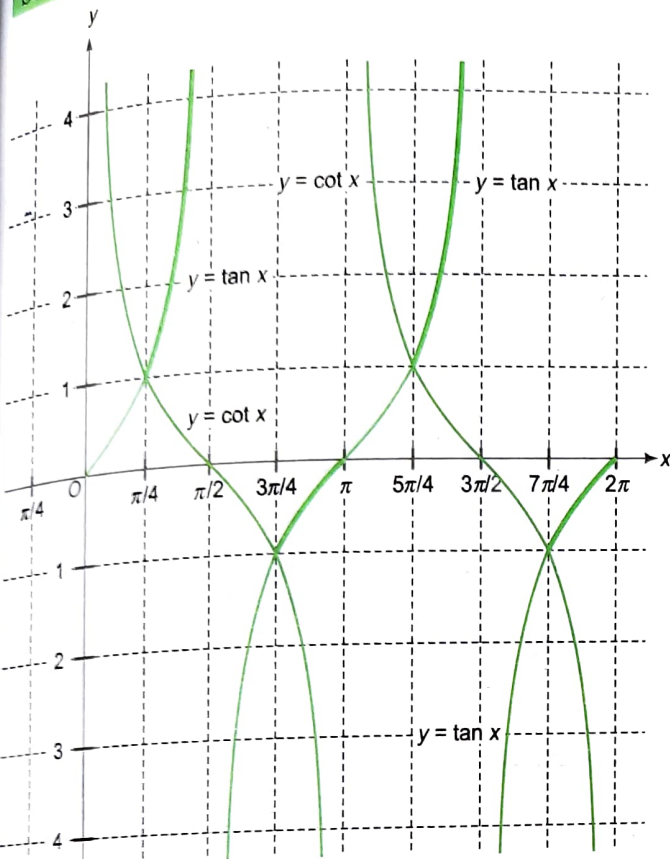
$$\text{as } \frac{1}{2} < \frac{3}{4}$$

$$\therefore \left(\frac{1}{2}\right)^{\ln(\sin x)} > \left(\frac{3}{4}\right)^{\ln(\sin x)}$$

Hence, the statement is correct.

ILLUSTRATION 2.52

 Solve $\tan x > \cot x$, where $x \in [0, 2\pi]$.

Sol.

 We find that $\tan x \geq \cot x$.

 Therefore, the value of $\tan x$ is more than the value of $\cot x$.

 That is, the values of x for which graph of $y = \tan x$ is above the graph of $y = \cot x$.

From the graph, it is clear that

$$x \in (\pi/4, \pi/2) \cup (3\pi/4, \pi) \cup (5\pi/4, 3\pi/2) \cup (7\pi/4, 2\pi).$$

CONCEPT APPLICATION EXERCISE 2.5

- Find the range of $f(x) = \frac{8}{\sin x + 3}$.
- Find the range of $f(x) = \sin(\cos x)$.
- Find the range of $12 \sin \theta - 9 \sin^2 \theta$.
- Find the minimum value of $9 \tan^2 \theta + 4 \cot^2 \theta$.
- Which of following is correct (where $n \in \mathbb{N}$)?

(a) $\sin \theta = \frac{n+1}{n}$

(b) $\sin \theta = \frac{n^2+1}{n+1}$
- If $\sin^2 \theta_1 + \sin^2 \theta_2 + \dots + \sin^2 \theta_n = 0$, then find the minimum value of $\cos \theta_1 + \cos \theta_2 + \dots + \cos \theta_n$.
- If $\sin^2 \theta = x^2 - 3x + 3$ is meaningful, then find the values of x .

 8. If $a, b, c \in \mathbb{R}$ then prove that $\sec^2 \theta = \frac{bc + ca + ab}{a^2 + b^2 + c^2}$ only if $a = b = c$.

 9. Find the range of $f(x) = \sqrt{4 - \sqrt{1 + \tan^2 x}}$.

 10. Find the range of $f(x) = \frac{1}{2|\cos x| - 3}$.

 11. Find the range of $f(x) = \cos^4 x + \sin^2 x - 1$.

 12. Find the minimum value of the function $f(x) = (1 + \sin x)(1 + \cos x)$, $\forall x \in \mathbb{R}$.

 13. Prove that $(\sin \theta + \csc \theta)^2 + (\cos \theta + \sec \theta)^2 \geq 9$.

 14. Find the range of $f(x) = \operatorname{cosec}^2 x + 25 \sec^2 x$.

 15. If $\cos^2 x + \cos x = a + 2$, then find the values of a for which equation has solution.

 16. If $a^2 + 2a + \operatorname{cosec}^2\left(\frac{\pi}{2}(a+x)\right) = 0$, then, find the values of a and x .

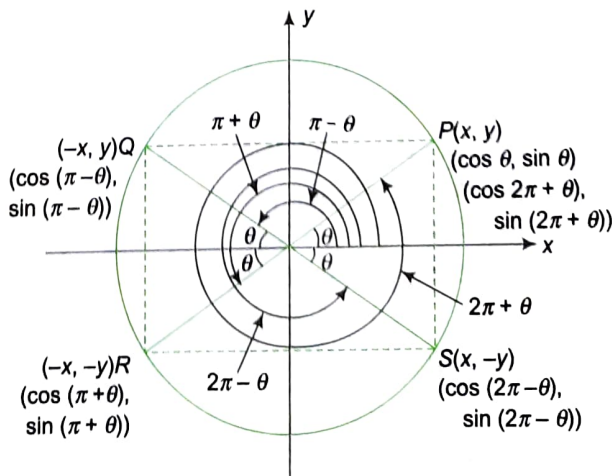
ANSWERS

- | | | |
|-----------------|---|--------------------|
| 1. $[2, 4]$ | 2. $[-\sin 1, \sin 1]$ | 3. $[-21, 4]$ |
| 4. 12 | 5. c. | 6. $-n$ |
| 7. $[1, 2]$ | 9. $[0, \sqrt{3}]$ | 10. $[-1, 1/3]$ |
| 11. $[-1/4, 0]$ | 12. 0 | 14. $[36, \infty)$ |
| 15. $[-9/4, 0]$ | 16. $a = -1$ and $x = 2n, n \in \mathbb{Z}$ | |

TRIGONOMETRIC RATIOS OF ALLIED ANGLES

When the sum or difference of two angles is either zero or a multiple of 90° , then they are called allied angles. e.g., 30° and 60° are allied angles because their sum is 90° . Also, 40° and 140° are allied angles because their sum is 180° . Angles $-\theta, 90^\circ \pm \theta, 180^\circ \pm \theta, 360^\circ \pm \theta$, are angles allied to the angle θ , if θ is measured in degrees. However, if θ is measured in radians, then the angles allied to θ are $-\theta, \frac{\pi}{2} \pm \theta, \pi \pm \theta, 2\pi \pm \theta$, etc..

In general, for angle θ , allied angles are $n \times 90^\circ \pm \theta$ or $n \frac{\pi}{2} \pm \theta$, $n \in \mathbb{I}$.

TRIGONOMETRIC RATIOS OF ALLIED ANGLES
 $n \times 90^\circ \pm \theta$ WHEN n IS EVEN


When n is even, for angles $n \times 90^\circ$ or $n \frac{\pi}{2}$, terminal side is either on positive x -axis or on negative x -axis.

Consider the unit circle with centre at origin. See the following figure to establish the required relations.

From the figure,

$$\begin{aligned} \text{(i)} \quad & \cos(\pi - \theta) = -x = -\cos \theta \\ & \sin(\pi - \theta) = y = \sin \theta \\ \text{and} \quad & \cos(\pi + \theta) = -x = -\cos \theta \\ & \sin(\pi + \theta) = -y = -\sin \theta \end{aligned}$$

If we consider angles in degree measure, we have

$$\begin{aligned} & \cos(180^\circ - \theta) = -x = -\cos \theta \\ & \sin(180^\circ - \theta) = y = \sin \theta \\ \text{and} \quad & \cos(180^\circ + \theta) = -\cos \theta \\ & \sin(180^\circ + \theta) = -\sin \theta \end{aligned}$$

Since angles π and $(2k+1)\pi$ ($k \in \mathbb{I}$) are coterminal (i.e., for both angles, terminal side lies on the negative x -axis), in general, we have

$$\begin{aligned} & \cos[(2k+1)\pi - \theta] = -\cos \theta, \\ & \sin[(2k+1)\pi - \theta] = \sin \theta \\ \text{and} \quad & \cos[(2k+1)\pi + \theta] = -\cos \theta, \\ & \sin[(2k+1)\pi + \theta] = -\sin \theta \\ \text{e.g.,} \quad & \cos(7\pi - \theta) = -\cos \theta, \\ & \sin(7\pi - \theta) = \sin \theta, \\ & \sin(-75\pi + \theta) = -\sin \theta \text{ etc..} \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad & \cos(-\theta) = x = \cos \theta \\ & \sin(-\theta) = -y = -\sin \theta \\ \text{or} \quad & \cos(2\pi - \theta) = x = \cos \theta \\ & \sin(2\pi - \theta) = -y = -\sin \theta \\ \text{and} \quad & \cos(2\pi + \theta) = x = \cos \theta \\ & \sin(2\pi + \theta) = y = \sin \theta \end{aligned}$$

If we consider angles in degree measure, we have

$$\begin{aligned} & \cos(360^\circ - \theta) = \cos \theta, \\ & \sin(360^\circ - \theta) = -\sin \theta \\ \text{and} \quad & \cos(360^\circ + \theta) = \cos \theta, \\ & \sin(360^\circ + \theta) = \sin \theta \end{aligned}$$

Since angles 2π and $2k\pi$ ($k \in \mathbb{I}$) are coterminal (i.e., for both angles, terminal side lies on the positive x -axis), in general, we have

$$\begin{aligned} & \cos(2k\pi - \theta) = \cos \theta, \\ & \sin(2k\pi - \theta) = -\sin \theta \\ \text{and} \quad & \cos(2k\pi + \theta) = \cos \theta, \\ & \sin(2k\pi + \theta) = \sin \theta \\ \text{e.g.,} \quad & \cos(6\pi + \theta) = \cos \theta, \\ & \sin(100\pi - \theta) = -\sin \theta, \\ & \cos(-68\pi - \theta) = \cos \theta \end{aligned}$$

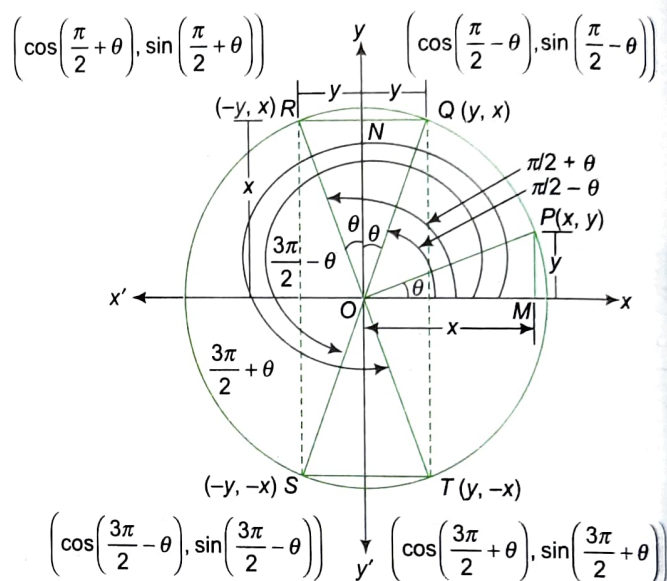
Note:

- $\cos(n\pi \pm \theta) = (-1)^n \cos \theta$
- $\sin(n\pi - \theta) = (-1)^{n-1} \sin \theta$
- $\sin(n\pi + \theta) = (-1)^n \sin \theta$

TRIGONOMETRIC RATIOS OF ALLIED ANGLES $n \times 90^\circ \pm \theta$ WHEN n IS ODD

When n is odd, for angles $n \times 90^\circ$ or $n \frac{\pi}{2}$, terminal ray is either on positive y -axis or negative y -axis.

Consider the unit circle with centre at origin. See the following figure to establish the required relations.



From the figure,

$$\begin{aligned} \text{(i)} \quad & \cos\left(\frac{\pi}{2} - \theta\right) = y = \sin \theta \\ & \sin\left(\frac{\pi}{2} - \theta\right) = x = \cos \theta \\ \text{and} \quad & \cos\left(\frac{\pi}{2} + \theta\right) = -y = -\sin \theta \\ & \sin\left(\frac{\pi}{2} + \theta\right) = x = \cos \theta \end{aligned}$$

If we consider angles in degree measure, we have

$$\begin{aligned} & \cos(90^\circ - \theta) = \sin \theta, \sin(90^\circ - \theta) = \cos \theta \\ \text{and} \quad & \cos(90^\circ + \theta) = -\sin \theta, \sin(90^\circ + \theta) = \cos \theta \end{aligned}$$

Since angles $\frac{\pi}{2}$ and $\frac{\pi}{2} + 2n\pi$ or $(4n+1)\frac{\pi}{2}$, ($n \in \mathbb{I}$) are coterminal (i.e., for both angles, terminal ray lies on the positive y -axis), in general, we have

$$\begin{aligned} & \cos\left((4n+1)\frac{\pi}{2} - \theta\right) = \sin \theta, \\ & \sin\left((4n+1)\frac{\pi}{2} - \theta\right) = \cos \theta \\ \text{and} \quad & \cos\left((4n+1)\frac{\pi}{2} + \theta\right) = -\sin \theta, \\ & \sin\left((4n+1)\frac{\pi}{2} + \theta\right) = \cos \theta \end{aligned}$$

e.g., $\sin\left(\frac{9\pi}{2} - \theta\right) = \cos \theta$, $\cos\left(\frac{13\pi}{2} + \theta\right) = -\sin \theta$,
 $\cos\left(-\frac{7\pi}{2} - \theta\right) = \sin \theta$ etc.

(ii) $\cos\left(\frac{3\pi}{2} - \theta\right) = -y = -\sin \theta$,

$\sin\left(\frac{3\pi}{2} - \theta\right) = -x = -\cos \theta$

$\cos\left(\frac{3\pi}{2} + \theta\right) = y = \sin \theta$,

$\sin\left(\frac{3\pi}{2} + \theta\right) = -x = -\cos \theta$

If we consider angles in degree measure, we have

$\cos(270^\circ - \theta) = -\sin \theta$, $\sin(270^\circ - \theta) = -\cos \theta$

and $\cos(270^\circ + \theta) = \sin \theta$, $\sin(270^\circ + \theta) = -\cos \theta$

Since angles $\frac{3\pi}{2}$ or $\frac{-\pi}{2}$ and $-\frac{\pi}{2} + 2n\pi$ or $(4n-1)\frac{\pi}{2}$,
 $(n \in \mathbb{I})$ are coterminal (i.e., for both angles, terminal ray lies
on the negative y-axis), in general, we have

$\cos\left((4n-1)\frac{\pi}{2} - \theta\right) = -\sin \theta$,

$\sin\left((4n-1)\frac{\pi}{2} - \theta\right) = -\cos \theta$

and $\cos\left((4n-1)\frac{\pi}{2} + \theta\right) = \sin \theta$,

$\sin\left((4n-1)\frac{\pi}{2} + \theta\right) = -\cos \theta$

e.g., $\sin\left(\frac{7\pi}{2} - \theta\right) = -\cos \theta$,

$\cos\left(\frac{11\pi}{2} + \theta\right) = \sin \theta$,

$\cos\left(-\frac{5\pi}{2} - \theta\right) = -\sin \theta$ etc..

In all the above proofs of formulas, angle θ is considered
to be acute, but these formulas also work if θ is not acute.

e.g., if angle $\theta \in (\pi/2, \pi)$ i.e., second quadrant, then

$\pi - \theta \in (0, \pi/2)$, i.e., first quadrant.

Clearly, $\cos \theta < 0$ and $\cos(\pi - \theta) > 0$, so formula

$\cos(\pi - \theta) = -\cos \theta$ holds.

If angle $\theta \in (3\pi/2, 2\pi)$ i.e., fourth quadrant, then

$\pi + \theta \in (5\pi/2, 3\pi)$ i.e., second quadrant.

Clearly, $\sin \theta < 0$ and $\sin(\pi + \theta) > 0$, so formula

$\sin(\pi + \theta) = -\sin \theta$ holds.

Similarly, all above formulas hold for any angle θ .

Algorithm to Find Trigonometric Ratios of Any Angle in Terms of Those of Positive Acute Angle

Step I: See whether the given angle α is positive or negative
if it is negative, make it positive as follows:

$\sin(-\theta) = -\sin \theta$, $\cos(-\theta) = \cos \theta$,

$\tan(-\theta) = -\tan \theta$ etc.

Step II: Express the positive angle α obtained in step I in the
form $\alpha = 90^\circ \times n \pm \theta$, where θ is an acute angle.

Step III: Determine the quadrant in which the terminal side of
the angle α lies from which determine the sign of the
given trigonometric function.

Step IV: In step II, if n is odd integer, then

$\sin \alpha = \pm \cos \theta$, $\cos \alpha = \pm \sin \theta$, $\tan \alpha = \pm \cot \theta$,

$\sec \alpha = \pm \operatorname{cosec} \theta$ etc. The sign on RHS will be the sign
obtained in step III.

In step II, if n is even integer, then $\sin \alpha = \pm \sin \theta$,
 $\cos \alpha = \pm \cos \theta$, $\tan \alpha = \pm \tan \theta$, $\sec \alpha = \pm \sec \theta$ etc. The
sign on RHS will be the sign obtained in step III.

ILLUSTRATION 2.53

Find the values of the following trigonometric ratios:

(i) $\cos 225^\circ$

(ii) $\sin 690^\circ$

(iii) $\tan(-390^\circ)$

(iv) $\sec 855^\circ$

Sol.

(i) For angle 225° , terminal side of angle lies in third quadrant
where cosine function is negative.

$\therefore \cos 225^\circ = \cos(90^\circ \times 2 + 45^\circ) = -\cos 45^\circ = -\frac{1}{\sqrt{2}}$

(ii) For angle 690° , terminal side of angle lies in fourth quadrant
where sine function is negative.

$\therefore \sin 690^\circ = \sin(90^\circ \times 7 + 60^\circ) = -\cos 60^\circ = -\frac{1}{2}$

(iii) For angle 390° , terminal side of angle lies in first quadrant
where tan function is positive.

$\therefore \tan(-390^\circ) = -\tan 390^\circ$
 $= -\tan(90^\circ \times 4 + 30^\circ)$
 $= -\tan 30^\circ = -\frac{1}{\sqrt{3}}$

(iv) For angle 855° , terminal side of angle lies in second quadrant
where secant is negative.

$\therefore \sec 855^\circ = \sec(90^\circ \times 9 + 45^\circ) = -\sec 45^\circ = -\sqrt{2}$

ILLUSTRATION 2.54

Prove that $\sin(-420^\circ)(\cos 390^\circ) + \cos(-660^\circ)(\sin 330^\circ) = -1$.

Sol. L.H.S. $= \sin(-420^\circ)(\cos 390^\circ) + \cos(-660^\circ)(\sin 330^\circ)$
 $= -\sin 420^\circ \cos 390^\circ + \cos 660^\circ \sin 330^\circ$
 $[\because \sin(-\theta) = -\sin \theta, \cos(-\theta) = \cos \theta]$
 $= -\sin(90^\circ \times 4 + 60^\circ) \cos(90^\circ \times 4 + 30^\circ)$
 $+ \cos(90^\circ \times 7 + 30^\circ) \sin(90^\circ \times 3 + 60^\circ)$
 $= -(\sin 60^\circ)(\cos 30^\circ) + (\sin 30^\circ)(-\cos 60^\circ)$
 $= -\frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \frac{1}{2} \left(-\frac{1}{2}\right) = -1 = \text{R.H.S.}$

ILLUSTRATION 2.55

Prove that

$$\frac{\cos(90^\circ + \theta) \sec(-\theta) \tan(180^\circ - \theta)}{\sec(360^\circ - \theta) \sin(180^\circ + \theta) \cot(90^\circ - \theta)} = -1.$$

Sol. L.H.S. = $\frac{\cos(90^\circ + \theta) \sec(-\theta) \tan(180^\circ - \theta)}{\sec(360^\circ - \theta) \sin(180^\circ + \theta) \cot(90^\circ - \theta)}$
 $= \frac{(-\sin \theta)(\sec \theta)(-\tan \theta)}{(\sec \theta)(-\sin \theta)(\tan \theta)} = -1 = \text{R.H.S.}$

ILLUSTRATION 2.56

If A, B, C, D are angles of a cyclic quadrilateral, then prove that $\cos A + \cos B + \cos C + \cos D = 0$.

Sol. We know that the opposite angles of a cyclic quadrilateral are supplementary, i.e., $A + C = \pi$ and $B + D = \pi$. Therefore,

$$A = \pi - C \text{ and } B = \pi - D$$

$$\cos A = \cos(\pi - C) = -\cos C$$

$$\text{and } \cos B = \cos(\pi - D) = -\cos D$$

$$\therefore \cos A + \cos B + \cos C + \cos D = -\cos C - \cos D + \cos C + \cos D = 0$$

ILLUSTRATION 2.57

Show that $\tan 1^\circ \tan 2^\circ \dots \tan 89^\circ = 1$.

Sol. L.H.S. = $(\tan 1^\circ \tan 89^\circ) (\tan 2^\circ \tan 88^\circ) \dots$
 $(\tan 44^\circ \tan 46^\circ) \tan 45^\circ$
 $= [\tan 1^\circ \tan(90^\circ - 1^\circ)] [\tan 2^\circ \tan(90^\circ - 2^\circ)] \dots$
 $[\tan 44^\circ \tan(90^\circ - 44^\circ)] \tan 45^\circ$
 $= (\tan 1^\circ \cot 1^\circ) (\tan 2^\circ \cot 2^\circ) \dots$
 $(\tan 44^\circ \cot 44^\circ) \tan 45^\circ = 1$
 $[\because \tan \theta \cot \theta = 1 \text{ and } \tan 45^\circ = 1]$
 $= \text{R.H.S.}$

ILLUSTRATION 2.58

Show that $\sin^2 5^\circ + \sin^2 10^\circ + \sin^2 15^\circ + \dots + \sin^2 90^\circ = 9 \frac{1}{2}$.

Sol. L.H.S. = $(\sin^2 5^\circ + \sin^2 85^\circ) + (\sin^2 10^\circ + \sin^2 80^\circ) + \dots$
 $+ (\sin^2 40^\circ + \sin^2 50^\circ) + \sin^2 45^\circ + \sin^2 90^\circ$
 $= (\sin^2 5^\circ + \cos^2 5^\circ) + (\sin^2 10^\circ + \cos^2 10^\circ) + \dots$
 $+ (\sin^2 40^\circ + \cos^2 40^\circ) + \sin^2 45^\circ + \sin^2 90^\circ$
 $= (1 + 1 + 1 + 1 + 1 + 1 + 1 + 1)$
 $+ \left(\frac{1}{\sqrt{2}}\right)^2 + 1 = 9 \frac{1}{2} = \text{R.H.S.}$

ILLUSTRATION 2.59

Find the value of

$$\cos^2 \frac{\pi}{16} + \cos^2 \frac{3\pi}{16} + \cos^2 \frac{5\pi}{16} + \cos^2 \frac{7\pi}{16}.$$

Sol. L.H.S. = $\cos^2 \frac{\pi}{16} + \cos^2 \frac{3\pi}{16} + \cos^2 \left(\frac{\pi}{2} - \frac{3\pi}{16}\right)$
 $+ \cos^2 \left(\frac{\pi}{2} - \frac{\pi}{16}\right)$
 $= \cos^2 \frac{\pi}{16} + \cos^2 \frac{3\pi}{16} + \sin^2 \frac{3\pi}{16} + \sin^2 \frac{\pi}{16}$
 $= \left(\cos^2 \frac{\pi}{16} + \sin^2 \frac{\pi}{16}\right) + \left(\cos^2 \frac{3\pi}{16} + \sin^2 \frac{3\pi}{16}\right)$
 $= 1 + 1 = 2$

ILLUSTRATION 2.60

If $\sin(120^\circ - \alpha) = \sin(120^\circ - \beta)$, $0^\circ < \alpha, \beta < 180^\circ$, then find the relation between α and β .

Sol. If $\sin A = \sin B$, where $A = 120^\circ - \alpha$ and $B = 120^\circ - \beta$, then
 $A = B$ or $A = \pi - B$, i.e., $A + B = \pi$
 $\Rightarrow 120^\circ - \alpha = 120^\circ - \beta$ or $120^\circ - \alpha + 120^\circ - \beta = 180^\circ$
 $\Rightarrow \alpha = \beta$ or $\alpha + \beta = 60^\circ$

ILLUSTRATION 2.61

Find the sign of the values of $\tan 113^\circ - \cos 107^\circ = a$ and $\tan 107^\circ - \cos 105^\circ = b$.

Sol. $a = \tan 113^\circ - \cos 107^\circ$
 $= \cos 73^\circ - \tan 67^\circ < \cos 73^\circ - \tan 45^\circ < 0$
 $b = \tan 107^\circ - \cos 105^\circ$
 $= \cos 75^\circ - \tan 73^\circ < \cos 75^\circ - \tan 45^\circ < 0$

ILLUSTRATION 2.62

In triangle ABC prove that

(i) $\sin A = \sin(B + C)$ (ii) $\sin 2A = -\sin(2B + 2C)$
 (iii) $\cos A = -\cos(A + B)$ (iv) $\tan\left(\frac{A+B}{2}\right) = \cot \frac{C}{2}$

Sol. (i) $\sin A = \sin(\pi - (B + C)) = \sin(B + C)$
 (ii) $\sin 2A = \sin(2\pi - (2B + 2C)) = -\sin(2B + 2C)$
 (iii) $\cos A = \cos(\pi - (B + C)) = -\cos(B + C)$
 (iv) $\tan\left(\frac{A+B}{2}\right) = \tan\left(\frac{\pi - C}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cot\left(\frac{C}{2}\right)$

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$$\frac{\cos(90^\circ + \theta) \sec(-\theta) \tan(180^\circ - \theta)}{\sec(360^\circ - \theta) \sin(180^\circ + \theta) \cot(90^\circ - \theta)} = -1.$$

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 $= \frac{(-\sin \theta)(\sec \theta)(-\tan \theta)}{(\sec \theta)(-\sin \theta)(\tan \theta)} = -1 = \text{R.H.S.}$

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 $= (\tan 1^\circ \cot 1^\circ) (\tan 2^\circ \cot 2^\circ) \dots$
 $(\tan 44^\circ \cot 44^\circ) \tan 45^\circ = 1$
 $[\because \tan \theta \cot \theta = 1 \text{ and } \tan 45^\circ = 1]$
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 $= \cos^2 \frac{\pi}{16} + \cos^2 \frac{3\pi}{16} + \sin^2 \frac{3\pi}{16} + \sin^2 \frac{\pi}{16}$
 $= \left(\cos^2 \frac{\pi}{16} + \sin^2 \frac{\pi}{16}\right) + \left(\cos^2 \frac{3\pi}{16} + \sin^2 \frac{3\pi}{16}\right)$
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$$\Rightarrow 120^\circ - \alpha = 120^\circ - \beta \text{ or } 120^\circ - \alpha + 120^\circ - \beta = 180^\circ$$

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Sol.

(i) $\sin A = \sin(\pi - (B + C)) = \sin(B + C)$

(ii) $\sin 2A = \sin(2\pi - (2B + 2C)) = -\sin(2B + 2C)$

(iii) $\cos A = \cos(\pi - (B + C)) = -\cos(B + C)$

(iv) $\tan\left(\frac{A+B}{2}\right) = \tan\left(\frac{\pi - C}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cot\left(\frac{C}{2}\right)$

CONCEPT APPLICATION EXERCISE 2.6

1. Prove that

$$(a) \tan 720^\circ - \cos 270^\circ - \sin 150^\circ \cos 120^\circ = \frac{1}{4}$$

$$(b) \sin 780^\circ \sin 480^\circ + \cos 120^\circ \sin 150^\circ = \frac{1}{2}$$

2. Find the value of the expression

$$\sec 610^\circ \operatorname{cosec} 160^\circ - \cot 380^\circ \tan 470^\circ.$$

3. If $\alpha = \frac{\pi}{3}$, prove that

$$\cos \alpha \cos 2\alpha \cos 3\alpha \cos 4\alpha \cos 5\alpha \cos 6\alpha = -\frac{1}{16}.$$

4. Find the value of $\tan \frac{\pi}{20} \tan \frac{3\pi}{20} \tan \frac{5\pi}{20} \tan \frac{7\pi}{20} \tan \frac{9\pi}{20}$.

5. Find the value of $\frac{\cot 54^\circ}{\tan 36^\circ} + \frac{\tan 20^\circ}{\cot 70^\circ}$.

6. Prove that $\sin^2 \frac{\pi}{18} + \sin^2 \frac{\pi}{9} + \sin^2 \frac{7\pi}{18} + \sin^2 \frac{4\pi}{9} = 2$.

7. Prove that $\sec\left(\frac{3\pi}{2} - \theta\right) \sec\left(\theta - \frac{5\pi}{2}\right) + \tan\left(\frac{5\pi}{2} + \theta\right) \tan\left(\theta - \frac{3\pi}{2}\right) = -1$.

8. If $\theta = \frac{\pi}{4n}$ then find the value of $\tan \theta \tan 2\theta \dots \tan(2n-2)\theta \tan(2n-1)\theta$.

9. In any quadrilateral $ABCD$, prove that

$$(a) \sin(A+B) + \sin(C+D) = 0$$

$$(b) \cos(A+B) = \cos(C+D)$$

ANSWERS

2. -1 4. 1 5. 2 8. 1

Solved Examples

EXAMPLE 2.1

If $2 \cos x + \sin x = 1$, then find the value of $7 \cos x + 6 \sin x$.

Sol. Given, $2 \cos x + \sin x = 1$

$$\text{or } 4 \cos^2 x = (1 - \sin x)^2$$

$$\text{or } 4 - 4 \sin^2 x = 1 + \sin^2 x - 2 \sin x$$

$$\text{or } 5 \sin^2 x - 2 \sin x - 3 = 0$$

$$\text{or } (\sin x - 1)(5 \sin x + 3) = 0$$

$$\text{or } \sin x = 1, \sin x = -\frac{3}{5}$$

For $\sin x = 1$, we have

$$7 \cos x + 6 \sin x = 0 + 6 = 6$$

and for $\sin x = -\frac{3}{5}$, we have

$$\begin{aligned} 7 \cos x + 6 \sin x &= 7 \left(\frac{1 + \frac{3}{5}}{2} \right) - \frac{6 \times 3}{5} \\ &= \frac{28 - 18}{5} = 2 \end{aligned}$$

EXAMPLE 2.2

If $u_n = \sin^n \theta + \cos^n \theta$, then prove that $\frac{u_5 - u_7}{u_3 - u_5} = \frac{u_3}{u_1}$.

Sol.

$$\begin{aligned} \frac{u_5 - u_7}{u_3 - u_5} &= \frac{(\sin^5 \theta + \cos^5 \theta) - (\sin^7 \theta + \cos^7 \theta)}{(\sin^3 \theta + \cos^3 \theta) - (\sin^5 \theta + \cos^5 \theta)} \\ &= \frac{\sin^5 \theta (1 - \sin^2 \theta) + \cos^5 \theta (1 - \cos^2 \theta)}{\sin^3 \theta (1 - \sin^2 \theta) + \cos^3 \theta (1 - \cos^2 \theta)} \\ &= \frac{\sin^2 \theta \cos^2 \theta [\sin^3 \theta + \cos^3 \theta]}{\sin^2 \theta \cos^2 \theta [\sin \theta + \cos \theta]} = \frac{u_3}{u_1} \end{aligned}$$

EXAMPLE 2.3

If $a^2 + b^2 + 2ab \cos \theta = 1$, $c^2 + d^2 + 2cd \cos \theta = 1$ and $ac + bd + (ad + bc) \cos \theta = 0$, then prove that $a^2 + c^2 = \operatorname{cosec}^2 \theta$.

Sol. $a^2 + b^2 + 2ab \cos \theta = 1$,

$$\Rightarrow (b + a \cos \theta)^2 = 1 - a^2 \sin^2 \theta$$

$$c^2 + d^2 + 2cd \cos \theta = 1$$

$$\Rightarrow (d + c \cos \theta)^2 = 1 - c^2 \sin^2 \theta$$

$$ac + bd + (ad + bc) \cos \theta = 0$$

$$\Rightarrow (b + a \cos \theta)(d + c \cos \theta) = -ac \sin^2 \theta$$

$$\Rightarrow (b + a \cos \theta)^2 (d + c \cos \theta)^2 = (-ac \sin^2 \theta)^2$$

$$\Rightarrow (1 - a^2 \sin^2 \theta)(1 - c^2 \sin^2 \theta) = a^2 c^2 \sin^4 \theta$$

$$\Rightarrow (a^2 + c^2) \sin^2 \theta = 1$$

$$\Rightarrow (a^2 + c^2) = \operatorname{cosec}^2 \theta$$

EXAMPLE 2.4

If $\frac{\sec^4 \theta}{a} + \frac{\tan^4 \theta}{b} = \frac{1}{a+b}$, then prove that $|b| \leq |a|$.

$$\text{Sol. } \frac{\sec^4 \theta}{a} + \frac{\tan^4 \theta}{b} = \frac{1}{a+b} = \frac{\sec^2 \theta - \tan^2 \theta}{a+b}$$

$$\Rightarrow \frac{\sec^2 \theta}{a(a+b)} [(a+b) \sec^2 \theta - a] + \frac{\tan^2 \theta}{b(a+b)} [(a+b) \tan^2 \theta + b] = 0$$

$$\Rightarrow a \tan^2 \theta + b \sec^2 \theta = 0$$

$$\Rightarrow \sin^2 \theta = -\frac{b}{a}$$

Since $\sin^2 \theta \leq 1$

$$\Rightarrow \left| \frac{b}{a} \right| \leq 1 \text{ or } |b| \leq |a|$$

EXAMPLE 2.5

Let $A = \sin x + \cos x$. Then find the value of $\sin^4 x + \cos^4 x$ in terms of A .

Sol. $A = \sin x + \cos x$

$$\therefore A^2 = 1 + 2 \sin x \cos x$$

$$\text{Now, } \sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2 \sin^2 x \cos^2 x$$

$$= 1 - 2 \sin^2 x \cos^2 x$$

$$= 1 - 2 \left(\frac{A^2 - 1}{2} \right)^2$$

$$= 1 - \frac{(A^2 - 1)^2}{2}$$

$$= \frac{2 - (A^4 - 2A^2 + 1)}{2}$$

$$= \frac{1 + 2A^2 - A^4}{2}$$

$$= \frac{1}{2} + A^2 - \frac{1}{2} A^4$$

EXAMPLE 2.6

If $x = \frac{\sin^3 p}{\cos^2 p}$, $y = \frac{\cos^3 p}{\sin^2 p}$ and $\sin p + \cos p = \frac{1}{2}$ then find the value of $x + y$.

Sol. $x + y = \frac{\sin^5 p + \cos^5 p}{\cos^2 p \sin^2 p}$

$$\begin{aligned} \text{Now, } \sin^5 p + \cos^5 p &= (\sin p + \cos p) (\sin^4 p - \sin^3 p \cos p \\ &\quad + \sin^2 p \cos^2 p - \sin p \cos^3 p + \cos^4 p) \\ &= (\sin p + \cos p) [(\sin^4 p + \cos^4 p) - \sin p \cos p \\ &\quad (\sin^2 p + \cos^2 p) + \sin^2 p \cos^2 p] \\ &= (\sin p + \cos p) [1 - 2 \sin^2 p \cos^2 p - \sin p \cos p \\ &\quad + \sin^2 p \cos^2 p] \\ &= (\sin p + \cos p) [1 - \sin^2 p \cos^2 p - \sin p \cos p] \end{aligned}$$

$$\text{Now, } \sin p + \cos p = \frac{1}{2}$$

$$\Rightarrow 1 + 2 \sin p \cos p = \frac{1}{4}$$

$$\Rightarrow \sin p \cos p = -\frac{3}{8}$$

Using these values, we get

$$\begin{aligned} x + y &= \frac{\frac{1}{2} \left[1 - \frac{9}{64} + \frac{3}{8} \right]}{\frac{9}{64}} \\ &= \frac{79}{18} \end{aligned}$$

EXAMPLE 2.7

If $\frac{\sin A}{\sin B} = \frac{\sqrt{3}}{2}$ and $\frac{\cos A}{\cos B} = \frac{\sqrt{5}}{2}$, $0 < A, B < \pi/2$, then find the value of $\tan A + \tan B$.

Sol. Given $\frac{\sin A}{\sin B} = \frac{\sqrt{3}}{2}$, $\frac{\cos A}{\cos B} = \frac{\sqrt{5}}{2}$, $0 < A, B < \pi/2$

On dividing, we get

$$\tan A = \frac{\sqrt{3}}{\sqrt{5}} \tan B$$

$$\text{Multiplying } \frac{\sin A \cos A}{\sin B \cos B} = \frac{\sqrt{15}}{4}$$

$$\Rightarrow \frac{\tan A \cdot \sec^2 B}{\tan B \cdot \sec^2 A} = \frac{\sqrt{15}}{4}$$

$$\text{or } \frac{\sqrt{3}}{\sqrt{5}} \frac{(1 + \tan^2 B)}{(1 + \tan^2 A)} = \frac{\sqrt{15}}{4}$$

$$\text{or } 4 + 4 \tan^2 B = 5 + 5 \tan^2 A$$

$$\text{or } -1 + 4 \tan^2 B = 5 \times \frac{3}{5} \tan^2 B$$

$$\text{or } \tan B = \pm 1$$

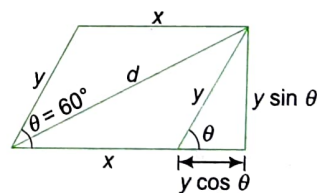
$$\text{or } \tan B = 1$$

$$\text{Now } \tan A + \tan B = \frac{\sqrt{3}}{\sqrt{5}} + 1 = \frac{\sqrt{3} + \sqrt{5}}{\sqrt{5}}$$

EXAMPLE 2.8

A parallelogram containing a 60° angle has perimeter p and longer diagonal is of length d . Find its area.

Sol.



Perimeter,

$$2(x + y) = p$$

$$\text{or } x + y = \frac{p}{2}$$

$$\begin{aligned} \text{Also, } d^2 &= (x + y \cos \theta)^2 + y^2 \sin^2 \theta \\ &= x^2 + y^2 + 2xy \cos \theta \\ &= x^2 + y^2 + xy \quad (\theta = 60^\circ) \\ &= (x + y)^2 - xy \\ &= \frac{p^2}{4} - xy \end{aligned}$$

$$\text{or } xy = \frac{p^2}{4} - d^2$$

Now,

Area of parallelogram = $xy \sin 60^\circ$

$$= \left(\frac{p^2}{4} - d^2 \right) \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{8} (p^2 - 4d^2)$$

EXAMPLE 2.9

For each natural number $n \geq 2$, prove that $\sin x_1 \cos x_2 + \sin x_2 \cos x_3 + \dots + \sin x_n \cos x_1 \leq n/2$ (where $x_1, x_2, \dots, x_n$ are arbitrary real numbers).

Sol. Let the required sum be S_n . We know that

$$(\sin x_1 - \cos x_2)^2 + (\sin x_2 - \cos x_3)^2 + \dots + (\sin x_{n-1} - \cos x_n)^2 + (\sin x_n - \cos x_1)^2 \geq 0$$

or

$$(\sin^2 x_1 + \cos^2 x_1) + (\sin^2 x_2 + \cos^2 x_2) + \dots + (\sin^2 x_n + \cos^2 x_n) \geq 2S_n$$

$$\Rightarrow n \geq 2S_n$$

$$\text{or } S_n \leq n/2$$

EXAMPLE 2.10

Find the range of $y = \sin^3 x - 6\sin^2 x + 11\sin x - 6$.

Sol. $y = \sin^3 x - 6\sin^2 x + 11\sin x - 6$

Put $\sin x = t$

$$\therefore y = t^3 - 6t^2 + 11t - 6, -1 \leq t \leq 1$$

$$\therefore \frac{dy}{dt} = 3t^2 - 12t + 11$$

$$= 3(t-2)^2 - 1 > 0 \text{ for } -1 \leq t \leq 1$$

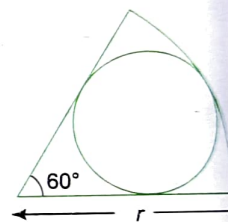
$\therefore$ Function is increasing.

Hence, range is $[f(-1), f(1)]$ or $[-24, 0]$.

Exercises

Single Correct Answer Type

- If $5 \tan \theta = 4$, then $\frac{5 \sin \theta - 3 \cos \theta}{5 \sin \theta + 2 \cos \theta}$ is equal to
 (1) 0 (2) 1 (3) $1/6$ (4) 6
- If $\tan \theta = -\frac{4}{3}$, then $\sin \theta$ is
 (1) $-\frac{4}{5}$ but not $\frac{4}{5}$ (2) $-\frac{4}{5}$ or $\frac{4}{5}$
 (3) $\frac{4}{5}$ but not $-\frac{4}{5}$ (4) none of these
- If $\sin x + \operatorname{cosec} x = 2$, then $\sin^n x + \operatorname{cosec}^n x$ is equal to
 (1) 2 (2) 2^n (3) 2^{n-1} (4) 2^{n-2}
- If $\tan \theta + \sin \theta = m$ and $\tan \theta - \sin \theta = n$, then
 (1) $m^2 - n^2 = 4mn$ (2) $m^2 + n^2 = 4mn$
 (3) $m^2 - n^2 = m^2 + n^2$ (4) $m^2 - n^2 = 4\sqrt{mn}$
- If $\operatorname{cosec} \theta - \cot \theta = q$, then the value of $\operatorname{cosec} \theta$ is
 (1) $q + \frac{1}{q}$ (2) $q - \frac{1}{q}$
 (3) $\frac{1}{2} \left(q + \frac{1}{q} \right)$ (4) none of these
- If $\frac{\sin x}{a} = \frac{\cos x}{b} = \frac{\tan x}{c} = k$, then $bc + \frac{1}{ck} + \frac{ak}{1+bk}$ is equal to
 (1) $k \left(a + \frac{1}{a} \right)$ (2) $\frac{1}{k} \left(a + \frac{1}{a} \right)$
 (3) $\frac{1}{k^2}$ (4) $\frac{a}{k}$
- If $\sec^4 \theta + \sec^2 \theta = 10 + \tan^4 \theta + \tan^2 \theta$, then $\sin^2 \theta =$
 (1) $\frac{2}{3}$ (2) $\frac{3}{4}$ (3) $\frac{4}{5}$ (4) $\frac{5}{6}$
- If $x = \frac{2 \sin \theta}{1 + \cos \theta + \sin \theta}$, then $\frac{1 - \cos \theta + \sin \theta}{1 + \sin \theta}$ is equal to
 (1) $1 + x$ (2) $1 - x$ (3) x (4) $1/x$
- If $\sec \alpha$ and $\operatorname{cosec} \alpha$ are the roots of $x^2 - px + q + 0$, then
 (1) $p^2 = q(q-2)$ (2) $p^2 = q(q+2)$
 (3) $p^2 + q^2 = 2q$ (4) none of these
- Which of the following is not the quadratic equation whose roots are $\operatorname{cosec}^2 \theta$ and $\sec^2 \theta$?
 (1) $x^2 - 6x + 6 = 0$ (2) $x^2 - 7x + 7 = 0$
 (3) $x^2 - 4x + 4 = 0$ (4) none of these
- If $\sin x + \sin^2 x = 1$, then the value of $\cos^{12} x + 3 \cos^{10} x + 3 \cos^8 x + \cos^6 x - 2$ is equal to
 (1) 0 (2) 1 (3) -1 (4) 2
- $3(\sin \theta - \cos \theta)^4 + 6(\sin \theta + \cos \theta)^2 + 4(\sin^6 \theta + \cos^6 \theta)$ is equal to
 (1) 11 (2) 12 (3) 13 (4) 14
- If $\sec x + \sec^2 x = 1$ then the value of $\tan^8 x - \tan^4 x + 2 \tan^2 x + 1$ will be equal to
 (1) 0 (2) 1 (3) 2 (4) 3
- $(1 + \tan \alpha \tan \beta)^2 + (\tan \alpha - \tan \beta)^2 =$
 (1) $\tan^2 \alpha \tan^2 \beta$ (2) $\sec^2 \alpha \sec^2 \beta$
 (3) $\tan^2 \alpha \cot^2 \beta$ (4) $\sec^2 \alpha \cos^2 \beta$
- Let $A_0 A_1 A_2 A_3 A_4 A_5$ be a regular hexagon inscribed in a circle of unit radius. Then the product of the lengths of the line segments $A_0 A_1$, $A_0 A_2$, and $A_0 A_4$ is
 (1) $3/4$ (2) $3\sqrt{3}$ (3) 3 (4) $3\sqrt{3}/2$
- A circle is drawn in a sector of a larger circle of radius r as shown in figure. The smaller circle is tangent to the two bounding radii and the arc of the sector. The radius of the smaller circle is
 (1) $\frac{r}{2}$
 (2) $\frac{r}{3}$
 (3) $\frac{2\sqrt{3}r}{5}$
 (4) $\frac{r}{\sqrt{2}}$



- A right triangle has perimeter of length 7 and hypotenuse of length 3. If θ is the larger non-right angle in the triangle then the value of $\cos \theta$ equals
 (1) $\frac{\sqrt{6} - \sqrt{2}}{4}$ (2) $\frac{4 + \sqrt{2}}{6}$
 (3) $\frac{4 - \sqrt{2}}{3}$ (4) $\frac{4 - \sqrt{2}}{6}$
- Given that the side length of a rhombus is the geometric mean of the lengths of its diagonals. The degree measure of the acute angle of the rhombus is
 (1) 15° (2) 30° (3) 45° (4) 60°
- Which of the following is correct?
 (1) $\sin 1^\circ > \sin 1$ (2) $\sin 1^\circ < \sin 1$
 (3) $\sin 1^\circ = \sin 1$ (4) $\sin 1^\circ = \frac{\pi}{180} \sin 1$

20. The equation $\sin^2 \theta = \frac{x^2 + y^2}{2xy}$, $x, y \neq 0$ is possible if
 (1) $x = y$ (2) $x = -y$
 (3) $2x = y$ (4) none of these
21. If $\sin^2 \theta = \frac{x^2 + y^2 + 1}{2x}$, then x must be
 (1) -3 (2) -2
 (3) 1 (4) none of these
22. $\sec^2 \theta = \frac{4xy}{(x+y)^2}$ is true if and only if
 (1) $x + y \neq 0$ (2) $x = y, x \neq 0$
 (3) $x = y$ (4) $x \neq 0, y \neq 0$
23. If $\sin \theta_1 + \sin \theta_2 + \sin \theta_3 = 3$, then $\cos \theta_1 + \cos \theta_2 + \cos \theta_3$ is equal to
 (1) 3 (2) 2 (3) 1 (4) 0
24. If $\sin x + \sin y + \sin z + \sin w = -4$, then the value of $\sin^{400} x + \sin^{300} y + \sin^{200} z + \sin^{100} w$ is
 (1) $\sin^{400} x \cdot \sin^{300} y \cdot \sin^{200} z + \sin^{100} w$
 (2) $\sin x \cdot \sin y \cdot \sin z \cdot \sin w$
 (3) 4
 (4) 3
25. The minimum value of the expression $\sin \alpha + \sin \beta + \sin \gamma$, where α, β, γ are real numbers satisfying $\alpha + \beta + \gamma = \pi$ is
 (1) positive (2) zero
 (3) negative (4) -3
26. If $1 + \sin x + \sin^2 x + \sin^3 x + \dots \infty$ is equal to $4 + 2\sqrt{3}$, $0 < x < \pi$, then x is equal to
 (1) $\frac{\pi}{6}$ (2) $\frac{\pi}{4}$
 (3) $\frac{\pi}{3}$ or $\frac{\pi}{6}$ (4) $\frac{\pi}{3}$ or $\frac{2\pi}{3}$
27. The value of the expression $(2 \sin^2 91^\circ - 1)(2 \sin^2 92^\circ - 1) \dots (2 \sin^2 180^\circ - 1)$ is equal to
 (1) 0 (2) 1 (3) 2^{90} (4) $2^{90} - 90$
28. If $\sin A = \sin^2 B$ and $2 \cos^2 A = 3 \cos^2 B$ then the triangle ABC is
 (1) right angled (2) obtuse angled
 (3) isosceles (4) equilateral
29. If $\sin \theta + \cos \theta = \frac{1}{5}$ and $0 \leq \theta < \pi$, then $\tan \theta$ is
 (1) $-4/3$ (2) $-3/4$ (3) $3/4$ (4) $4/3$
30. If $\pi < \alpha < \frac{3\pi}{2}$, then $\sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} + \sqrt{\frac{1 + \cos \alpha}{1 - \cos \alpha}}$ is equal to
 (1) $\frac{2}{\sin \alpha}$ (2) $-\frac{2}{\sin \alpha}$
 (3) $\frac{1}{\sin \alpha}$ (4) $-\frac{1}{\sin \alpha}$
31. If $0 < \alpha < \frac{\pi}{6}$, then α ($\operatorname{cosec} \alpha$) is
 (1) less than $\pi/6$ (2) greater than $\pi/6$
 (3) less than $\pi/3$ (4) greater than $\pi/3$
32. The least value of $2 \sin^2 \theta + 3 \cos^2 \theta$ is
 (1) 1 (2) 2 (3) 3 (4) 5
33. The greatest value of $\sin^4 \theta + \cos^4 \theta$ is
 (1) $1/2$ (2) 1 (3) 2 (4) 3
34. If $f(x) = \sin^6 x + \cos^6 x$, then range of $f(x)$ is
 (1) $\left[\frac{1}{4}, 1\right]$ (2) $\left[\frac{1}{4}, \frac{3}{4}\right]$
 (3) $\left[\frac{3}{4}, 1\right]$ (4) none of these
35. The minimum value of $a \tan^2 x + b \cot^2 x$ equals the maximum value of $a \sin^2 \theta + b \cos^2 \theta$ where $a > b > 0$. Then a/b is
 (1) 2 (2) 4 (3) 6 (4) 8
36. Range of $f(\theta) = \cos^2 \theta (\cos^2 \theta + 1) + 2 \sin^2 \theta$ is
 (1) $[3/4, 1]$ (2) $[3/16, 1]$
 (3) $[3/4, 7/4]$ (4) $[7/4, 2]$
37. If $0 < \theta < \pi$, then minimum value of $3 \sin \theta + \operatorname{cosec}^3 \theta$ is
 (1) 4 (2) 3 (3) 5 (4) 6
38. If $\theta_i > 0$ for $1 \leq i \leq n$ and $\theta_1 + \theta_2 + \theta_3 + \dots + \theta_n = \pi$, then the greatest value of $\sum \sin \theta_i + \sin \theta_2 + \sin \theta_3 + \dots + \sin \theta_n$ is equal to
 (1) n (2) $n \sin \left(\frac{\pi}{n}\right)$
 (3) π (4) none of these
39. The set of values of $\lambda \in R$ such that $\sin^2 \theta + \cos \theta = \lambda \cos^2 \theta$ holds for some θ , is
 (1) $(-\infty, 1]$ (2) $(-\infty, -1]$
 (3) ϕ (4) $[-1, \infty)$
40. Let $A = \sin^8 \theta + \cos^{14} \theta$; then $A_{\max}$ is
 (1) 1 (2) $\frac{1}{2}$
 (3) $\frac{3}{2}$ (4) none of these
41. Minimum value of $y = 256 \sin^2 x + 324 \operatorname{cosec}^2 x \forall x \in R$ is
 (1) 432 (2) 504 (3) 576 (4) 776
42. If a and b are positive quantities such that $a > b$, the minimum value of $a \sec \theta - b \tan \theta$ is
 (1) $2ab$ (2) $\sqrt{a^2 - b^2}$
 (3) $a - b$ (4) $\sqrt{a^2 + b^2}$
43. If $y = (\sin x + \operatorname{cosec} x)^2 + (\cos x + \sec x)^2$, then the minimum value of $y, \forall x \in R$, is
 (1) 7 (2) 3 (3) 9 (4) 0
44. The variable x satisfying the equation $|\sin x \cos x| + \sqrt{2 + \tan^2 x + \cot^2 x} = \sqrt{3}$ belongs to the interval

(1) $\left[0, \frac{\pi}{3}\right]$

(2) $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$

(3) $\left[\frac{3\pi}{4}, \pi\right)$

(4) non-existent

45. If the equation $\cot^4 x - 2 \operatorname{cosec}^2 x + a^2 = 0$ has at least one solution, then the sum of all possible integral values of a is equal to

(1) 4 (2) 3 (3) 2 (4) 0

46. If $\cos^2 x - (c-1) \cos x + 2c \geq 6$ for every $x \in R$, then the true set of values of c is

(1) $[2, \infty)$ (2) $[4, \infty)$
(3) $(-\infty, -2]$ (4) $(-\infty, -4]$

47. If the inequality $\sin^2 x + a \cos x + a^2 > 1 + \cos x$ holds for any $x \in R$, then the largest negative integral value of a is

(1) -4 (2) -3 (3) -2 (4) -1

48. If $\frac{3\pi}{4} < \alpha < \pi$, then $\sqrt{2 \cot \alpha + \frac{1}{\sin^2 \alpha}}$ is equal to

(1) $1 + \cot \alpha$ (2) $-1 - \cot \alpha$
(3) $1 - \cot \alpha$ (4) $-1 + \cot \alpha$

49. The value of $\frac{\sec \theta}{\sqrt{1 + \tan^2 \theta}} + \frac{\operatorname{cosec} \theta}{\sqrt{1 + \cot^2 \theta}}$ for $\theta \in \left(\pi, \frac{3\pi}{2}\right)$ is

(1) 0 (2) -2 (3) 2 (4) 1

50. The minimum value of the function $f(x) =$

$$\frac{\sin x}{\sqrt{1 - \cos^2 x}} + \frac{\cos x}{\sqrt{1 - \sin^2 x}} + \frac{\tan x}{\sqrt{\sec^2 x - 1}} + \frac{\cot x}{\sqrt{\operatorname{cosec}^2 x - 1}}$$

whenever it is defined is

(1) 4 (2) -2 (3) 0 (4) 2

51. If $\left| \cos \theta \left\{ \sin \theta + \sqrt{\sin^2 \theta + \sin^2 \alpha} \right\} \right| \leq k$, then the value of k is

(1) $\sqrt{1 + \cos^2 \alpha}$ (2) $\sqrt{1 + \sin^2 \alpha}$

(3) $\sqrt{2 + \sin^2 \alpha}$ (4) $\sqrt{2 + \cos^2 \alpha}$

52. In which one of the following intervals the inequality $\sin x < \cos x < \tan x < \cot x$ can hold good?

(1) $\left(\frac{7\pi}{4}, 2\pi\right)$ (2) $\left(\frac{3\pi}{4}, \pi\right)$

(3) $\left(\frac{5\pi}{4}, \frac{3\pi}{2}\right)$ (4) $\left(0, \frac{\pi}{4}\right)$

53. The values of k for which the inequality $k \cos^2 x - k \cos x + 1 \geq 0, \forall x \in (-\infty, \infty)$ holds is

(1) $k < -\frac{1}{2}$ (2) $k > 4$

(3) $-\frac{1}{2} \leq k \leq 4$ (4) $\frac{1}{2} \leq k \leq 5$

54. The value of $\cos \frac{\pi}{7} + \cos \frac{2\pi}{7} + \cos \frac{3\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{5\pi}{7} + \cos \frac{6\pi}{7} + \cos \frac{7\pi}{7}$ is

(1) 1 (2) -1
(3) 0 (4) none of these

55. The numerical value of $\tan \frac{\pi}{3} + 2 \tan \frac{2\pi}{3} + 4 \tan \frac{4\pi}{3} + 8 \tan \frac{8\pi}{3}$ is equal to

(1) $-5\sqrt{3}$ (2) $-\frac{5}{\sqrt{3}}$
(3) $5\sqrt{3}$ (4) $\frac{5}{\sqrt{3}}$

56. The expression $3 \left[\sin^4 \left(\frac{3}{2} \pi - \alpha \right) + \sin^4 (3\pi + \alpha) \right] - 2 \left[\sin^6 \left(\frac{1}{2} \pi + \alpha \right) + \sin^6 (5\pi - \alpha) \right]$ is equal to

(1) 0 (2) 1
(3) 3 (4) none of these

57. The value of the expression $\log_{10}(\tan 6^\circ) + \log_{10}(\tan 12^\circ) + \log_{10}(\tan 18^\circ) + \dots + \log_{10}(\tan 84^\circ)$ is

(1) -1 (2) 0 (3) 1 (4) 2

Multiple Correct Answers Type

- If $0 \leq \theta \leq \pi$ and $81^{\sin^2 \theta} + 81^{\cos^2 \theta} = 30$, then θ is
(1) 30° (2) 60° (3) 120° (4) 150°
- Suppose $ABCD$ (in order) is a quadrilateral inscribed in circle. Which of the following is/are always true?
(1) $\sec B = \sec D$ (2) $\cot A + \cot C = 0$
(3) $\operatorname{cosec} A = \operatorname{cosec} C$ (4) $\tan B + \tan D = 0$
- Which of the following is/are correct?
(1) $(\tan x)^{\ln(\sin x)} > (\cot x)^{\ln(\sin x)}, \forall x \in (0, \pi/4)$
(2) $4^{\ln \operatorname{cosec} x} < 5^{\ln \operatorname{cosec} x}, \forall x \in (0, \pi/2)$
(3) $(1/2)^{\ln(\cos x)} < (1/3)^{\ln(\cos x)}, \forall x \in (0, \pi/2)$
(4) $2^{\ln(\tan x)} > 2^{\ln(\sin x)}, \forall x \in (0, \pi/2)$
- If $3 \tan A + 4 = 0$, then the value of $2 \cot A - 5 \cos A + \sin A$ is equal to
(1) $\frac{23}{10}$ if $\frac{\pi}{2} < A < \pi$ (2) $\frac{23}{10}$ if $\frac{3\pi}{2} < A < 2\pi$
(3) $-\frac{53}{10}$ if $\frac{\pi}{2} < A < \pi$ (4) $-\frac{53}{10}$ if $\frac{3\pi}{2} < A < 2\pi$
- A circle centered at O has radius 1 and contains point A . Segment AB is tangent to the circle at A and $\angle AOB = \theta$. If point C lies on OA , and BC bisects the angle ABO , then OC equals

- (1) $\sec \theta (\sec \theta - \tan \theta)$ (2) $\frac{\cos^2 \theta}{1 + \sin \theta}$
- (3) $\frac{1}{1 + \sin \theta}$ (4) $\frac{1 - \sin \theta}{\cos^2 \theta}$
6. $(a+2) \sin \alpha + (2a-1) \cos \alpha = (2a+1)$ if $\tan \alpha$ is
 (1) $3/4$ (2) $4/3$
 (3) $2a/(a^2+1)$ (4) $2a/(a^2-1)$
7. Let $f(x) = \log(\log_{1/3}(\log_7(\sin x + a)))$ be defined for every real value of x , then the possible value of a is
 (1) 3 (2) 4 (3) 5 (4) 6
8. If $b > 1$, $\sin t > 0$, $\cos t > 0$ and $\log_b(\sin t) = x$, then $\log_b(\cos t)$ is equal to
 (1) $\frac{1}{2} \log_b(1 - b^{2x})$ (2) $2 \log(1 - b^{x/2})$
 (3) $\log_b \sqrt{1 - b^{2x}}$ (4) $\sqrt{1 - x^2}$
9. Which of the following is possible in $\triangle ABC$?
 (1) $\cos A + \cos B + \cos C = \frac{3}{2}$
 (2) $\cos A \cos B \cos C = 0$
 (3) $\sin A + \sin B + \sin C = \sqrt{2} + 1$
 (4) $\sin A \sin B \sin C = -\frac{3}{8}$
10. If $2 \sec^2 \alpha - \sec^4 \alpha - 2 \operatorname{cosec}^2 \alpha + \operatorname{cosec}^4 \alpha = 15/4$, then $\tan \alpha$ is equal to
 (1) $1/\sqrt{2}$ (2) $1/2$
 (3) $1/2\sqrt{2}$ (4) $-1/\sqrt{2}$
11. If $\cot \theta + \tan \theta = x$ and $\sec \theta - \cos \theta = y$, then
 (1) $x \sin \theta \cdot \cos \theta = 1$ (2) $\sin^2 \theta = y \cos \theta$
 (3) $(x^2 y)^{1/3} + (xy^2)^{1/3} = 1$ (4) $(x^2 y)^{2/3} - (xy^2)^{2/3} = 1$
12. If $x = \sec \phi - \tan \phi$ and $y = \operatorname{cosec} \phi + \cot \phi$, then
 (1) $x = \frac{y+1}{y-1}$ (2) $x = \frac{y-1}{y+1}$
 (3) $y = \frac{1+x}{1-x}$ (4) $xy + x - y + 1 = 0$
13. The value of $f(\alpha) = \sqrt{\operatorname{cosec}^2 \alpha - 2 \cot \alpha} + \sqrt{\operatorname{cosec}^2 \alpha + 2 \cot \alpha}$ can be
 (1) $2 \cot \alpha$ (2) $-2 \cot \alpha$
 (3) 2 (4) -2
14. If $\frac{y+3}{2y+5} = \sin^2 x + 2 \cos x + 1$, then the value of y lies in the interval
 (1) $\left[-\infty, -\frac{8}{3}\right]$ (2) $\left[-\frac{12}{5}, \infty\right)$
 (3) $\left[-\frac{8}{3}, -\frac{12}{5}\right]$ (4) $\left[-\frac{8}{3}, \infty\right)$

15. If $\cos \alpha = \frac{1}{2} \left(x + \frac{1}{x} \right)$ and $\cos \beta = \frac{1}{2} \left(y + \frac{1}{y} \right)$, ($xy > 0$);

$x, y, \alpha, \beta \in R$ then

- (1) $\sin(\alpha + \beta + \gamma) = \sin \gamma \forall \gamma \in R$
 (2) $\cos \alpha \cos \beta = 1 \forall \alpha, \beta \in R$
 (3) $(\cos \alpha + \cos \beta)^2 = 4 \forall \alpha, \beta \in R$
 (4) $\sin(\alpha + \beta + \gamma) = \sin \alpha + \sin \beta + \sin \gamma \forall \alpha, \beta, \gamma \in R$

16. Four numbers n_1, n_2, n_3 and n_4 are given as
 $n_1 = \sin 15^\circ - \cos 15^\circ, n_2 = \cos 93^\circ + \sin 93^\circ, n_3 = \tan 27^\circ - \cot 27^\circ, n_4 = \cot 127^\circ + \tan 127^\circ$. Then

- (1) $n_1 < 0$ (2) $n_2 < 0$
 (3) $n_3 < 0$ (4) $n_4 < 0$

17. For $0 < \phi < \pi/2$, if $x = \sum_{n=0}^{\infty} \cos^{2n} \phi, y = \sum_{n=0}^{\infty} \sin^{2n} \phi,$
 $z = \sum_{n=0}^{\infty} \cos^{2n} \phi \sin^{2n} \phi$, then

- (1) $xyz = xz + y$ (2) $xyz = xy + z$
 (3) $xyz = x + y + z$ (4) $xyz = yz + x$

Linked Comprehension Type

For Problems 1-3

Let us consider the equation $\frac{\cos^4 x}{a} + \frac{\sin^4 x}{b} = \frac{1}{a+b}, x \in \left[0, \frac{\pi}{2}\right];$
 $a, b > 0$

1. For the given equation,

- (1) $\frac{\sin^4 x}{b} = \frac{\cos^4 x}{a}$ (2) $\frac{\sin x}{a} = \frac{\cos x}{b}$
 (3) $\frac{\sin^4 x}{b^2} = \frac{\cos^4 x}{a^2}$ (4) $\frac{\sin^2 x}{a} = \frac{\cos^2 x}{b}$

2. The value of $\sin^2 x$ in terms of a and b is

- (1) $\sqrt{ab}$ (2) $\frac{b}{a+b}$
 (3) $\frac{b^2 - a^2}{a^2 + b^2}$ (4) $\frac{a^2 + b^2}{b^2 - a^2}$

3. The value of $\frac{\sin^8 x}{b^3} + \frac{\cos^8 x}{a^3}$ is

- (1) $\frac{1}{(a+b)^2}$ (2) $\frac{1}{(a+b)^3}$
 (3) $\frac{1}{(a+b)^4}$ (4) $\frac{1}{a^3 + b^3}$

For Problems 4-6

α, β, γ and δ are angles in I, II, III and IV quadrants, respectively and none of them is an integral multiple of $\pi/2$. They form an increasing arithmetic progression.

4. Which of the following holds?

- (1) $\cos(\alpha + \delta) > 0$
 (2) $\cos(\alpha + \delta) = 0$

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- (3) $\cos(\alpha + \delta) < 0$
 (4) $\cos(\alpha + \delta) > 0$ or $\cos(\alpha + \delta) < 0$
5. Which of the following does not hold?
 (1) $\sin(\beta + \gamma) = \sin(\alpha + \delta)$
 (2) $\sin(\beta - \gamma) = \sin(\alpha - \delta)$
 (3) $\tan 2(\alpha - \beta) = \tan(\beta - \delta)$
 (4) $\cos(\alpha + \gamma) = \cos 2\beta$
6. If $\alpha + \beta + \gamma + \delta = \theta$ and $\alpha = 70^\circ$, then
 (1) $400^\circ < \theta < 580^\circ$ (2) $470^\circ < \theta < 650^\circ$
 (3) $680^\circ < \theta < 860^\circ$ (4) $540^\circ < \theta < 900^\circ$

For Problems 7 and 8

In $\triangle ABC$, $BC = 1$, $\sin \frac{A}{2} = x_1$, $\sin \frac{B}{2} = x_2$, $\cos \frac{A}{2} = x_3$ and

$$\cos \frac{B}{2} = x_4 \text{ with } \left(\frac{x_1}{x_2}\right)^{2007} - \left(\frac{x_3}{x_4}\right)^{2006} = 0.$$

7. Length of side AC is equal to
 (1) $1/2$ (2) 1
 (3) 2 (4) can't be determined
8. If $\angle A = 90^\circ$, then area of $\triangle ABC$ is
 (1) $1/2$ sq. units (2) $1/3$ sq. units
 (3) 1 sq. units (4) 2 sq. units

For Problems 9–11

Let $f(x) = \sin^6 x + \cos^6 x + k(\sin^4 x + \cos^4 x)$ for some real number k .

9. Value of k for which $f(x)$ is constant for all values of x is
 (1) $-1/2$ (2) $1/2$ (3) $1/4$ (4) $-3/2$
10. All real numbers k for which the equation $f(x) = 0$ has solution lie in
 (1) $[-1, 0]$ (2) $\left[0, \frac{1}{2}\right]$
 (3) $\left[-1, -\frac{1}{2}\right]$ (4) None of these
11. Number of values of k for which $f(x) = 0$ is an identity is
 (1) 0 (2) 1
 (3) infinite (4) none of these

Matrix Match Type

1.

List I	List II
a. If $x = \sin \theta$ $\sin \theta$ and $y = \cos \theta$ $\cos \theta$ and $\frac{99\pi}{2} < \theta < 50\pi$, then $(y - x)$ is equal to	p. -1
b. $\frac{\sin(270^\circ + x)\cos^3(720^\circ - x) - \sin(270^\circ - x)\sin^3(540^\circ + x)}{\sin(90^\circ + x)\sin(-x) - \cos^2(180^\circ - x)} + \frac{\cot(270^\circ - x)}{\operatorname{cosec}^2(450^\circ + x)} =$	q. 0

c. $\sin(-870^\circ) + \operatorname{cosec}(-660^\circ) + \tan(-855^\circ) + 2 \cot(840^\circ) + \cos(480^\circ) + \sec(900^\circ) =$	r. -2
d. $2 \frac{\cos^3\left(\frac{\pi}{2} + x\right) \cot(3\pi + x) \sec(x - 3\pi) \operatorname{cosec}\left(\frac{3\pi}{2} - x\right)}{\cot x \tan^2(x - \pi) \sin(x - 2\pi)}$ is equal to	s. 1

2.

List I	List II
a. Suppose ABC is a triangle with three acute angles A , B and C . The point whose coordinates are $(\cos B - \sin A, \sin B - \cos A)$ can be in the	p. 1st quadrant
b. If $2^{\sin \theta} > 1$ and $3^{\cos \theta} < 1$, then $\theta \in$	q. 2nd quadrant
c. For $ \cos x + \sin x = \sin x + \cos x $, x belongs to	r. 3rd quadrant
d. If $\sqrt{\frac{1 - \sin A}{1 + \sin A}} + \frac{\sin A}{\cos A} = \frac{1}{\cos A}$, for all permissible values of A , then A can belong to	s. 4th quadrant

3. For all real values of θ , choose the correct option.

List I	List II
a. $A = \sin^2 \theta + \cos^4 \theta$	p. $A \in [-1, 1]$
b. $A = 3 \cos^2 \theta + \sin^4 \theta$	q. $A \in \left[\frac{3}{4}, 1\right]$
c. $A = \sin^2 \theta - \cos^4 \theta$	r. $A \in [2\sqrt{2}, \infty)$
d. $A = \tan^2 \theta + 2 \cot^2 \theta$	s. $A \in [1, 3]$

Codes:

a	b	c	d
(1) p	q	r	s
(2) s	r	p	q
(3) r	p	s	q
(4) q	s	p	r

Numerical Value Type

1. The value of the expression $\frac{\tan^2 20^\circ - \sin^2 20^\circ}{\tan^2 20^\circ \cdot \sin^2 20^\circ}$ is _____.
2. Suppose that for some angles x and y , the equations $\sin^2 x + \cos^2 y = \frac{3a}{2}$ and $\cos^2 x + \sin^2 y = \frac{a^2}{2}$ hold simultaneously. The possible value of a is _____.
3. If $0 < x < \frac{\pi}{4}$ and $\cos x + \sin x = \frac{5}{4}$, then the value $16(\cos x - \sin x)^2$ is _____.
4. The value of $3 \frac{\sin^4 t + \cos^4 t - 1}{\sin^6 t + \cos^6 t - 1}$ is equal to _____.

5. If $\sin \theta - \cos \theta = 1$, then the value of $\sin^3 \theta - \cos^3 \theta$ is _____.

6. If $\sin \theta, \tan \theta, \cos \theta$ are in G.P. then $4 \sin^2 \theta - 3 \sin^4 \theta + \sin^6 \theta =$ _____.

7. Let $f(\theta) = \frac{1}{1 + (\cot \theta)^x}$, and $S = \sum_{\theta=1^\circ}^{89^\circ} f(\theta)$, then the value of $\sqrt{5}$ is _____.

8. The minimum value of $\sqrt{(3 \sin x - 4 \cos x - 10)(3 \sin x + 4 \cos x - 10)}$ is _____.

9. If $a \in (0, 1)$, and $f(a) = (a^2 - a + 1) + \frac{8 \sin^2 a}{\sqrt{a^2 - a + 1}} + \frac{27 \operatorname{cosec}^2 a}{\sqrt{a^2 - a + 1}}$

$\frac{8 \sin^2 a}{\sqrt{a^2 - a + 1}} + \frac{27 \operatorname{cosec}^2 a}{\sqrt{a^2 - a + 1}}$, then the least value of $f(a)$ is _____.

10. Minimum value of $\frac{\sec^4 \alpha}{\tan^2 \beta} + \frac{\sec^4 \beta}{\tan^2 \alpha}$,

where $\alpha \neq \frac{\pi}{2}, \beta \neq \frac{\pi}{2}, 0 < \alpha, \beta < \frac{\pi}{2}$, is _____.

11. If $p \operatorname{cosec} \theta + q \cot \theta = 2$ and $p^2 \operatorname{cosec}^2 \theta - q^2 \cot^2 \theta = 5$ then the value of $81p^{-2} - q^{-2}$ is _____.

Archives

JEE MAIN

Single Correct Answer Type

1. The expression $\frac{\tan A}{1 - \cot A} + \frac{\cot A}{1 - \tan A}$ can be written as:

- (1) $\sin A \cos A + 1$ (2) $\sec A \operatorname{cosec} A + 1$
(3) $\tan A + \cot A$ (4) $\sec A + \operatorname{cosec} A$

(JEE Main 2013)

2. Let $f_k(x) = \frac{1}{k} (\sin^k x + \cos^k x)$ where $x \in R$ and $k \geq 1$.

Then $f_4(x) - f_6(x)$ equals

- (1) $\frac{1}{6}$ (2) $\frac{1}{3}$
(3) $\frac{1}{4}$ (4) $\frac{1}{12}$

(JEE Main 2014)

JEE ADVANCED

Single Correct Answer Type

1. Let $P = \{\theta : \sin \theta - \cos \theta = \sqrt{2} \cos \theta\}$ and $Q = \{\theta : \sin \theta + \cos \theta = \sqrt{2} \sin \theta\}$ be two sets. Then

- (1) $P \subset Q$ and $Q - P \neq \emptyset$ (2) $Q \subset P$
(3) $P \not\subset Q$ (4) $P = Q$ (IIT-JEE 2011)

Multiple Correct Answers Type

1. If $\frac{\sin^4 x}{2} + \frac{\cos^4 x}{3} = \frac{1}{5}$, then

- (1) $\tan^2 x = \frac{2}{3}$ (2) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{1}{125}$
(3) $\tan^2 x = \frac{1}{3}$ (4) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{2}{125}$

(IIT-JEE 2009)

Answers Key

EXERCISES

Single Correct Answer Type

- | | | | | |
|---------|---------|---------|---------|---------|
| 1. (3) | 2. (2) | 3. (1) | 4. (4) | 5. (3) |
| 6. (2) | 7. (3) | 8. (3) | 9. (2) | 10. (4) |
| 11. (3) | 12. (3) | 13. (3) | 14. (2) | 15. (3) |
| 16. (2) | 17. (4) | 18. (2) | 19. (2) | 20. (1) |
| 21. (3) | 22. (2) | 23. (4) | 24. (3) | 25. (3) |
| 26. (4) | 27. (1) | 28. (2) | 29. (1) | 30. (2) |
| 31. (3) | 32. (2) | 33. (2) | 34. (1) | 35. (2) |
| 36. (4) | 37. (1) | 38. (2) | 39. (4) | 40. (2) |
| 41. (3) | 42. (2) | 43. (3) | 44. (4) | 45. (4) |
| 46. (2) | 47. (2) | 48. (2) | 49. (2) | 50. (2) |

Multiple Correct Answers Type

- | | |
|------------------------|-------------------|
| 1. (1), (2), (3), (4) | 2. (2), (3), (4) |
| 3. (1), (2), (3), (4) | 4. (1), (4) |
| 5. (1), (3), (4) | 6. (2), (4) |
| 7. (1), (2), (3) | 8. (1), (3) |
| 9. (1), (2), (3) | 10. (1), (4) |
| 11. (1), (2), (4) | 12. (2), (3), (4) |
| 13. (1), (2), (3) | 14. (1), (2) |
| 15. (1), (2), (3), (4) | 16. (1), (3), (4) |
| 17. (2), (3) | |

2.32 Trigonometry

Linked Comprehension Type

- | | | | | |
|---------|--------|--------|--------|---------|
| 1. (3) | 2. (2) | 3. (2) | 4. (1) | 5. (2) |
| 6. (3) | 7. (2) | 8. (1) | 9. (4) | 10. (3) |
| 11. (1) | | | | |

Matrix Match Type

1. $a \rightarrow s; b \rightarrow s; c \rightarrow p; d \rightarrow r$
2. $a \rightarrow q; b \rightarrow q; c \rightarrow p; r; d \rightarrow p, s$
3. (4)

Numerical Value Type

- | | | | | |
|----------|-----------|--------|---------|---------|
| 1. (1) | 2. (1) | 3. (7) | 4. (2) | 5. (1) |
| 6. (1) | 7. (44.5) | 8. (7) | 9. (18) | 10. (8) |
| 11. (16) | | | | |

ARCHIVES

JEE MAIN

Single Correct Answer Type

1. (2)
2. (4)

JEE ADVANCED

Single Correct Answer Type

1. (4)

Multiple Correct Answers Type

1. (1), (2)

3

Trigonometric Ratios and Transformation Formulas

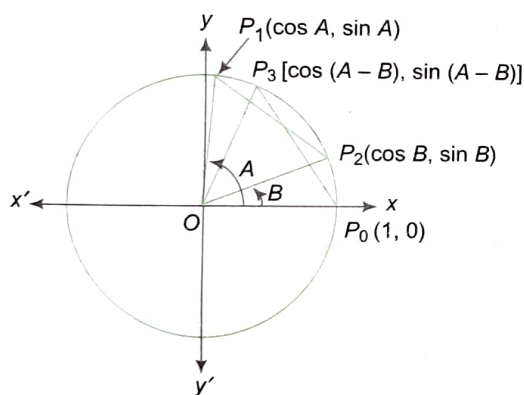
COMPOUND ANGLE FORMULAS FOR COSINE AND SINE

COSINE OF THE DIFFERENCE AND SUM OF THE ANGLES

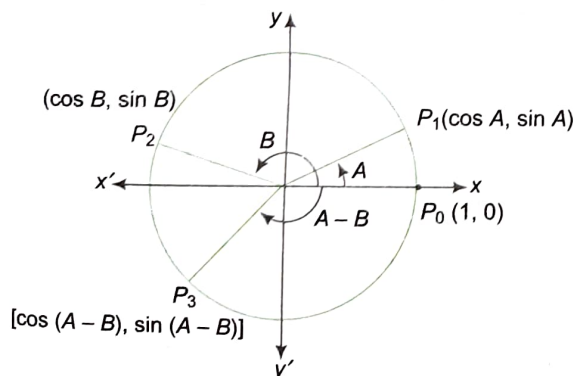
1. $\cos(A - B) = \cos A \cos B + \sin A \sin B$
2. $\cos(A + B) = \cos A \cos B - \sin A \sin B$
for all angles A and B .

Proof:

1. Let $X'OX$ and YOY' be the coordinate axes. Consider a unit circle with O as the center (Figures (a) and (b)). Let P_1 , P_2 , and P_3 be the three points on the circle such that $\angle XOP_1 = A$, $\angle XOP_2 = B$, and $\angle XOP_3 = A - B$. As we know that the terminal side of any angle intersects the circle with center at O and unit radius at a point whose coordinates are the cosine and sine of the angle. Therefore, coordinates of P_1 , P_2 , and P_3 are $(\cos A, \sin A)$, $(\cos B, \sin B)$, and $(\cos(A - B), \sin(A - B))$, respectively.



(a)



(b)

We know that equal chords of a circle make equal angles at its center, so chords P_0P_3 and P_1P_2 subtend equal angles at O . Therefore,

$$\text{Chord } P_0P_3 = \text{Chord } P_1P_2$$

$$\Rightarrow \sqrt{\{\cos(A - B) - 1\}^2 + \{\sin(A - B) - 0\}^2}$$

$$= \sqrt{(\cos B - \cos A)^2 + (\sin B - \sin A)^2}$$

$$\text{or } \{\cos(A - B) - 1\}^2 + \sin^2(A - B)$$

$$= (\cos B - \cos A)^2 + (\sin B - \sin A)^2$$

$$\text{or } \cos^2(A - B) - 2 \cos(A - B) + 1 + \sin^2(A - B)$$

$$= \cos^2 B + \cos^2 A - 2 \cos A \cos B + \sin^2 B$$

$$+ \sin^2 A - 2 \sin A \sin B$$

$$\text{or } 2 - 2 \cos(A - B)$$

$$= 2 - 2 \cos A \cos B - 2 \sin A \sin B$$

$$\text{or } \cos(A - B) = \cos A \cos B + \sin A \sin B \quad (i)$$

$$2. \cos(A + B) = \cos(A - (-B))$$

$$= \cos A \cos(-B) + \sin A \sin(-B) \quad [\text{Using (i)}]$$

$$= \cos A \cos B - \sin A \sin B$$

$$[\because \cos(-B) = \cos B, \sin(-B) = -\sin B]$$

$$\text{Hence, } \cos(A + B) = \cos A \cos B - \sin A \sin B$$

Note:

This method of proof of the above formula is true for all values of angles A and B whether the value is positive, zero, or negative.

SINE OF THE DIFFERENCE AND SUM OF THE ANGLES

$$1. \sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$2. \sin(A + B) = \sin A \cos B + \cos A \sin B$$

Proof.

$$1. \sin(A - B) = \cos(90^\circ - (A - B)) \quad [\because \cos(90^\circ - \theta) = \sin \theta]$$

$$= \cos((90^\circ - A) + B)$$

$$= \cos(90^\circ - A) \cos B - \sin(90^\circ - A) \sin B$$

$$= \sin A \cos B - \cos A \sin B \quad (i)$$

$$2. \sin(A + B) = \sin(A - (-B))$$

$$= \sin A \cos(-B) - \cos A \sin(-B) \quad [\text{Using (i)}]$$

$$= \sin A \cos B + \cos A \sin B$$

$$[\because \sin(-B) = -\sin B]$$

SOME MORE RESULTS

$$1. \sin(A + B) \sin(A - B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$$

$$2. \cos(A + B) \cos(A - B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$$

3.2 Trigonometry

$$3. \sin(A+B+C) = \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C$$

$$4. \cos(A+B+C) = \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C - \sin A \sin B \cos C$$

Proof:

$$1. \sin(A+B) \sin(A-B)$$

$$= (\sin A \cos B + \cos A \sin B) (\sin A \cos B - \cos A \sin B)$$

$$= \sin^2 A \cos^2 B - \cos^2 A \sin^2 B$$

$$= \sin^2 A (1 - \sin^2 B) - (1 - \sin^2 A) \sin^2 B$$

$$= \sin^2 A - \sin^2 A \sin^2 B - \sin^2 B + \sin^2 A \sin^2 B$$

$$= \sin^2 A - \sin^2 B$$

$$= (1 - \cos^2 A) - (1 - \cos^2 B) = \cos^2 B - \cos^2 A$$

$$2. \cos(A+B) \cos(A-B)$$

$$= (\cos A \cos B - \sin A \sin B) (\cos A \cos B + \sin A \sin B)$$

$$= \cos^2 A \cos^2 B - \sin^2 A \sin^2 B$$

$$= \cos^2 A (1 - \sin^2 B) - (1 - \cos^2 A) \sin^2 B = \cos^2 A - \sin^2 B$$

$$= (1 - \sin^2 A) - (1 - \cos^2 B) = \cos^2 B - \sin^2 A$$

ILLUSTRATION 3.1

Prove that $\frac{\sin(B-C)}{\cos B \cos C} + \frac{\sin(C-A)}{\cos C \cos A} + \frac{\sin(A-B)}{\cos A \cos B} = 0$.

Sol. The first term of the L.H.S. is

$$\frac{\sin(B-C)}{\cos B \cos C} = \frac{\sin B \cos C - \cos B \sin C}{\cos B \cos C}$$

$$= \frac{\sin B \cos C}{\cos B \cos C} - \frac{\cos B \sin C}{\cos B \cos C} = \tan B - \tan C$$

Similarly, the second term of the L.H.S. is $(\tan C - \tan A)$

and the third term of the L.H.S. is $(\tan A - \tan B)$

Now L.H.S. = $(\tan B - \tan C) + (\tan C - \tan A)$
 $+ (\tan A - \tan B) = 0$.

ILLUSTRATION 3.2

Eliminate x from equations $\sin(a+x) = 2b$ and $\sin(a-x) = 2c$.

Sol. On adding we get,

$$\sin(a+x) + \sin(a-x) = 2(b+c),$$

$$2 \sin a \cos x = 2(b+c)$$

or $\cos x = \frac{b+c}{\sin a}$... (i)

On subtracting, we get

$$\sin(a+x) - \sin(a-x) = 2(b-c)$$

or $2 \cos a \sin x = 2(b-c)$

or $\sin x = \frac{b-c}{\cos a}$... (ii)

Squaring and adding Eqs. (i) and (ii), we get

$$\frac{(b+c)^2}{\sin^2 a} + \frac{(b-c)^2}{\cos^2 a} = 1$$

ILLUSTRATION 3.3

Let A, B, C be the three angles such that $A+B+C = \pi$.
 $\tan A \cdot \tan B = 2$, then find the value of $\frac{\cos A \cos B}{\cos C}$.

Sol. Given, $\tan A \cdot \tan B = 2$

Let $y = \frac{\cos A \cos B}{\cos C} = -\frac{\cos A \cdot \cos B}{\cos(A+B)}$

$$(\because \cos C = \cos(\pi - (A+B)) = -\cos(A+B))$$

$$= \frac{\cos A \cdot \cos B}{\sin A \sin B - \cos A \cos B}$$

$$= \frac{1}{\tan A \tan B - 1} = \frac{1}{2-1} = 1$$

ILLUSTRATION 3.4

If $\sin \alpha \sin \beta - \cos \alpha \cos \beta + 1 = 0$, then prove that $1 + \cos \tan \beta = 0$.

Sol. Given, $\sin \alpha \sin \beta - \cos \alpha \cos \beta + 1 = 0$

or $\cos \alpha \cos \beta - \sin \alpha \sin \beta = 1$

or $\cos(\alpha + \beta) = 1$

Now $1 + \cot \alpha \tan \beta = 1 + \frac{\cos \alpha}{\sin \alpha} \times \frac{\sin \beta}{\cos \beta}$

$$= \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\sin \alpha \cos \beta}$$

$$= \frac{\sin(\alpha + \beta)}{\sin \alpha \cos \beta}$$

$$= \frac{0}{\sin \alpha \cos \beta} = 0$$

$$[\because \sin^2(\alpha + \beta) = 1 - \cos^2(\alpha + \beta) = 1 - 1 = 0]$$

ILLUSTRATION 3.5

If $\sin \alpha \cos \beta = -\frac{1}{2}$ then find the range of values of $\cos \alpha \sin \beta$.

Sol. $\sin \alpha \cos \beta + \cos \alpha \sin \beta = \sin(\alpha + \beta)$

Now, $-1 \leq \sin(\alpha + \beta) \leq 1$

$$\Rightarrow -1 \leq \sin \alpha \cos \beta + \cos \alpha \sin \beta \leq 1$$

$$\Rightarrow -1 \leq -0.5 + \cos \alpha \sin \beta \leq 1$$

$$\Rightarrow -0.5 \leq \cos \alpha \sin \beta \leq 0.5$$

ILLUSTRATION 3.6

Show that $\cos^2 \theta + \cos^2(\alpha + \theta) - 2 \cos \alpha \cos \theta \cos(\alpha + \theta)$ independent of θ .

Sol. $\cos^2 \theta + \cos^2(\alpha + \theta) - 2 \cos \alpha \cos \theta \cos(\alpha + \theta)$

$$= \cos^2 \theta + \cos(\alpha + \theta)[\cos(\alpha + \theta) - 2 \cos \alpha \cos \theta]$$

$$= \cos^2 \theta + \cos(\alpha + \theta)$$

$$[\cos \alpha \cos \theta - \sin \alpha \sin \theta - 2 \cos \alpha \cos \theta]$$

$$\begin{aligned}
&= \cos^2 \theta - \cos(\alpha + \theta)[\cos \alpha \cos \theta + \sin \alpha \sin \theta] \\
&= \cos^2 \theta - \cos(\alpha + \theta) \cos(\alpha - \theta) \\
&= \cos^2 \theta - [\cos^2 \alpha - \sin^2 \theta] = \cos^2 \theta + \sin^2 \theta - \cos^2 \alpha \\
&= 1 - \cos^2 \alpha, \text{ which is independent of } \theta.
\end{aligned}$$

ILLUSTRATION 3.7

If $3 \tan \theta \tan \varphi = 1$, then prove that $2 \cos(\theta + \varphi) = \cos(\theta - \varphi)$.

Sol. Given, $3 \tan \theta \tan \varphi = 1$ or $\cot \theta \cot \varphi = 3$

or $\frac{\cos \theta \cos \varphi}{\sin \theta \sin \varphi} = \frac{3}{1}$

By componendo and dividendo, we get

$$\frac{\cos \theta \cos \varphi + \sin \theta \sin \varphi}{\cos \theta \cos \varphi - \sin \theta \sin \varphi} = \frac{3 + 1}{3 - 1}$$

or $\frac{\cos(\theta - \varphi)}{\cos(\theta + \varphi)} = 2$

or $2 \cos(\theta + \varphi) = \cos(\theta - \varphi)$

ILLUSTRATION 3.8

In $\triangle ABC$, if $\cot A + \cot B + \cot C = 0$ then find the value of $\cos A \cos B \cos C$.

Sol. $\cot A + \cot B + \cot C = 0$

$$\Rightarrow \cot B + \cot C = -\cot A$$

$$\Rightarrow \frac{\sin B \cos C + \sin C \cos B}{\sin B \sin C} = \frac{-\cos A}{\sin A}$$

$$\Rightarrow \frac{\sin(B + C)}{\sin B \sin C} = \frac{-\cos A}{\sin A}$$

$$\Rightarrow \frac{\sin^2 A}{\sin B \sin C} = -\cos A \quad (\because \sin(B + C) = \sin(\pi - A) = \sin A) \quad (i)$$

Similarly $\frac{\sin^2 B}{\sin A \sin C} = -\cos B \quad (ii)$

and $\frac{\sin^2 C}{\sin A \sin B} = -\cos C \quad (iii)$

Multiplying Eqs. (i), (ii) and (iii), we get

$$\cos A \cos B \cos C = -1$$

ILLUSTRATION 3.9

If $\alpha, \beta, \gamma \in \left(0, \frac{\pi}{2}\right)$, then prove that $\frac{\sin(\alpha + \beta + \gamma)}{\sin \alpha + \sin \beta + \sin \gamma} < 1$.

Sol. $\sin(\alpha + \beta + \gamma) = \sin \alpha \cos \beta \cos \gamma + \cos \alpha \sin \beta \cos \gamma + \cos \alpha \cos \beta \sin \gamma - \sin \alpha \sin \beta \sin \gamma$

or $\sin(\alpha + \beta + \gamma) - \sin \alpha - \sin \beta - \sin \gamma$
 $= \sin \alpha (\cos \beta \cos \gamma - 1) + \sin \beta (\cos \alpha \cos \gamma - 1)$
 $+ \sin \gamma (\cos \alpha \cos \beta - 1) - \sin \alpha \sin \beta \sin \gamma$

$\therefore \sin(\alpha + \beta + \gamma) - \sin \alpha - \sin \beta - \sin \gamma < 0$

or $\sin(\alpha + \beta + \gamma) < \sin \alpha + \sin \beta + \sin \gamma$

or $\frac{\sin(\alpha + \beta + \gamma)}{\sin \alpha + \sin \beta + \sin \gamma} < 1$

ILLUSTRATION 3.10

Let α, β and γ satisfy $0 < \alpha < \beta < \gamma < 2\pi$. If $\cos(x + \alpha) + \cos(x + \beta) + \cos(x + \gamma) = 0$ for all $x \in R$, then find the possible values of $(\gamma - \alpha)$.

Sol. Given, $\cos(x + \alpha) + \cos(x + \beta) + \cos(x + \gamma) = 0 \quad \forall x \in R$
 $\therefore \cos x(\cos \alpha + \cos \beta + \cos \gamma) - \sin x(\sin \alpha + \sin \beta + \sin \gamma) = 0 \quad \forall x \in R$

$$\therefore \cos \alpha + \cos \beta + \cos \gamma = 0 = \sin \alpha + \sin \beta + \sin \gamma$$

$$\therefore \cos \alpha + \cos \gamma = -\cos \beta \quad \dots(1)$$

and $\sin \alpha + \sin \gamma = -\sin \beta \quad \dots(2)$

Squaring Eqs. (1) and (2) and then adding then, we get

$$2 + 2 \cos(\alpha - \gamma) = 1$$

$$\Rightarrow \cos(\gamma - \alpha) = \frac{-1}{2}$$

$$\Rightarrow \gamma - \alpha = \frac{2\pi}{3} \text{ or } \frac{4\pi}{3}$$

ILLUSTRATION 3.11

If in triangle ABC , $\angle C = 45^\circ$ then find the range of the values of $\sin^2 A + \sin^2 B$.

Sol. $E = \sin^2 A + \sin^2 B$
 $= \sin^2 A + \sin^2(135^\circ - A)$
 $= \sin^2 A + \cos^2(45^\circ - A)$
 $= 1 + \sin^2 A - \sin^2(45^\circ - A)$
 $= 1 + \sin(2A - 45^\circ) \sin 45^\circ$
 $= 1 + \frac{1}{\sqrt{2}} \sin(2A - 45^\circ)$

Now $0 < A < 135^\circ$

$$\Rightarrow 0 < 2A < 270^\circ$$

$$\Rightarrow -45^\circ < 2A - 45^\circ < 225^\circ$$

$$\Rightarrow -\frac{1}{\sqrt{2}} < \sin(2A - 45^\circ) \leq 1$$

$$\Rightarrow -\frac{1}{2} < \frac{1}{\sqrt{2}} \sin(2A - 45^\circ) \leq \frac{1}{\sqrt{2}}$$

$$\Rightarrow \frac{1}{2} < 1 + \frac{1}{\sqrt{2}} \sin(2A - 45^\circ) \leq 1 + \frac{1}{\sqrt{2}}$$

ILLUSTRATION 3.12

Prove that $\sum_{k=1}^{100} \sin(kx) \cos(101-k)x = 50 \sin(101x)$

Sol. Let

$$S = \sum_{k=1}^{100} \sin(kx) \cos(101-k)x$$

$$\Rightarrow S = \sin x \cos 100x + \sin 2x \cos 99x + \dots + \sin 100x \cos x \quad \dots (i)$$

3.4 Trigonometry

$$S = \cos x \sin 100x + \cos 2x \sin 99x + \dots + \sin x \cos 100x \dots \text{(ii)}$$

(on writing in reverse order)

Adding Eqs. (i) and (ii), we get

$$\begin{aligned} 2S &= (\sin x \cos 100x + \cos x \sin 100x) \\ &\quad + (\sin 2x \cos 99x + \cos 2x \sin 99x) \\ &\quad \dots \\ &\quad \dots \\ &\quad + (\sin 100x \cos x + \sin x \cos 100x) \\ &= \sin 101x + \sin 101x + \dots + \sin 101x \text{ (100 times)} \end{aligned}$$

$$\text{Hence, } S = 50 \sin(101x)$$

ILLUSTRATION 3.13

Find the maximum value of $4 \sin^2 x + 3 \cos^2 x + \sin \frac{x}{2} + \cos \frac{x}{2}$.

Sol. We have

$$\begin{aligned} S &= 4 \sin^2 x + 3 \cos^2 x + \sin \frac{x}{2} + \cos \frac{x}{2} \\ &= 3 + \sin^2 x + \sin \frac{x}{2} + \cos \frac{x}{2} \\ &= 3 + \sin^2 x + \sqrt{2} \sin \left(\frac{\pi}{4} + \frac{x}{2} \right) \end{aligned}$$

Clearly, maximum value of S occurs when $x = \frac{\pi}{2}$.

$$\text{So, } S_{\max} = 3 + 1 + \sqrt{2} = 4 + \sqrt{2}$$

EXPRESSION $a \cos x + b \sin x$

We have expression

$$\begin{aligned} f(x) &= a \cos x + b \sin x \\ \therefore f(x) &= \sqrt{a^2 + b^2} \left[\frac{a}{\sqrt{a^2 + b^2}} \cos x + \frac{b}{\sqrt{a^2 + b^2}} \sin x \right] \dots(1) \end{aligned}$$

$$\text{Let } \frac{a}{\sqrt{a^2 + b^2}} = \sin \alpha, \frac{b}{\sqrt{a^2 + b^2}} = \cos \alpha$$

$$\begin{aligned} \therefore f(x) &= \sqrt{a^2 + b^2} (\sin \alpha \cos x + \cos \alpha \sin x) \\ &= \sqrt{a^2 + b^2} \sin(x + \alpha) \end{aligned}$$

Since $-1 \leq \sin(x + \alpha) \leq 1$, we have

$$-\sqrt{a^2 + b^2} \leq \sqrt{a^2 + b^2} \sin(x + \alpha) \leq \sqrt{a^2 + b^2}$$

$$\therefore a \cos x + b \sin x \in \left[-\sqrt{a^2 + b^2}, \sqrt{a^2 + b^2} \right]$$

In (1), if we let $\frac{a}{\sqrt{a^2 + b^2}} = \cos \alpha, \frac{b}{\sqrt{a^2 + b^2}} = \sin \alpha$, then

$$\begin{aligned} f(x) &= \sqrt{a^2 + b^2} (\cos \alpha \cos x + \sin \alpha \sin x) \\ &= \sqrt{a^2 + b^2} \cos(x - \alpha) \end{aligned}$$

$$\text{Also, } -\sqrt{a^2 + b^2} \leq \sqrt{a^2 + b^2} \cos(x - \alpha) \leq \sqrt{a^2 + b^2}$$

ILLUSTRATION 3.14

Prove that $5 \cos \theta + 3 \cos \left(\theta + \frac{\pi}{3} \right) + 3$ lies between -4 and

$$\begin{aligned} \text{Sol. We have } 5 \cos \theta + 3 \cos \left(\theta + \frac{\pi}{3} \right) + 3 &= 5 \cos \theta + 3 \cos \theta \cos \frac{\pi}{3} - 3 \sin \theta \sin \frac{\pi}{3} + 3 \\ &= \frac{13}{2} \cos \theta - \frac{3\sqrt{3}}{2} \sin \theta + 3 \end{aligned}$$

$$\begin{aligned} \text{Now, } -\sqrt{\left(\frac{13}{2} \right)^2 + \left(\frac{3\sqrt{3}}{2} \right)^2} &\leq \frac{13}{2} \cos \theta - \frac{3\sqrt{3}}{2} \sin \theta \\ &\leq \sqrt{\left(\frac{13}{2} \right)^2 + \left(\frac{3\sqrt{3}}{2} \right)^2} \end{aligned}$$

$$\Rightarrow -7 \leq \frac{13}{2} \cos \theta - \frac{3\sqrt{3}}{2} \sin \theta \leq 7$$

$$\Rightarrow -4 \leq \frac{13}{2} \cos \theta - \frac{3\sqrt{3}}{2} \sin \theta + 3 \leq 10$$

ILLUSTRATION 3.15

Find the maximum vertical distance between the graphs of $y = 2 + 3 \sin x$ and $y = 4 \cos x - 3$.

Sol. We have $y_1 = 2 + 3 \sin x$ and $y_2 = 4 \cos x - 3$

$$\therefore y_1 - y_2 = 3 \sin x - 4 \cos x + 5$$

$$\therefore (y_1 - y_2)_{\max} = \sqrt{3^2 + (-4)^2} + 5 = 10$$

So, maximum vertical distance between the graphs is 10.

ILLUSTRATION 3.16

Find the range of the expression $27^{\cos 2x} 81^{\sin 2x}$.

Sol. Let

$$\begin{aligned} y &= 27^{\cos 2x} \times 81^{\sin 2x} \\ &= 3^{3 \cos 2x} \times 3^{4 \sin 2x} \\ &= 3^{3 \cos 2x + 4 \sin 2x} \end{aligned}$$

$$\text{Now, } -\sqrt{3^2 + 4^2} \leq 3 \cos 2x + 4 \sin 2x \leq \sqrt{3^2 + 4^2}$$

$$\therefore -5 \leq 3 \cos 2x + 4 \sin 2x \leq 5$$

$$\Rightarrow 3^{-5} \leq 3^{3 \cos 2x + 4 \sin 2x} \leq 3^5$$

ILLUSTRATION 3.17

Find the range of $f(x) = \frac{1}{(\cos x - 3)^2 + (\sin x + 4)^2}$

$$\text{Sol. } f(x) = \frac{1}{(\cos x - 3)^2 + (\sin x + 4)^2}$$

$$= \frac{1}{\cos^2 x - 6 \cos x + 9 + \sin^2 x + 8 \sin x + 16}$$

$$= \frac{1}{8 \sin x - 6 \cos x + 26}$$

Now $8 \sin x - 6 \cos x \in \left[-\sqrt{(8)^2 + (-6)^2}, \sqrt{(8)^2 + (-6)^2} \right]$

$$\therefore 8 \sin x - 6 \cos x \in [-10, 10]$$

$$\therefore 8 \sin x - 6 \cos x + 26 \in [16, 36]$$

$$\therefore \frac{1}{36} \leq \frac{1}{(\cos x - 3)^2 + (\sin x + 4)^2} \leq \frac{1}{16}$$

Thus range of $f(x)$ is $\left[\frac{1}{36}, \frac{1}{16} \right]$.

ILLUSTRATION 3.18

Find the range of function $f(x) = \sin\left(x + \frac{\pi}{6}\right) + \cos\left(x - \frac{\pi}{6}\right)$.

Sol. $f(x) = \sin\left(x + \frac{\pi}{6}\right) + \cos\left(x - \frac{\pi}{6}\right)$
 $= \sin x \cos \frac{\pi}{6} + \cos x \sin \frac{\pi}{6} + \cos x \cos \frac{\pi}{6} + \sin x \sin \frac{\pi}{6}$
 $= \frac{\sqrt{3}+1}{2} (\sin x + \cos x)$

Now, $\sin x + \cos x \in [-\sqrt{2}, \sqrt{2}]$

$$\therefore f(x) = \frac{\sqrt{3}+1}{2} (\sin x + \cos x) \in \left[-\sqrt{2} \times \frac{\sqrt{3}+1}{2}, \sqrt{2} \times \frac{\sqrt{3}+1}{2} \right]$$

or $f(x) \in \left[-\frac{\sqrt{3}+1}{\sqrt{2}}, \frac{\sqrt{3}+1}{\sqrt{2}} \right]$

ILLUSTRATION 3.19

If $\sin^2(\theta - \alpha) \cos \alpha = \cos^2(\theta - \alpha) \sin \alpha = m \sin \alpha \cos \alpha$, then prove that $|m| \geq \frac{1}{\sqrt{2}}$.

Sol. $\sin^2(\theta - \alpha) \cos \alpha = \cos^2(\theta - \alpha) \sin \alpha = m \sin \alpha \cos \alpha$

or $\sin^2(\theta - \alpha) = m \sin \alpha$

$\cos^2(\theta - \alpha) = m \cos \alpha$

Adding, we get

$$1 = m(\sin \alpha + \cos \alpha)$$

or $\sin \alpha + \cos \alpha = \frac{1}{m}$

or $\sin\left(\alpha + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}m}$

since $\left| \sin\left(\alpha + \frac{\pi}{4}\right) \right| \leq 1$

$$\therefore \left| \frac{1}{m\sqrt{2}} \right| \leq 1 \quad \text{or} \quad |m| \geq \frac{1}{\sqrt{2}}$$

ILLUSTRATION 3.20

In $\triangle ABC$, if $\sqrt{3} \sin C = 2 \sec A - \tan A$, then prove that triangle is right angled.

Sol. We have $\sqrt{3} \sin C = 2 \sec A - \tan A$

$$\therefore (\sqrt{3} \sin C) \cos A + \sin A = 2$$

Now, $(\sqrt{3} \sin C) \cos A + \sin A \leq \sqrt{(\sqrt{3} \sin C)^2 + 1}$

$$\Rightarrow 2 \leq \sqrt{3 \sin^2 C + 1}$$

$$\Rightarrow \sin^2 C \geq 1$$

$$\Rightarrow \sin C = 1$$

$$\Rightarrow \angle C = \frac{\pi}{2}$$

CONCEPT APPLICATION EXERCISE 3.1

- In $\triangle ABC$, if $\cos A + \sin A - \frac{2}{\cos B + \sin B} = 0$ then prove that triangle is isosceles right angled.
- If x is A.M. of $\tan \pi/9$ and $\tan 5\pi/18$ and y is A.M. of $\tan \pi/9$ and $\tan 7\pi/18$, then relate x and y .
- Find the value of $\cos \frac{\pi}{12} \left(\sin \frac{5\pi}{12} + \cos \frac{\pi}{4} \right) + \sin \frac{\pi}{12} \left(\cos \frac{5\pi}{12} - \sin \frac{\pi}{4} \right)$.
- If $\cos(\alpha + \beta) + \sin(\alpha - \beta) = 0$ and $\tan \beta \neq 1$, then find the value of $\tan \alpha$.
- If $\sin A + \cos 2A = 1/2$ and $\cos A + \sin 2A = 1/3$, then find the value of $\sin 3A$.
- If $\sin x + \sin y + \sin z = 0 = \cos x + \cos y + \cos z$, then find the value of $\cos(\theta - x) + \cos(\theta - y) + \cos(\theta - z)$.
- In a triangle ABC , if $\sin A \sin(B - C) = \sin C \sin(A - B)$, then prove that $\cot A, \cot B, \cot C$ are in A.P.
- Find the value of
$$\frac{(\cos 1^\circ + \sin 1^\circ)(\cos 2^\circ + \sin 2^\circ)(\cos 3^\circ + \sin 3^\circ) \dots (\cos 45^\circ + \sin 45^\circ)}{\cos 1^\circ \cos 2^\circ \cos 3^\circ \dots \cos 45^\circ}$$
- Find the maximum value of $\sqrt{3} \sin x + \cos x$ and corresponding value of x for which a maximum value occurs.
- Find the maximum value of $1 + \sin\left(\frac{\pi}{4} + \theta\right) + 2 \sin\left(\frac{\pi}{4} - \theta\right)$ for all real values of θ .
- Show that $2^{\sin x} + 2^{\cos x} \geq 2^{1 - \frac{1}{\sqrt{2}}}$.

ANSWERS

- $y = 2x$
- $3/2$
- -1
- $-59/72$
- 0
- 2^{23}
- Max. value = 2 when $x = 60^\circ$
- $1 + \sqrt{5}$

COMPOUND ANGLE FORMULAS FOR TANGENT

We have following formulas for tangent of compound angles.

$$1. \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\begin{aligned} \text{Proof: } \tan(A+B) &= \frac{\sin(A+B)}{\cos(A+B)} \\ &= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B} \\ &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \end{aligned}$$

[On dividing the numerator and denominator by $\cos A \cos B$]

$$2. \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\begin{aligned} \text{Proof: } \tan(A-B) &= \tan(A+(-B)) \\ &= \frac{\tan A + \tan(-B)}{1 - \tan A \tan(-B)} \\ &= \frac{\tan A - \tan B}{1 + \tan A \tan B} \end{aligned}$$

Similarly, it can be proved that:

$$\cot(A+B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}$$

$$\text{and } \cot(A-B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$$

$$3. \tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$$

$$\begin{aligned} \text{Proof: } \tan(A+B+C) &= \tan((A+B)+C) \\ &= \frac{\tan(A+B) + \tan C}{1 - \tan(A+B) \tan C} \\ &= \frac{\frac{\tan A + \tan B}{1 - \tan A \tan B} + \tan C}{1 - \left(\frac{\tan A + \tan B}{1 - \tan A \tan B} \right) \tan C} \\ &= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A} \end{aligned}$$

If $A+B+C = n\pi$, $n \in \mathbb{Z}$ then $\tan(A+B+C) = 0$.

So, $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

If $A+B+C = \pi$ then $\frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2}$

Therefore, $\tan\left(\frac{A}{2} + \frac{B}{2} + \frac{C}{2}\right)$ approaches to infinity.

$$\text{So, } \tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} = 1$$

In general,

$$\tan(A_1 + A_2 + \dots + A_n) = \frac{S_1 - S_3 + S_5 - S_7 + \dots}{1 - S_2 + S_4 - S_6 + \dots}$$

where,

$$S_1 = \tan A_1 + \tan A_2 + \dots + \tan A_n$$

= Sum of the tangents of angles,

$$S_2 = \tan A_1 \tan A_2 + \tan A_1 \tan A_3 + \dots$$

= Sum of the product of tangents taken two at a time,

$$S_3 = \tan A_1 \tan A_2 \tan A_3 + \tan A_2 \tan A_3 \tan A_4 + \dots$$

= Sum of the product of tangents taken three at a time, and so on.

ILLUSTRATION 3.21

If $\tan \alpha = \frac{m}{m+1}$ and $\tan \beta = \frac{1}{2m+1}$, find the least possible value of $(\alpha + \beta)$.

Sol. We have

$$\begin{aligned} \tan(\alpha + \beta) &= \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \\ &= \frac{\frac{m}{m+1} + \frac{1}{2m+1}}{1 - \frac{m}{m+1} \times \frac{1}{2m+1}} \\ &= \frac{2m^2 + 2m + 1}{2m^2 + 2m + 1} = 1 \end{aligned}$$

$$\Rightarrow (\alpha + \beta)_{\text{least}} = \frac{\pi}{4}$$

ILLUSTRATION 3.22

If $\sin(A-B) = \frac{1}{\sqrt{10}}$, $\cos(A+B) = \frac{2}{\sqrt{29}}$, find the value of $\tan 2A$ where A and B lie between 0 and $\pi/4$.

$$\begin{aligned} \text{Sol. } \tan 2A &= \tan[(A+B) + (A-B)] \\ &= \frac{\tan(A+B) + \tan(A-B)}{1 - \tan(A+B) \tan(A-B)} \end{aligned}$$

Given that, $0 < A < \pi/4$ and $0 < B < \pi/4$. Therefore,

$$0 < A+B < \frac{\pi}{2}$$

$$\text{Also, } -\frac{\pi}{4} < A-B < \frac{\pi}{4}$$

$$\text{and } \sin(A-B) = \frac{1}{\sqrt{10}}$$

$$\therefore 0 < A-B < \frac{\pi}{4}$$

$$\text{Now, } \sin(A-B) = \frac{1}{\sqrt{10}}$$

$$\Rightarrow \tan(A-B) = \frac{1}{3}$$

$$\cos(A+B) = \frac{2}{\sqrt{29}}$$

$$\Rightarrow \tan(A+B) = \frac{5}{2} \quad \text{(iii)}$$

From Eqs. (i), (ii), and (iii), we get

$$\tan 2A = \frac{\frac{5}{2} + \frac{1}{3}}{1 - \frac{5}{2} \times \frac{1}{3}} = \frac{17}{6} \times \frac{6}{1} = 17$$

ILLUSTRATION 3.23

Prove that $\frac{\cos 10^\circ + \sin 10^\circ}{\cos 10^\circ - \sin 10^\circ} = \tan 55^\circ$.

Sol.
$$\frac{\cos 10^\circ + \sin 10^\circ}{\cos 10^\circ - \sin 10^\circ} = \frac{1 + \tan 10^\circ}{1 - \tan 10^\circ}$$

$$= \frac{\tan 45^\circ + \tan 10^\circ}{1 - \tan 45^\circ \tan 10^\circ}$$

(dividing by $\cos 10^\circ$)

$$= \tan(45^\circ + 10^\circ) = \tan 55^\circ$$

ILLUSTRATION 3.24

Prove that $\tan 70^\circ = 2 \tan 50^\circ + \tan 20^\circ$.

Sol.
$$\tan 70^\circ = \tan(50^\circ + 20^\circ)$$

$$= \frac{\tan 50^\circ + \tan 20^\circ}{1 - \tan 50^\circ \tan 20^\circ} \quad \text{(i)}$$

or $\tan 70^\circ (1 - \tan 50^\circ \tan 20^\circ) = \tan 50^\circ + \tan 20^\circ$

or $\tan 70^\circ - \tan 50^\circ \tan 20^\circ \tan 70^\circ = \tan 50^\circ + \tan 20^\circ$

or $\tan 70^\circ = \tan 70^\circ \tan 50^\circ \tan 20^\circ + \tan 50^\circ + \tan 20^\circ$

$$= \tan(90^\circ - 20^\circ) \tan 50^\circ \tan 20^\circ + \tan 50^\circ + \tan 20^\circ$$

$$= \cot 20^\circ \tan 50^\circ \tan 20^\circ + \tan 50^\circ + \tan 20^\circ$$

$$= \tan 50^\circ + \tan 50^\circ + \tan 20^\circ$$

$$= 2 \tan 50^\circ + \tan 20^\circ$$

ILLUSTRATION 3.25

Find the value of

$$\frac{\cot 25^\circ + \cot 55^\circ}{\tan 25^\circ + \tan 55^\circ} + \frac{\cot 55^\circ + \cot 100^\circ}{\tan 55^\circ + \tan 100^\circ} + \frac{\cot 100^\circ + \cot 25^\circ}{\tan 100^\circ + \tan 25^\circ}$$

Sol.

$$E = \frac{\cot 25^\circ + \cot 55^\circ}{\tan 25^\circ + \tan 55^\circ} + \frac{\cot 55^\circ + \cot 100^\circ}{\tan 55^\circ + \tan 100^\circ} + \frac{\cot 100^\circ + \cot 25^\circ}{\tan 100^\circ + \tan 25^\circ}$$

$$= \frac{1}{\tan 55^\circ \tan 100^\circ} + \frac{1}{\tan 55^\circ \tan 100^\circ} + \frac{1}{\tan 100^\circ \tan 25^\circ}$$

$$= \frac{\tan 25^\circ + \tan 55^\circ + \tan 100^\circ}{\tan 25^\circ \cdot \tan 55^\circ \cdot \tan 100^\circ}$$

Since $25^\circ + 55^\circ + 100^\circ = 180^\circ$

$$\tan 25^\circ + \tan 55^\circ + \tan 100^\circ = \tan 25^\circ \tan 55^\circ \tan 100^\circ$$

$$\Rightarrow E = 1$$

ILLUSTRATION 3.26

Prove that $(1 + \tan 1^\circ)(1 + \tan 2^\circ) \dots (1 + \tan 45^\circ) = 2^{23}$.

Sol.
$$(1 + \tan x^\circ)(1 + \tan(45^\circ - x^\circ))$$

$$= (1 + \tan x^\circ) \left(1 + \frac{1 - \tan x^\circ}{1 + \tan x^\circ} \right) = 2$$

$$\therefore (1 + \tan 1^\circ)(1 + \tan 44^\circ) = (1 + \tan 2^\circ)(1 + \tan 43^\circ)$$

$$= (1 + \tan 3^\circ)(1 + \tan 42^\circ)$$

$$\dots$$

$$\dots$$

$$= (1 + \tan 22^\circ)(1 + \tan 23^\circ)$$

$$= 2$$

$$(1 + \tan 1^\circ)(1 + \tan 2^\circ) \dots (1 + \tan 45^\circ) = 2^{23}$$

(as $1 + \tan 45^\circ = 2$)

ILLUSTRATION 3.27

If $A = \frac{\pi}{5}$, then find the value of $\sum_{r=1}^8 \tan(rA) \cdot \tan((r+1)A)$.

Sol.
$$\tan((r+1)A - (rA)) = \frac{\tan(r+1)A - \tan(rA)}{1 + \tan(r+1)A \cdot \tan(rA)}$$

$$\Rightarrow S = \sum_{r=1}^8 \tan(rA) \cdot \tan(r+1)A$$

$$= \sum_{r=1}^8 (-1) + \frac{1}{\tan A} \sum_{r=1}^8 (\tan(r+1)A - \tan(rA))$$

$$= -8 + \frac{1}{\tan A} \cdot (\tan 9A - \tan A)$$

Now $\tan 9A = \tan \frac{9\pi}{5} = \tan \left(2\pi - \frac{\pi}{5} \right) = -\tan \frac{\pi}{5}$

$$\Rightarrow S = -8 + \frac{1}{\tan A} (-2 \tan A) = -8 - 2 = -10$$

ILLUSTRATION 3.28

In $\triangle ABC$, if $\angle A = \frac{\pi}{4}$, then find all possible values of $\tan B \tan C$.

Sol. $A + B + C = \pi$

$$\angle A = \frac{\pi}{4} \Rightarrow B + C = \frac{3\pi}{4}$$

Let $\tan B \tan C = x$

$$\Rightarrow \tan B \tan \left(\frac{3\pi}{4} - B \right) = x$$

$$\Rightarrow \tan B \cdot \frac{\tan \frac{3\pi}{4} - \tan B}{1 + \tan \frac{3\pi}{4} \tan B} = x$$

$$\Rightarrow \frac{-\tan B - \tan^2 B}{1 - \tan B} = x$$

$$\Rightarrow \tan^2 B + (1-x) \tan B + x = 0$$

Now, $\tan B$ is real.

(1)

3.8 Trigonometry

- $\therefore$ Discriminant of Eq. (1) ≥ 0
 $\therefore (1-x)^2 - 4x \geq 0$
 $\therefore x^2 - 6x + 1 \geq 0$
 $\therefore [x - (3 - 2\sqrt{2})][x - (3 + 2\sqrt{2})] \geq 0$
 $\therefore x \in (-\infty, 3 - 2\sqrt{2}] \cup [3 + 2\sqrt{2}, \infty)$

ILLUSTRATION 3.29

If $\tan^3 A + \tan^3 B + \tan^3 C = 3 \tan A \cdot \tan B \cdot \tan C$, then prove that triangle ABC is an equilateral triangle.

Sol. $\tan^3 A + \tan^3 B + \tan^3 C = 3 \tan A \cdot \tan B \cdot \tan C$
 $\Rightarrow (\tan A + \tan B + \tan C) \cdot ((\tan A - \tan B)^2 + (\tan B - \tan C)^2 + (\tan C - \tan A)^2) = 0$

In $\triangle ABC$

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C$$

So, $\tan A + \tan B + \tan C \neq 0$

$\therefore \tan A = \tan B = \tan C$

$\Rightarrow A = B = C = 60^\circ$

CONCEPT APPLICATION EXERCISE 3.2

- If $A + B = 225^\circ$, then find the value of $\frac{\cot A}{1 + \cot A} \times \frac{\cot B}{1 + \cot B}$.
- If $\tan A - \tan B = x$ and $\cot B - \cot A = y$, then find the value of $\cot(A - B)$.
- Prove that $\frac{\tan^2 2\theta - \tan^2 \theta}{1 - \tan^2 2\theta \tan^2 \theta} = \tan 3\theta \tan \theta$.
- If $A + B = 45^\circ$, show that $(1 + \tan A)(1 + \tan B) = 2$.
- If $\tan A = 1/2$, $\tan B = 1/3$, then prove that $\cos 2A = \sin 2B$.
- If $P + Q = \frac{7\pi}{6}$, then find the value of $(\sqrt{2} + \tan P) \times (\sqrt{3} + \tan Q)$.
- If $\tan \beta = \frac{n \sin \alpha \cos \alpha}{1 - n \sin^2 \alpha}$, prove that $\tan(\alpha - \beta) = (1 - n) \times \tan \alpha$.

ANSWERS

1. $1/2$ 2. $\frac{1}{x} + \frac{1}{y}$ 6. 4

TRANSFORMATION FORMULA

FORMULA TO TRANSFORM THE PRODUCT INTO SUM OR DIFFERENCE

We know that

$$\sin A \cos B + \cos A \sin B = \sin(A + B) \quad (i)$$

$$\sin A \cos B - \cos A \sin B = \sin(A - B) \quad (ii)$$

$$\cos A \cos B - \sin A \sin B = \cos(A + B) \quad (iii)$$

$$\cos A \cos B + \sin A \sin B = \cos(A - B) \quad (iv)$$

Adding Eqs. (i) and (ii), we obtain

$$2 \sin A \cos B = \sin(A + B) + \sin(A - B) \quad (v)$$

Subtracting Eq. (ii) from (i), we get

$$2 \cos A \sin B = \sin(A + B) - \sin(A - B) \quad (vi)$$

Adding Eq. (iii) and (iv), we get

$$2 \cos A \cos B = \cos(A + B) + \cos(A - B)$$

Subtracting Eq. (iii) from (iv), we get

$$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$$

Above four formulas are used to convert product of two sines and cosines into the sum or difference of two sines and cosines.

FORMULAS TO TRANSFORM THE SUM OR DIFFERENCE INTO PRODUCT

Let $A + B = C$ and $A - B = D$. Then, $A = \frac{C + D}{2}$ and $B = \frac{C - D}{2}$

Substituting the values of A , B , C , and D in Eqs. (v), (vi), (vii), and (viii), we get

$$\sin C + \sin D = 2 \sin\left(\frac{C + D}{2}\right) \cos\left(\frac{C - D}{2}\right)$$

$$\sin C - \sin D = 2 \sin\left(\frac{C - D}{2}\right) \cos\left(\frac{C + D}{2}\right)$$

$$\cos C + \cos D = 2 \cos\left(\frac{C + D}{2}\right) \cos\left(\frac{C - D}{2}\right)$$

$$\cos D - \cos C = 2 \sin\left(\frac{C + D}{2}\right) \sin\left(\frac{C - D}{2}\right)$$

or $\cos C - \cos D = -2 \sin\left(\frac{C + D}{2}\right) \sin\left(\frac{C - D}{2}\right)$

or $\cos C - \cos D = 2 \sin\left(\frac{C + D}{2}\right) \sin\left(\frac{D - C}{2}\right)$

These four formulas are used to convert the sum or difference of two sines or two cosines into the product of sines and cosines.

ILLUSTRATION 3.30

If $\sin A = \sin B$ and $\cos A = \cos B$, then prove that $\sin \frac{A - B}{2} = 0$

Sol. We have

$$\sin A = \sin B \text{ and } \cos A = \cos B$$

or $\sin A - \sin B = 0$ and $\cos A - \cos B = 0$

or $2 \sin\left(\frac{A - B}{2}\right) \cos\left(\frac{A + B}{2}\right) = 0$

and $-2 \sin\left(\frac{A - B}{2}\right) \sin\left(\frac{A + B}{2}\right) = 0$

or $\sin \frac{A - B}{2} = 0$, which is common for both the equations

ILLUSTRATION 3.31

Prove that $\cos 55^\circ + \cos 65^\circ + \cos 175^\circ = 0$.

Sol. L.H.S. = $\cos 55^\circ + \cos 65^\circ + \cos 175^\circ$

$$= 2 \cos \frac{55^\circ + 65^\circ}{2} \cos \frac{55^\circ - 65^\circ}{2} + \cos 175^\circ$$

$$= 2 \cos 60^\circ \cos(-5^\circ) + \cos 175^\circ$$

$$= 2 \times \frac{1}{2} \cos 5^\circ + \cos(180^\circ - 5^\circ)$$

$$= \cos 5^\circ - \cos 5^\circ = 0$$

ILLUSTRATION 3.32

Prove that $\cos 18^\circ - \sin 18^\circ = \sqrt{2} \sin 27^\circ$.

Sol. L.H.S. = $\cos 18^\circ - \sin 18^\circ$
 $= \cos 18^\circ - \sin(90^\circ - 72^\circ) = \cos 18^\circ - \cos 72^\circ$
 $= 2 \sin \frac{18^\circ + 72^\circ}{2} \sin \frac{72^\circ - 18^\circ}{2}$
 $= 2 \sin 45^\circ \sin 27^\circ$
 $= 2 \frac{1}{\sqrt{2}} \sin 27^\circ$
 $= \sqrt{2} \sin 27^\circ$

ILLUSTRATION 3.33

Prove that

(i) $\frac{\sin 5A - \sin 3A}{\cos 5A + \cos 3A} = \tan A$ (ii) $\frac{\sin A + \sin 3A}{\cos A + \cos 3A} = \tan 2A$

Sol.

(i) L.H.S. = $\frac{\sin 5A - \sin 3A}{\cos 5A + \cos 3A}$
 $= \frac{2 \sin \left(\frac{5A - 3A}{2} \right) \cos \left(\frac{5A + 3A}{2} \right)}{2 \cos \left(\frac{5A + 3A}{2} \right) \cos \left(\frac{5A - 3A}{2} \right)}$
 $= \frac{2 \sin A \cos 4A}{2 \cos 4A \cos A} = \tan A = \text{R.H.S.}$

(ii) L.H.S. = $\frac{\sin 3A + \sin A}{\cos 3A + \cos A}$
 $= \frac{2 \sin \left(\frac{3A + A}{2} \right) \cos \left(\frac{3A - A}{2} \right)}{2 \cos \left(\frac{3A + A}{2} \right) \cos \left(\frac{3A - A}{2} \right)}$
 $= \frac{\sin 2A \cos A}{\cos 2A \cos A} = \tan 2A = \text{R.H.S.}$

ILLUSTRATION 3.34

Prove that $\frac{\sin A + \sin 2A + \sin 4A + \sin 5A}{\cos A + \cos 2A + \cos 4A + \cos 5A} = \tan 3A$.

Sol.

$$\begin{aligned} & \frac{\sin A + \sin 2A + \sin 4A + \sin 5A}{\cos A + \cos 2A + \cos 4A + \cos 5A} \\ &= \frac{(\sin 5A + \sin A) + (\sin 4A + \sin 2A)}{(\cos 5A + \cos A) + (\cos 4A + \cos 2A)} \\ &= \frac{2 \sin 3A \cos 2A + 2 \sin 3A \cos A}{2 \cos 3A \cos 2A + 2 \cos 3A \cos A} \\ &= \frac{2 \sin 3A (\cos 2A + \cos A)}{2 \cos 3A (\cos 2A + \cos A)} = \tan 3A \end{aligned}$$

ILLUSTRATION 3.35

Prove that $\cos \alpha + \cos \beta + \cos \gamma + \cos(\alpha + \beta + \gamma)$

$$= 4 \cos \frac{\alpha + \beta}{2} \cos \frac{\beta + \gamma}{2} \cos \frac{\gamma + \alpha}{2}.$$

Sol. L.H.S. = $\cos \alpha + \cos \beta + \cos \gamma + \cos(\alpha + \beta + \gamma)$

$$\begin{aligned} &= (\cos \alpha + \cos \beta) + [\cos \gamma + \cos(\alpha + \beta + \gamma)] \\ &= 2 \cos \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right) \\ &\quad + 2 \cos \left(\frac{\alpha + \beta + \gamma}{2} \right) \cos \left(\frac{\alpha + \beta + \gamma - \gamma}{2} \right) \\ &= 2 \cos \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right) \\ &\quad + 2 \cos \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha + \beta + 2\gamma}{2} \right) \\ &= 2 \cos \left(\frac{\alpha + \beta}{2} \right) \left\{ \cos \left(\frac{\alpha - \beta}{2} \right) + \cos \left(\frac{\alpha + \beta + 2\gamma}{2} \right) \right\} \\ &= 2 \cos \left(\frac{\alpha + \beta}{2} \right) \left\{ 2 \cos \left(\frac{\frac{\alpha - \beta}{2} + \frac{\alpha + \beta + 2\gamma}{2}}{2} \right) \right. \\ &\quad \left. \cos \left(\frac{\frac{\alpha - \beta}{2} - \frac{\alpha + \beta + 2\gamma}{2}}{2} \right) \right\} \\ &= 2 \cos \left(\frac{\alpha + \beta}{2} \right) \left\{ 2 \cos \left(\frac{\alpha + \gamma}{2} \right) \cos \left(\frac{\beta + \gamma}{2} \right) \right\} \\ &= 4 \cos \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\beta + \gamma}{2} \right) \cos \left(\frac{\gamma + \alpha}{2} \right) = \text{R.H.S.} \end{aligned}$$

ILLUSTRATION 3.36

Prove that $\left(\frac{\cos A + \cos B}{\sin A - \sin B} \right)^n + \left(\frac{\sin A + \sin B}{\cos A - \cos B} \right)^n$
 $= 2 \cot^n \frac{A-B}{2}$ or 0, accordingly as n is even or odd.

Sol. L.H.S. = $\left(\frac{2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}}{2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}} \right)^n$
 $+ \left(\frac{2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}}{2 \sin \frac{A+B}{2} \sin \frac{B-A}{2}} \right)^n$

$$\begin{aligned}
 &= \left(\cot \frac{A-B}{2} \right)^n + \left(-\cot \frac{A-B}{2} \right)^n \\
 &\quad [\because \sin(-\theta) = -\sin \theta] \\
 &= \cot^n \frac{A-B}{2} + (-1)^n \cot^n \frac{A-B}{2} \\
 &= \cot^n \frac{A-B}{2} [1 + (-1)^n] \\
 &= \begin{cases} 0, & \text{if } n \text{ is odd} \\ 2 \cot^n \frac{A-B}{2}, & \text{if } n \text{ is even} \end{cases}
 \end{aligned}$$

ILLUSTRATION 3.37

Prove that $(\cos \alpha + \cos \beta)^2 + (\sin \alpha + \sin \beta)^2 = 4 \cos^2 \left(\frac{\alpha - \beta}{2} \right)$.

Sol. L.H.S. = $(\cos \alpha + \cos \beta)^2 + (\sin \alpha + \sin \beta)^2$

$$\begin{aligned}
 &= \left\{ 2 \cos \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right) \right\}^2 + \\
 &\quad \left\{ 2 \sin \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right) \right\}^2 \\
 &= 4 \cos^2 \left(\frac{\alpha - \beta}{2} \right) \left\{ \cos^2 \frac{\alpha + \beta}{2} + \sin^2 \frac{\alpha + \beta}{2} \right\} \\
 &= 4 \cos^2 \left(\frac{\alpha - \beta}{2} \right) = \text{R.H.S.}
 \end{aligned}$$

ILLUSTRATION 3.38

In quadrilateral $ABCD$, if

$$\sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) + \sin \left(\frac{C+D}{2} \right) \cos \left(\frac{C-D}{2} \right) = 2,$$

then find the value of $\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \sin \frac{D}{2}$.

Sol. $\sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) + \sin \left(\frac{C+D}{2} \right) \cos \left(\frac{C-D}{2} \right) = 2$

or $\frac{1}{2} [\sin A + \sin B + \sin C + \sin D] = 2$

or $\sin A + \sin B + \sin C + \sin D = 4$

$\Rightarrow A = B = C = D = 90^\circ$

$\Rightarrow \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \sin \frac{D}{2} = \frac{1}{4}$

ILLUSTRATION 3.39

In $\triangle ABC$, $\sin C + \cos C + \sin(2B + C) - \cos(2B + C) = 2\sqrt{2}$.
Prove that $\triangle ABC$ is right-angled isosceles.

Sol. $\sin C + \cos C + \sin(2B + C) - \cos(2B + C) = 2\sqrt{2}$
 $\Rightarrow \sin C + \sin(2B + C) + \cos C - \cos(2B + C) = 2\sqrt{2}$
 $\Rightarrow 2 \sin(B + C) \cos B + 2 \sin(B + C) \sin B = 2\sqrt{2}$
 $\Rightarrow 2 \sin A (\cos B + \sin B) = 2\sqrt{2}$ [$\because \sin(B + C) = \sin A$]

$$\Rightarrow 2\sqrt{2} \sin A \cos \left(\frac{\pi}{4} - B \right) = 2\sqrt{2}$$

$$\Rightarrow \sin A \cos \left(B - \frac{\pi}{4} \right) = 1$$

$$\Rightarrow \sin A = 1 \text{ and } \cos \left(B - \frac{\pi}{4} \right) = 1$$

$$\Rightarrow A = \frac{\pi}{2} \text{ and } B = \frac{\pi}{4}.$$

Hence, $C = \frac{\pi}{4}$

ILLUSTRATION 3.40

If α and β are acute angles such that $\alpha + \beta = \lambda$, where λ is constant, find the maximum possible value of the expression $\sin \alpha + \sin \beta + \cos \alpha + \cos \beta$.

Sol. $\sin \alpha + \sin \beta + \cos \alpha + \cos \beta$
 $= 2 \sin \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right) + 2 \cos \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)$
 $= 2 \cos \left(\frac{\alpha - \beta}{2} \right) \left[\sin \frac{\lambda}{2} + \cos \frac{\lambda}{2} \right] \leq 2 \left[\sin \frac{\lambda}{2} + \cos \frac{\lambda}{2} \right]$
 $= 2\sqrt{2} \sin \left(\frac{\lambda}{2} + \frac{\pi}{4} \right)$

This is the maximum value of expression.

ILLUSTRATION 3.41

Prove that

$$\sum_{r=1}^n \left(\frac{1}{\cos \theta + \cos(2r+1)\theta} \right) = \frac{\sin n\theta}{2 \sin \theta \cdot \cos \theta \cos(n+1)\theta},$$

(where $n \in \mathbb{N}$)

Sol. $S = \sum_{r=1}^n \left(\frac{1}{\cos \theta + \cos(2r+1)\theta} \right)$
 $= \sum_{r=1}^n \left(\frac{\sin \theta}{2 \cos(r+1)\theta \cos r\theta \sin \theta} \right)$
 $= \frac{1}{2 \sin \theta} \left(\sum_{r=1}^n \frac{\sin((r+1)\theta - r\theta)}{\cos(r+1)\theta \cos r\theta} \right)$
 $= \frac{1}{2 \sin \theta} \left(\sum_{r=1}^n \frac{\sin(r+1)\theta \cos r\theta - \sin r\theta \cos(r+1)\theta}{\cos(r+1)\theta \cos r\theta} \right)$
 $= \frac{1}{2 \sin \theta} \left(\sum_{r=1}^n (\tan(r+1)\theta - \tan r\theta) \right)$
 $= \frac{1}{2 \sin \theta} (\tan(n+1)\theta - \tan \theta)$
 $= \frac{\sin n\theta}{2 \sin \theta \cdot \cos \theta \cos(n+1)\theta}$

CONCEPT APPLICATION EXERCISE 3.3

- (a) Prove that $\sin 65^\circ + \cos 65^\circ = \sqrt{2} \cos 20^\circ$.
(b) Prove that $\sin 47^\circ + \cos 77^\circ = \cos 17^\circ$.
- Prove that $\cos 80^\circ + \cos 40^\circ - \cos 20^\circ = 0$.
- Prove that $\sin 10^\circ + \sin 20^\circ + \sin 40^\circ + \sin 50^\circ = \sin 70^\circ + \sin 80^\circ$.
- Prove that $\cos \frac{\pi}{5} + \cos \frac{2\pi}{5} + \cos \frac{6\pi}{5} + \cos \frac{7\pi}{5} = 0$.
- If $\sin \alpha - \sin \beta = 1/3$ and $\cos \beta - \cos \alpha = 1/2$, show that $\cot \frac{\alpha + \beta}{2} = \frac{2}{3}$.
- If $\operatorname{cosec} A + \sec A = \operatorname{cosec} B + \sec B$, prove that $\tan A \tan B = \cot \frac{A+B}{2}$.
- Prove that $\sin 25^\circ \cos 115^\circ = (1/2)(\sin 40^\circ - 1)$.
- If $x \cos \theta = y \cos \left(\theta + \frac{2\pi}{3} \right) = z \cos \left(\theta + \frac{4\pi}{3} \right)$, prove that $xy + yz + zx = 0$.
- If $y \sin \phi = x \sin(2\theta + \phi)$, show that $(x + y) \cot(\theta + \phi) = (y - x) \cot \theta$.
- If $\cos(A + B) \sin(C + D) = \cos(A - B) \sin(C - D)$, prove that $\cot A \cot B \cot C = \cot D$.
- If $\tan(A + B) = 3 \tan A$, prove that
(a) $\sin(2A + B) = 2 \sin B$
(b) $\sin 2(A + B) + \sin 2A = 4 \sin B \cos B$
- If $\frac{x}{y} = \frac{\cos A}{\cos B}$, then prove that $\frac{x \tan A + y \tan B}{x + y} = \tan \frac{A + B}{2}$.
- If $\frac{\cos 6x + 6 \cos 4x + 15 \cos 2x + 10}{\cos 5x + 5 \cos 3x + 10 \cos x} = 1$, then find the smallest positive value of x .

ANSWERS

13. $x = 60^\circ$

TRIGONOMETRIC RATIOS OF MULTIPLE ANGLES

FORMULAS FOR SINE AND COSINE OF MULTIPLE ANGLES

1. $\cos 2A = \cos^2 A - \sin^2 A$

Proof: We know that

$$\cos(A + B) = \cos A \cos B - \sin A \sin B.$$

Putting $B = A$, we get

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$= 2 \cos^2 A - 1$$

$$= 1 - 2 \sin^2 A$$

Also, $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$,

$$\cos^2 A = \frac{1}{2}(1 + \cos 2A)$$

Similarly, $\cos A = 2 \cos^2 \frac{A}{2} - 1$, $\cos \frac{A}{2} = 1 - 2 \sin^2 \frac{A}{4}$ etc.

2. $\sin 2A = 2 \sin A \cos A$

Proof: We know that

$$\sin(A + B) = \sin A \cos B + \sin B \cos A$$

Putting $B = A$, we get

$$\sin 2A = \sin A \cos A + \sin A \cos A = 2 \sin A \cos A$$

Similarly,

$$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}, \sin \frac{A}{2} = 2 \sin \frac{A}{4} \cos \frac{A}{4} \text{ etc.}$$

3. $\cos 3A = 4 \cos^3 A - 3 \cos A$

Proof:

$$\cos 3A = \cos(2A + A)$$

$$= \cos 2A \cos A - \sin 2A \sin A$$

$$= (2 \cos^2 A - 1) \cos A - 2 \sin A \cos A \sin A$$

$$= 2 \cos^3 A - \cos A - 2 \cos A (1 - \cos^2 A)$$

$$= 2 \cos^3 A - \cos A - 2 \cos A + 2 \cos^3 A$$

$$= 4 \cos^3 A - 3 \cos A$$

4. $\sin 3A = 3 \sin A - 4 \sin^3 A$

Proof:

$$\sin 3A = \sin(2A + A)$$

$$= \sin 2A \cos A + \cos 2A \sin A$$

$$= 2 \sin A \cos A \cos A + (1 - 2 \sin^2 A) \sin A$$

$$= 2 \sin A \cos^2 A + \sin A - 2 \sin^3 A$$

$$= 2 \sin A (1 - \sin^2 A) + \sin A - 2 \sin^3 A$$

$$= 2 \sin A - 2 \sin^3 A + \sin A - 2 \sin^3 A$$

$$= 3 \sin A - 4 \sin^3 A$$

5. (i) $\sin 2A = \frac{2 \tan A}{1 + \tan^2 A}$ (ii) $\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$

Proof:

$$\sin 2A = 2 \sin A \cos A = \frac{2 \sin A \cos A}{\cos^2 A + \sin^2 A} = \frac{2 \tan A}{1 + \tan^2 A}$$

[Dividing numerator and denominator by $\cos^2 A$]

$$\cos 2A = \cos^2 A - \sin^2 A = \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A} = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

[Dividing numerator and denominator by $\cos^2 A$]

Also, $\tan^2 A = \frac{1 - \cos 2A}{1 + \cos 2A}$

6. (i) $\cos nA = {}^nC_0 \cos^n A - {}^nC_2 \cos^{n-2} A \sin^2 A + {}^nC_4 \cos^{n-4} A \sin^4 A - \dots$
(ii) $\sin nA = {}^nC_1 \cos^{n-1} A \sin A - {}^nC_3 \cos^{n-3} A \sin^3 A + {}^nC_5 \cos^{n-5} A \sin^5 A - \dots$

Proof:For complex number $z = \cos A + i \sin A$, we have

$$\cos nA + i \sin nA = (\cos A + i \sin A)^n$$

Now, expanding R.H.S. and comparing real and imaginary parts of both sides, we get the above results.

ILLUSTRATION 3.42

Prove that

$$(i) \frac{\sin 2\theta}{1 + \cos 2\theta} = \tan \theta \quad (ii) \frac{\sin 2\theta}{1 - \cos 2\theta} = \cot \theta$$

$$(iii) \frac{1 + \sin 2\theta + \cos 2\theta}{1 + \sin 2\theta - \cos 2\theta} = \cot \theta$$

$$(iv) \frac{1 + \sin \theta - \cos \theta}{1 + \sin \theta + \cos \theta} = \tan \theta/2$$

$$(v) \frac{\cos 2\theta}{1 + \sin 2\theta} = \tan(\pi/4 - \theta)$$

$$(vi) \frac{\cos \theta}{1 + \sin \theta} = \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$

Sol.

$$(i) \text{ L.H.S.} = \frac{\sin 2\theta}{1 + \cos 2\theta} = \frac{2 \sin \theta \cos \theta}{2 \cos^2 \theta} = \tan \theta = \text{R.H.S.}$$

$$(ii) \text{ L.H.S.} = \frac{\sin 2\theta}{1 - \cos 2\theta} = \frac{2 \sin \theta \cos \theta}{2 \sin^2 \theta} = \cot \theta = \text{R.H.S.}$$

$$\begin{aligned} (iii) \text{ L.H.S.} &= \frac{1 + \sin 2\theta + \cos 2\theta}{1 + \sin 2\theta - \cos 2\theta} \\ &= \frac{(1 + \cos 2\theta) + \sin 2\theta}{(1 - \cos 2\theta) + \sin 2\theta} \\ &= \frac{2 \cos^2 \theta + 2 \sin \theta \cos \theta}{2 \sin^2 \theta + 2 \sin \theta \cos \theta} \\ &= \frac{2 \cos \theta (\cos \theta + \sin \theta)}{2 \sin \theta (\cos \theta + \sin \theta)} = \frac{\cos \theta}{\sin \theta} = \cot \theta = \text{R.H.S.} \end{aligned}$$

$$\begin{aligned} (iv) \text{ L.H.S.} &= \frac{1 + \sin \theta - \cos \theta}{1 + \sin \theta + \cos \theta} = \frac{(1 - \cos \theta) + \sin \theta}{(1 + \cos \theta) + \sin \theta} \\ &= \frac{2 \sin^2 \frac{\theta}{2} + 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \cos^2 \frac{\theta}{2} + 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} = \frac{2 \sin \frac{\theta}{2} \left(\sin \frac{\theta}{2} + \cos \frac{\theta}{2} \right)}{2 \cos \frac{\theta}{2} \left(\sin \frac{\theta}{2} + \cos \frac{\theta}{2} \right)} \\ &= \tan \frac{\theta}{2} = \text{R.H.S.} \end{aligned}$$

$$\begin{aligned} (v) \text{ L.H.S.} &= \frac{\cos 2\theta}{1 + \sin 2\theta} = \frac{\sin\left(\frac{\pi}{2} - 2\theta\right)}{1 + \cos\left(\frac{\pi}{2} - 2\theta\right)} \\ &= \frac{2 \sin\left(\frac{\pi}{4} - \theta\right) \cos\left(\frac{\pi}{4} - \theta\right)}{2 \cos^2\left(\frac{\pi}{4} - \theta\right)} \\ &= \tan\left(\frac{\pi}{4} - \theta\right) = \text{R.H.S.} \end{aligned}$$

$$\begin{aligned} (vi) \text{ L.H.S.} &= \frac{\cos \theta}{1 + \sin \theta} = \frac{\sin\left(\frac{\pi}{2} - \theta\right)}{1 + \cos\left(\frac{\pi}{2} - \theta\right)} \\ &= \frac{2 \sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)}{2 \cos^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right)} = \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \text{R.H.S.} \end{aligned}$$

ILLUSTRATION 3.43

$$\text{Prove that } \frac{1 + \sin 2\theta}{1 - \sin 2\theta} = \left(\frac{1 + \tan \theta}{1 - \tan \theta} \right)^2$$

$$\text{Sol. L.H.S.} = \frac{1 + \sin 2\theta}{1 - \sin 2\theta}$$

$$\begin{aligned} &= \frac{\sin^2 \theta + \cos^2 \theta + 2 \sin \theta \cos \theta}{\sin^2 \theta + \cos^2 \theta - 2 \sin \theta \cos \theta} \\ &= \left(\frac{\sin \theta + \cos \theta}{\sin \theta - \cos \theta} \right)^2 \\ &= \left(\frac{1 + \tan \theta}{1 - \tan \theta} \right)^2 \end{aligned}$$

(Dividing numerator and denominator by $\cos \theta$)**ILLUSTRATION 3.44**If $\alpha + \beta = 90^\circ$, find the maximum value of $\sin \alpha \sin \beta$.**Sol.** Let

$$\begin{aligned} y &= \sin \alpha \sin \beta \\ &= \sin \alpha \sin(90^\circ - \alpha) \\ &= \sin \alpha \cos \alpha = (1/2) \sin 2\alpha \end{aligned}$$

which has the maximum value $1/2$ when $\sin 2\alpha = 1$.**ILLUSTRATION 3.45**If $\sin A = 3/5$ and $0^\circ < A < 90^\circ$, find the values of $\sin 2A$, $\cos 2A$, $\tan 2A$, and $\sin 4A$.**Sol.** Given $\sin A = 3/5$ and A is an acute angle. Therefore,

$$\cos A = \frac{4}{5}$$

[$\because A$ is acute]

$$\text{and } \tan A = \frac{3}{4}$$

$$\text{Now, } \sin 2A = 2 \sin A \cos A = 2 \times \frac{3}{5} \times \frac{4}{5} = \frac{24}{25}$$

$$\cos 2A = 1 - 2 \sin^2 A = 1 - 2 \times \frac{9}{25} = \frac{7}{25}$$

$$\tan 2A = \frac{\sin 2A}{\cos 2A} = \frac{24}{7}$$

$$\sin 4A = 2 \sin 2A \cos 2A = 2 \times \frac{24}{25} \times \frac{7}{25} = \frac{336}{625}$$

ILLUSTRATION 3.46

Show that $\sqrt{2 + \sqrt{2 + \sqrt{2 + 2 \cos 8\theta}}} = 2 \cos \theta, 0 < \theta < \pi/16$.

Sol. L.H.S. = $\sqrt{2 + \sqrt{2 + \sqrt{2(1 + \cos 8\theta)}}}$

$$= \sqrt{2 + \sqrt{2 + \sqrt{2(2 \cos^2 4\theta)}}}$$

$$\left[\because 1 + \cos 8\theta = 2 \cos^2 \frac{8\theta}{2} \right]$$

$$= \sqrt{2 + \sqrt{2 + \sqrt{4 \cos^2 4\theta}}}$$

$$= \sqrt{2 + \sqrt{2 + 2 \cos 4\theta}}$$

$$= \sqrt{2 + \sqrt{2(1 + \cos 4\theta)}}$$

$$= \sqrt{2 + \sqrt{2(2 \cos^2 2\theta)}} \quad [\because 1 + \cos 4\theta = 2 \cos^2 2\theta]$$

$$= \sqrt{2 + 2 \cos 2\theta}$$

$$= \sqrt{2(1 + \cos 2\theta)}$$

$$= \sqrt{2(2 \cos^2 \theta)}$$

$$= 2 \cos \theta = \text{R.H.S.}$$

ILLUSTRATION 3.47

Prove that $\frac{\sec 8\theta - 1}{\sec 4\theta - 1} = \frac{\tan 8\theta}{\tan 2\theta}$.

Sol. L.H.S. = $\frac{\sec 8\theta - 1}{\sec 4\theta - 1}$

$$= \frac{\frac{1}{\cos 8\theta} - 1}{\frac{1}{\cos 4\theta} - 1} = \frac{1 - \cos 8\theta}{\cos 8\theta} \times \frac{\cos 4\theta}{1 - \cos 4\theta}$$

$$= \frac{2 \sin^2 4\theta}{\cos 8\theta} \times \frac{\cos 4\theta}{2 \sin^2 2\theta}$$

$$\left[\begin{array}{l} \because 1 - \cos 8\theta = 2 \sin^2 \frac{8\theta}{2} = 2 \sin^2 4\theta \\ \text{and } 1 - \cos 4\theta = 2 \sin^2 \frac{4\theta}{2} = 2 \sin^2 2\theta \end{array} \right]$$

$$= \frac{(2 \sin 4\theta \cos 4\theta)}{\cos 8\theta} \times \frac{\sin 4\theta}{2 \sin^2 2\theta}$$

$$= \left(\frac{2 \sin 4\theta \cos 4\theta}{\cos 8\theta} \right) \times \left(\frac{2 \sin 2\theta \cos 2\theta}{2 \sin^2 2\theta} \right)$$

$$= \left(\frac{\sin 2(4\theta)}{\cos 8\theta} \right) \times \left(\frac{\cos 2\theta}{\sin 2\theta} \right)$$

$$= \left(\frac{\sin 8\theta}{\cos 8\theta} \right) \times \left(\frac{\cos 2\theta}{\sin 2\theta} \right)$$

$$= \tan 8\theta \cot 2\theta = \frac{\tan 8\theta}{\tan 2\theta} = \text{R.H.S.}$$

ILLUSTRATION 3.48

Prove that $\tan \frac{\pi}{16} + 2 \tan \frac{\pi}{8} + 4 = \cot \frac{\pi}{16}$.

Sol. $\cot \theta - \tan \theta = \frac{\cos \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta}$

$$= \frac{\cos^2 \theta - \sin^2 \theta}{\sin \theta \cos \theta}$$

$$= \frac{2 \cos 2\theta}{\sin 2\theta}$$

$$= 2 \cot 2\theta$$

$$\therefore \cot \theta - 2 \cot 2\theta = \tan \theta$$

Now, $\tan \frac{\pi}{16} + 2 \tan \frac{\pi}{8} + 4$

$$= \left(\cot \frac{\pi}{16} - 2 \cot \frac{\pi}{8} \right) + 2 \left(\cot \frac{\pi}{8} - 2 \cot \frac{\pi}{4} \right) + 4$$

$$= \cot \frac{\pi}{16} - 4 \cot \frac{\pi}{4} + 4 = \cot \frac{\pi}{16}$$

ILLUSTRATION 3.49

Prove that $\cos^4 \frac{\pi}{8} + \cos^4 \frac{3\pi}{8} + \cos^4 \frac{5\pi}{8} + \cos^4 \frac{7\pi}{8} = \frac{3}{2}$.

Sol. We have $\frac{7\pi}{8} = \pi - \frac{\pi}{8}$ and $\frac{5\pi}{8} = \pi - \frac{3\pi}{8}$

$$\Rightarrow \cos \frac{7\pi}{8} = -\cos \frac{\pi}{8} \text{ and } \cos \frac{5\pi}{8} = -\cos \frac{3\pi}{8}$$

$$\Rightarrow \cos^4 \frac{7\pi}{8} = \cos^4 \frac{\pi}{8} \text{ and } \cos^4 \frac{5\pi}{8} = \cos^4 \frac{3\pi}{8}$$

$$\therefore \text{L.H.S.} = 2 \cos^4 \frac{\pi}{8} + 2 \cos^4 \frac{3\pi}{8}$$

$$= 2 \left[\left(\cos^2 \frac{\pi}{8} \right)^2 + \left(\cos^2 \frac{3\pi}{8} \right)^2 \right]$$

$$= 2 \left\{ \frac{1 + \cos \frac{\pi}{4}}{2} \right\}^2 + \left\{ \frac{1 + \cos \frac{3\pi}{4}}{2} \right\}^2$$

$$\left[\because \frac{1 + \cos 2\theta}{2} = \cos^2 \theta \right]$$

$$= \frac{1}{2} \left\{ \left(1 + \cos \frac{\pi}{4} \right)^2 + \left(1 + \cos \frac{3\pi}{4} \right)^2 \right\}$$

$$= \frac{1}{2} \left\{ \left(1 + \frac{1}{\sqrt{2}} \right)^2 + \left(1 - \frac{1}{\sqrt{2}} \right)^2 \right\}$$

$$= \frac{1}{2} \left\{ \left(1 + \frac{1}{2} + \sqrt{2} \right) + \left(1 + \frac{1}{2} - \sqrt{2} \right) \right\}$$

$$= \frac{3}{2} = \text{R.H.S.}$$

ILLUSTRATION 3.50

If $\pi < x < 2\pi$, prove that

$$\frac{\sqrt{1+\cos x} + \sqrt{1-\cos x}}{\sqrt{1+\cos x} - \sqrt{1-\cos x}} = \cot\left(\frac{x}{2} + \frac{\pi}{4}\right).$$

Sol. L.H.S. =
$$\frac{\sqrt{1+\cos x} + \sqrt{1-\cos x}}{\sqrt{1+\cos x} - \sqrt{1-\cos x}}$$

$$= \frac{\sqrt{2\cos^2 \frac{x}{2}} + \sqrt{2\sin^2 \frac{x}{2}}}{\sqrt{2\cos^2 \frac{x}{2}} - \sqrt{2\sin^2 \frac{x}{2}}}$$

$$= \frac{\sqrt{2}\left|\cos \frac{x}{2}\right| + \sqrt{2}\left|\sin \frac{x}{2}\right|}{\sqrt{2}\left|\cos \frac{x}{2}\right| - \sqrt{2}\left|\sin \frac{x}{2}\right|} = \frac{\left|\cos \frac{x}{2}\right| + \left|\sin \frac{x}{2}\right|}{\left|\cos \frac{x}{2}\right| - \left|\sin \frac{x}{2}\right|}$$

$$= \frac{-\cos \frac{x}{2} + \sin \frac{x}{2}}{-\cos \frac{x}{2} - \sin \frac{x}{2}} \left[\because \pi < x < 2\pi, \therefore \frac{\pi}{2} < \frac{x}{2} < \pi \right]$$

Dividing numerator and denominator by $\sin(x/2)$, we get

$$\text{L.H.S.} = \frac{\cot \frac{x}{2} - 1}{\cot \frac{x}{2} + 1} = \cot\left(\frac{x}{2} + \frac{\pi}{4}\right) = \text{R.H.S.}$$

ILLUSTRATION 3.51

If $\sin \alpha + \sin \beta = a$ and $\cos \alpha + \cos \beta = b$, prove that

$$\tan \frac{\alpha - \beta}{2} = \pm \sqrt{\frac{4 - a^2 - b^2}{a^2 + b^2}}.$$

Sol. Given, $\sin \alpha + \sin \beta = a$ (i)

and $\cos \alpha + \cos \beta = b$ (ii)

Now $(\cos \alpha + \cos \beta)^2 + (\sin \alpha + \sin \beta)^2 = b^2 + a^2$

or $\cos^2 \alpha + \cos^2 \beta + 2 \cos \alpha \cos \beta + \sin^2 \alpha + \sin^2 \beta + 2 \sin \alpha \sin \beta = b^2 + a^2$

or $(\cos^2 \alpha + \sin^2 \alpha) + (\cos^2 \beta + \sin^2 \beta) + 2(\cos \alpha \cos \beta + \sin \alpha \sin \beta) = a^2 + b^2$

or $2 + 2 \cos(\alpha - \beta) = a^2 + b^2$

or $\cos(\alpha - \beta) = \frac{a^2 + b^2 - 2}{2}$

Now, $\tan \frac{\alpha - \beta}{2} = \pm \sqrt{\frac{1 - \cos(\alpha - \beta)}{1 + \cos(\alpha - \beta)}}$

$$= \pm \sqrt{\frac{1 - \frac{a^2 + b^2 - 2}{2}}{1 + \frac{a^2 + b^2 - 2}{2}}} = \pm \sqrt{\frac{4 - a^2 - b^2}{a^2 + b^2}}$$

ILLUSTRATION 3.52

Prove that
$$\frac{1 - \tan^2\left(\frac{\pi}{4} - A\right)}{1 + \tan^2\left(\frac{\pi}{4} - A\right)} = \sin 2A.$$

Sol.

$$\frac{1 - \tan^2\left(\frac{\pi}{4} - A\right)}{1 + \tan^2\left(\frac{\pi}{4} - A\right)} = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \quad \left(\text{where } \frac{\pi}{4} - A = \theta\right)$$

$$= \cos 2\theta = \cos\left(\frac{\pi}{2} - 2A\right) = \sin 2A$$

ILLUSTRATION 3.53

If $\tan \frac{\theta}{2} = \sqrt{\frac{a-b}{a+b}} \tan \frac{\phi}{2}$, prove that $\cos \alpha = \frac{a \cos \phi + b}{a + b \cos \phi}$.

Sol. Given,

$$\tan \frac{\theta}{2} = \sqrt{\frac{a-b}{a+b}} \tan \frac{\phi}{2}$$

Now, $\cos \theta = \frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}} = \frac{1 - \frac{a-b}{a+b} \tan^2 \frac{\phi}{2}}{1 + \frac{a-b}{a+b} \tan^2 \frac{\phi}{2}}$

$$= \frac{1 - \frac{a-b}{a+b} \frac{\sin^2 \frac{\phi}{2}}{\cos^2 \frac{\phi}{2}}}{1 + \frac{a-b}{a+b} \frac{\sin^2 \frac{\phi}{2}}{\cos^2 \frac{\phi}{2}}}$$

$$= \frac{(a+b) \cos^2 \frac{\phi}{2} - (a-b) \sin^2 \frac{\phi}{2}}{(a+b) \cos^2 \frac{\phi}{2} + (a-b) \sin^2 \frac{\phi}{2}}$$

$$= \frac{a \left(\cos^2 \frac{\phi}{2} - \sin^2 \frac{\phi}{2} \right) + b \left(\cos^2 \frac{\phi}{2} + \sin^2 \frac{\phi}{2} \right)}{a \left(\cos^2 \frac{\phi}{2} + \sin^2 \frac{\phi}{2} \right) + b \left(\cos^2 \frac{\phi}{2} - \sin^2 \frac{\phi}{2} \right)}$$

$$= \frac{a \cos \phi + b}{a + b \cos \phi}$$

ILLUSTRATION 3.54

If $\cos \theta = \cos \alpha \cos \beta$, prove that

$$\tan \frac{\theta + \alpha}{2} \tan \frac{\theta - \alpha}{2} = \tan^2 \frac{\beta}{2}.$$

Sol. Given,

$$\cos \theta = \cos \alpha \cos \beta. \text{ We have}$$

$$\cos \beta = \frac{\cos \theta}{\cos \alpha}$$

(i)

$$\begin{aligned} \text{Now, } \tan^2 \frac{\beta}{2} &= \frac{1 - \cos \beta}{1 + \cos \beta} = \frac{1 - \frac{\cos \theta}{\cos \alpha}}{1 + \frac{\cos \theta}{\cos \alpha}} \\ &= \frac{\cos \alpha - \cos \theta}{\cos \alpha + \cos \theta} \\ &= \frac{2 \sin \frac{\alpha + \theta}{2} \sin \frac{\theta - \alpha}{2}}{2 \cos \frac{\theta + \alpha}{2} \cos \frac{\theta - \alpha}{2}} \\ &= \tan \frac{\theta + \alpha}{2} \tan \frac{\theta - \alpha}{2} \end{aligned}$$

ILLUSTRATION 3.55

If $\tan \beta = \frac{\tan \alpha + \tan \gamma}{1 + \tan \alpha \tan \gamma}$, prove that $\sin 2\beta = \frac{\sin 2\alpha + \sin 2\gamma}{1 + \sin 2\alpha \sin 2\gamma}$.

$$\text{Sol. } \tan \beta = \frac{\frac{\sin \alpha}{\cos \alpha} + \frac{\sin \gamma}{\cos \gamma}}{1 + \frac{\sin \alpha}{\cos \alpha} \cdot \frac{\sin \gamma}{\cos \gamma}} = \frac{\sin(\alpha + \gamma)}{\cos(\alpha - \gamma)}$$

$$\begin{aligned} \sin 2\beta &= \frac{2 \tan \beta}{1 + \tan^2 \beta} = \frac{2 \frac{\sin(\alpha + \gamma)}{\cos(\alpha - \gamma)}}{1 + \frac{\sin^2(\alpha + \gamma)}{\cos^2(\alpha - \gamma)}} \\ &= \frac{2 \sin(\alpha + \gamma) \cos(\alpha - \gamma)}{\cos^2(\alpha - \gamma) + \sin^2(\alpha + \gamma)} \\ &= \frac{\sin 2\alpha + \sin 2\gamma}{\frac{1 + \cos 2(\alpha - \gamma)}{2} + \frac{1 - \cos 2(\alpha + \gamma)}{2}} \\ &= \frac{\sin 2\alpha + \sin 2\gamma}{1 + \frac{1}{2} \times 2 \sin 2\alpha \sin 2\gamma} = \frac{\sin 2\alpha + \sin 2\gamma}{1 + \sin 2\alpha \sin 2\gamma} \end{aligned}$$

ILLUSTRATION 3.56

Prove that $(4 \cos^2 9^\circ - 3)(4 \cos^2 27^\circ - 3) = \tan 9^\circ$.

Sol. We have $\cos 3x = 4 \cos^3 x - 3 \cos x$. Hence,

$$4 \cos^2 x - 3 = \frac{\cos 3x}{\cos x} \text{ for all } x \neq (2k+1) \cdot \frac{\pi}{2}, k \in \mathbb{Z}.$$

$$\begin{aligned} \therefore (4 \cos^2 9^\circ - 3)(4 \cos^2 27^\circ - 3) &= \frac{\cos 27^\circ}{\cos 9^\circ} \cdot \frac{\cos 81^\circ}{\cos 27^\circ} \\ &= \frac{\cos 81^\circ}{\cos 9^\circ} \\ &= \frac{\sin 9^\circ}{\cos 9^\circ} = \tan 9^\circ \end{aligned}$$

ILLUSTRATION 3.57

Prove that $4 \cos \frac{2\pi}{7} \cdot \cos \frac{\pi}{7} - 1 = 2 \cos \frac{2\pi}{7}$.

$$\begin{aligned} \text{Sol. L.H.S.} &= 4 \cos \frac{2\pi}{7} \cdot \cos \frac{\pi}{7} - 1 \\ &= \frac{\sin \frac{\pi}{7} \cdot 4 \cos \frac{\pi}{7} \cdot \cos \left(\frac{2\pi}{7} \right)}{\sin \frac{\pi}{7}} - 1 \\ &= \frac{2 \sin \frac{2\pi}{7} \cos \frac{2\pi}{7}}{\sin \frac{\pi}{7}} - 1 \\ &= \frac{\sin \frac{4\pi}{7}}{\sin \frac{\pi}{7}} - 1 \\ &= \frac{\sin \frac{3\pi}{7}}{\sin \frac{\pi}{7}} - 1 \\ &= \frac{3 \sin \frac{\pi}{7} - 4 \sin^3 \frac{\pi}{7}}{\sin \frac{\pi}{7}} - 1 \\ &= 3 - 4 \sin^2 \frac{\pi}{7} - 1 \\ &= 2 \left[1 - 2 \sin^2 \frac{\pi}{7} \right] \\ &= 2 \cos \frac{2\pi}{7} = \text{R.H.S.} \end{aligned}$$

ILLUSTRATION 3.58

Evaluate $\cos a \cos 2a \cos 3a \cdots \cos 999a$, where $a = \frac{2\pi}{1999}$.

Sol. Let

$$P = \cos a \cos 2a \cos 3a \cdots \cos 999a;$$

$$Q = \sin a \sin 2a \sin 3a \cdots \sin 999a.$$

$$\begin{aligned} \text{Then } 2^{999} PQ &= (2 \sin a \cos a)(2 \sin 2a \cos 2a) \cdots (2 \sin 999a \cos 999a) \\ &= \sin 2a \sin 4a \cdots \sin 1998a \\ &= (\sin 2a \sin 4a \cdots \sin 998a) [-\sin(2\pi - 1000a)] \\ &\quad \cdot [-\sin(2\pi - 1002a)] \cdots [-\sin(2\pi - 1998a)] \\ &\quad (\because 2\pi = 1999a) \\ &= \sin 2a \sin 4a \cdots \sin 998a \sin 999a \sin 997a \\ &\quad \cdots \sin a = Q. \end{aligned}$$

It is easy to see that $Q \neq 0$. Hence, the desired product is $P = \frac{1}{2^{999}}$.

ILLUSTRATION 3.59

Prove that $\sin \theta \cdot \sec 3\theta + \sin 3\theta \cdot \sec 3^2\theta + \sin 3^2\theta \cdot \sec 3^3\theta + \dots$ upto n terms $= \frac{1}{2} [\tan 3^n \theta - \tan \theta]$

Sol. $\sin \theta \cdot \sec 3\theta + \sin 3\theta \cdot \sec 3^2\theta + \sin 3^2\theta \cdot \sec 3^3\theta + \dots$ upto n terms

$$\begin{aligned}
 &= \sum_{r=1}^n \sin 3^{r-1} \theta \cdot \sec 3^r \theta \\
 &= \sum_{r=1}^n \frac{2 \cos 3^{r-1} \theta \sin 3^{r-1} \theta}{2 \cos 3^{r-1} \theta \cdot \cos 3^r \theta} \\
 &= \frac{1}{2} \sum_{r=1}^n \frac{\sin (2 \cdot 3^{r-1} \theta)}{\cos 3^{r-1} \theta \cdot \cos 3^r \theta} \\
 &= \frac{1}{2} \sum_{r=1}^n \frac{\sin (3^r \theta - 3^{r-1} \theta)}{\cos 3^{r-1} \theta \cdot \cos 3^r \theta} \\
 &= \frac{1}{2} \sum_{r=1}^n \frac{\sin 3^r \theta \cdot \cos 3^{r-1} \theta - \cos 3^r \theta \cdot \sin 3^{r-1} \theta}{\cos 3^{r-1} \theta \cdot \cos 3^r \theta} \\
 &= \frac{1}{2} \sum_{r=1}^n (\tan 3^r \theta - \tan 3^{r-1} \theta) \\
 &= \frac{1}{2} [\tan 3^n \theta - \tan \theta]
 \end{aligned}$$

ILLUSTRATION 3.60

Let $f(x) = 2 \operatorname{cosec} 2x + \sec x + \operatorname{cosec} x$. Then find the minimum value of $f(x)$ for $x \in \left(0, \frac{\pi}{2}\right)$.

Sol. $f(x) = 2 \operatorname{cosec} 2x + \sec x + \operatorname{cosec} x$

$$= 2 \operatorname{cosec} 2x + \left(\sqrt{\sec x} - \sqrt{\operatorname{cosec} x} \right)^2 + \frac{2\sqrt{2}}{\sqrt{2 \cos x \cdot \sin x}}$$

$$\therefore = 2 \operatorname{cosec} 2x + \left(\sqrt{\sec x} - \sqrt{\operatorname{cosec} x} \right)^2 + 2\sqrt{2} \sqrt{\operatorname{cosec} 2x}$$

Clearly, minimum value occurs for $x = \frac{\pi}{4}$, as for this value of x , $\sqrt{\sec x} - \sqrt{\operatorname{cosec} x} = 0$ and $\operatorname{cosec} 2x = 1$.

$$\text{So, } f_{\min} \left(x = \frac{\pi}{4} \right) = 2(\sqrt{2} + 1)$$

ILLUSTRATION 3.61

Find the maximum and minimum values of $\cos^2 \theta - 6 \sin \theta \cos \theta + 3 \sin^2 \theta + 2$.

Sol. $\cos^2 \theta - 6 \sin \theta \cos \theta + 3 \sin^2 \theta + 2$

$$\begin{aligned}
 &= \frac{1 + \cos 2\theta}{2} - 3 \sin 2\theta + 3 \frac{(1 - \cos 2\theta)}{2} + 2 \\
 &= 4 - \cos 2\theta - 3 \sin 2\theta
 \end{aligned}$$

Now, $-\cos 2\theta - 3 \sin 2\theta \in [-\sqrt{10}, \sqrt{10}]$

$$\Rightarrow 4 - \cos 2\theta - 3 \sin 2\theta \in [4 - \sqrt{10}, 4 + \sqrt{10}]$$

FORMULAS FOR TANGENT AND COTANGENT OF MULTIPLE ANGLES

$$1. \text{ (i) } \tan 2A = \frac{2 \tan A}{1 - \tan^2 A} \quad \text{(ii) } \cot 2A = \frac{\cot^2 A - 1}{2 \cot A}$$

Proof:

$$\tan 2A = \tan(A + A) = \frac{\tan A + \tan A}{1 - \tan A \tan A} = \frac{2 \tan A}{1 - \tan^2 A}$$

$$\therefore \frac{1}{\cot 2A} = \frac{2}{1 - \frac{1}{\cot^2 A}}$$

$$\Rightarrow \cot 2A = \frac{\cot^2 A - 1}{2 \cot A}$$

$$2. \tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$$

Proof:

$$\tan 3A = \tan(2A + A) = \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A}$$

$$\begin{aligned}
 &= \frac{\frac{2 \tan A}{1 - \tan^2 A} + \tan A}{1 - \frac{2 \tan A}{1 - \tan^2 A} \cdot \tan A} = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}
 \end{aligned}$$

Similarly, we can prove that

$$\cot 3A = \frac{\cot^3 A - 3 \cot A}{3 \cot^2 A - 1}$$

$$3. \tan nA =$$

$$\frac{{}^n C_1 \tan A - {}^n C_3 \tan^3 A + {}^n C_5 \tan^5 A - {}^n C_7 \tan^7 A + \dots}{1 - {}^n C_2 \tan^2 A + {}^n C_4 \tan^4 A - {}^n C_6 \tan^6 A + \dots}$$

Proof:

$$\begin{aligned}
 \tan nA &= \frac{\sin nA}{\cos nA} \\
 &= \frac{{}^n C_1 \cos^{n-1} A \sin A - {}^n C_3 \cos^{n-3} A \sin^3 A + {}^n C_5 \cos^{n-5} A \sin^5 A - {}^n C_7 \cos^{n-7} A \sin^7 A + \dots}{({}^n C_0 \cos^n A - {}^n C_2 \cos^{n-2} A \sin^2 A + {}^n C_4 \cos^{n-4} A \sin^4 A - {}^n C_6 \cos^{n-6} A \sin^6 A + \dots)} \\
 &= \frac{({}^n C_1 \tan A - {}^n C_3 \tan^3 A + {}^n C_5 \tan^5 A - {}^n C_7 \tan^7 A + \dots)}{(1 - {}^n C_2 \tan^2 A + {}^n C_4 \tan^4 A - {}^n C_6 \tan^6 A + \dots)}
 \end{aligned}$$

[Dividing Nr. and Dr. by $\cos^n A$]

ILLUSTRATION 3.62

If $\tan \alpha = \frac{1}{7}$, $\sin \beta = \frac{1}{\sqrt{10}}$, prove that $\alpha + 2\beta = \frac{\pi}{4}$, where

$$0 < \alpha < \frac{\pi}{2} \text{ and } 0 < \beta < \frac{\pi}{2}$$

Sol. $\tan(\alpha + 2\beta) = \frac{\tan \alpha + \tan 2\beta}{1 - \tan \alpha \tan 2\beta} = \frac{\frac{1}{7} + \tan 2\beta}{1 - \frac{1}{7} \tan 2\beta}$ (i)

Now, $\tan 2\beta = \frac{2 \tan \beta}{1 - \tan^2 \beta} = \frac{2 \times \frac{1}{3}}{1 - \frac{1}{9}} = \frac{3}{4}$

[$\tan \beta > 0$ as $0 < \beta < \pi/2$]

Substituting the value of $\tan 2\beta$ in Eq. (i), we get

$$\tan(\alpha + 2\beta) = \frac{\frac{1}{7} + \frac{3}{4}}{1 - \frac{1}{7} \times \frac{3}{4}} = \frac{25}{25} = 1$$

Now, $0 < \alpha < \frac{\pi}{2}$ and $0 < \beta < \frac{\pi}{2}$

$\therefore 0 < 2\beta < \pi$, but $\tan 2\beta = \frac{3}{4} > 0$

$\Rightarrow 0 < 2\beta < \frac{\pi}{2}$

Hence, $0 < \alpha + 2\beta < \pi$.

In the interval $(0, \pi)$, $\tan \theta$ takes value 1 at $\pi/4$ only. Therefore,

$$\alpha + 2\beta = \frac{\pi}{4}$$

ILLUSTRATION 3.63

Prove that $\tan \frac{\pi}{10}$ is a root of polynomial equation $5x^4 - 10x^2 + 1 = 0$.

Sol. If $\theta = 18^\circ$ then $5\theta = 90^\circ$

$\Rightarrow \tan 5\theta = \infty$

$\Rightarrow \frac{5 \tan \theta - {}^5C_3 \tan^3 \theta + {}^5C_5 \tan^5 \theta}{1 - {}^5C_2 \tan^2 \theta + {}^5C_4 \tan^4 \theta} = \infty$

$\Rightarrow 5x^4 - 10x^2 + 1 = 0$, where $x = \tan \theta$

Thus $x = \tan \theta$ is root of the equation $5x^4 - 10x^2 + 1 = 0$.

ILLUSTRATION 3.64

If $x + y + z = xyz$, prove that

$$\frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{2x}{1-x^2} \frac{2y}{1-y^2} \frac{2z}{1-z^2}$$

Sol. Let

$$x = \tan A, y = \tan B, z = \tan C$$

Now $x + y + z = xyz$

$\Rightarrow \tan A + \tan B + \tan C = \tan A \tan B \tan C$

$\Rightarrow A + B + C = n\pi$

or $2A + 2B + 2C = 2n\pi$

$\Rightarrow \tan 2A + \tan 2B + \tan 2C = \tan 2A \tan 2B \tan 2C$

$$\begin{aligned} \Rightarrow \frac{2 \tan A}{1 - \tan^2 A} + \frac{2 \tan B}{1 - \tan^2 B} + \frac{2 \tan C}{1 - \tan^2 C} \\ = \frac{2 \tan A}{1 - \tan^2 A} \frac{2 \tan B}{1 - \tan^2 B} \frac{2 \tan C}{1 - \tan^2 C} \\ \Rightarrow \frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{2x}{1-x^2} \frac{2y}{1-y^2} \frac{2z}{1-z^2} \end{aligned}$$

ILLUSTRATION 3.65

Prove that $1 + \cot \theta \leq \cot \frac{\theta}{2}$ for $0 < \theta < \pi$. Find θ when equality sign holds.

Sol. We have

$$\begin{aligned} 1 + \cot \theta - \cot \frac{\theta}{2} &= 1 + \frac{\cot^2 \frac{\theta}{2} - 1}{2 \cot \frac{\theta}{2}} - \cot \frac{\theta}{2} \\ &= \frac{2 \cot \frac{\theta}{2} + \cot^2 \frac{\theta}{2} - 1 - 2 \cot^2 \frac{\theta}{2}}{2 \cot \frac{\theta}{2}} \\ &= \frac{-\left(\cot \frac{\theta}{2} - 1\right)^2}{2 \cot \frac{\theta}{2}} \leq 0 \text{ for } 0 < \theta < \pi \end{aligned}$$

or $1 + \cot \theta \leq \cot \frac{\theta}{2}$

Equality holds when $\cot \frac{\theta}{2} - 1 = 0$ or $\theta = \frac{\pi}{2}$.

CONCEPT APPLICATION EXERCISE 3.4

1. Prove that $\frac{1 + \sin 2A - \cos 2A}{1 + \sin 2A + \cos 2A} = \tan A$.
2. Prove that $\frac{1 + \sin 2A}{\cos 2A} = \frac{\cos A + \sin A}{\cos A - \sin A} = \tan \left(\frac{\pi}{4} + A \right)$.
3. Prove that $\cot \theta - \tan \theta = 2 \cot 2\theta$.
4. Prove that $\frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} = \sec 2\theta - \tan 2\theta$.
5. Prove that $\tan \left(\frac{\pi}{4} + \theta \right) - \tan \left(\frac{\pi}{4} - \theta \right) = 2 \tan 2\theta$.
6. Prove that $\operatorname{cosec} A - 2 \cot 2A \cos A = 2 \sin A$.
7. Prove that $\cos^3 \theta \sin 3\theta + \sin^3 \theta \cos 3\theta = \frac{3}{4} \sin 4\theta$.
8. Prove that $\frac{\sin^2 3A}{\sin^2 A} - \frac{\cos^2 3A}{\cos^2 A} = 8 \cos 2A$.
9. Prove that $(1 + \sec 2\theta)(1 + \sec 4\theta)(1 + \sec 8\theta) = \frac{\tan 8\theta}{\tan \theta}$.
10. In an isosceles triangle with base a , vertical angle 20° and lateral side each of length b , prove that $a^3 + b^3 = 3ab^2$.

11. In triangle ABC , $a = 3$, $b = 4$ and $c = 5$. Then find the value of $\sin A + \sin 2B + \sin 3C$.
12. If $\cos A = 3/4$, then find the value of $32 \sin\left(\frac{A}{2}\right) \sin\left(\frac{5A}{2}\right)$.
13. Find the value of $(4 \cos^2 9^\circ - 1)(4 \cos^2 27^\circ - 1)(4 \cos^2 81^\circ - 1)(4 \cos^2 243^\circ - 1)$.
14. If θ is an acute angle and $\sin \frac{\theta}{2} = \sqrt{\frac{x-1}{2x}}$, find $\tan \theta$ in terms of x .
15. In a triangle ABC , if $\sin A \sin(B-C) = \sin C \sin(A-B)$, then prove that $\cos 2A$, $\cos 2B$ and $\cos 2C$ are in A.P.
16. Let $a = \frac{\pi}{7}$. Then
 (a) show that $\sin^2 3a - \sin^2 a = \sin 2a \sin 3a$.
 (b) show that $\operatorname{cosec} a = \operatorname{cosec} 2a + \operatorname{cosec} 4a$.
 (c) prove that $\cos a$ is a root of the equation $8x^3 + 4x^2 - 4x + 1 = 0$.
17. Show that $\frac{1}{\sin 10^\circ} - \frac{\sqrt{3}}{\cos 10^\circ} = 4$.
18. Prove that $\cos 2\alpha = 2 \sin^2 \beta + 4 \cos(\alpha + \beta) \sin \alpha \sin \beta + \cos 2(\alpha + \beta)$.
19. If $\tan x = \frac{a}{b}$ and $\tan 2x = \frac{b}{a+b}$ then find the smallest positive value of x .
20. Prove that $\tan \theta + \tan(60^\circ + \theta) + \tan(120^\circ + \theta) = 3 \tan 3\theta$.
21. If $A = 110^\circ$, then prove that $\frac{1 + \sqrt{1 + \tan^2 2A}}{\tan 2A} = -\tan A$.
22. If α and β are the two different roots of equation $a \cos \theta + b \sin \theta = c$, prove that
 (a) $\tan(\alpha + \beta) = \frac{2ab}{a^2 - b^2}$ (b) $\cos(\alpha + \beta) = \frac{a^2 - b^2}{a^2 + b^2}$
23. If $\tan \beta = \cos \theta \tan \alpha$, then prove that $\tan^2 \frac{\theta}{2} = \frac{\sin(\alpha - \beta)}{\sin(\alpha + \beta)}$.
24. If $\cos \theta = \frac{a}{b+c}$, $\cos \phi = \frac{b}{a+c}$, $\cos \psi = \frac{c}{a+b}$, where $\theta, \phi, \psi \in (0, \pi)$ and a, b, c are sides of triangle ABC then find the value of $\tan^2 \frac{\theta}{2} + \tan^2 \frac{\phi}{2} + \tan^2 \frac{\psi}{2}$.
25. If $\cos \theta = \frac{\cos \alpha - \cos \beta}{1 - \cos \alpha \cos \beta}$, prove that one of the values of $\tan \frac{\theta}{2}$ is $\tan \frac{\alpha}{2} \cot \frac{\beta}{2}$.
26. If $\tan \theta \tan \phi = \sqrt{\frac{a-b}{a+b}}$, prove that $(a - b \cos 2\theta)(a - b \cos 2\phi)$ is independent of θ and ϕ .

ANSWERS

11. 14/25 12. 11 13. 1 14. $\tan \theta = \sqrt{x^2 - 1}$
 19. $\tan^{-1} \frac{1}{3}$ 24. 1

VALUES OF TRIGONOMETRIC RATIOS OF TYPICAL ANGLES

1. Value of $\sin 15^\circ$, $\cos 15^\circ$, $\sin 75^\circ$, $\cos 75^\circ$, $\tan 15^\circ$, $\tan 75^\circ$
 $\sin 15^\circ = \sin(45^\circ - 30^\circ) = \sin 45^\circ \cos 30^\circ - \sin 30^\circ \cos 45^\circ$
 $= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$

Also, $\sin 15^\circ = \cos 75^\circ = -\cos 105^\circ$.

Similarly, we can prove that $\cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$.

Also, $\cos 15^\circ = \sin 75^\circ = \sin 105^\circ$.

Now, $\tan 15^\circ = \tan(60^\circ - 45^\circ) = \frac{\tan 60^\circ - \tan 45^\circ}{1 + \tan 60^\circ \tan 45^\circ}$

$$= \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2 - \sqrt{3}$$

and $\tan 75^\circ = \tan(30^\circ + 45^\circ) = \frac{\tan 30^\circ + \tan 45^\circ}{1 - \tan 30^\circ \tan 45^\circ}$

$$= \frac{\frac{1}{\sqrt{3}}+1}{\sqrt{3}-1} = 2 + \sqrt{3}$$

2. Value of $\sin 18^\circ$:

Let $\theta = 18^\circ$, then $5\theta = 90^\circ$

or $2\theta + 3\theta = 90^\circ$

or $2\theta = 90^\circ - 3\theta$

or $\sin 2\theta = \sin(90^\circ - 3\theta)$

or $\sin 2\theta = \cos 3\theta$

or $2 \sin \theta \cos \theta = 4 \cos^3 \theta - 3 \cos \theta$

or $2 \sin \theta = 4 \cos^2 \theta - 3$ [Dividing by $\cos \theta$]

or $2 \sin \theta = 4(1 - \sin^2 \theta) - 3 = 1 - 4 \sin^2 \theta$

or $4 \sin^2 \theta + 2 \sin \theta - 1 = 0$

or $\sin \theta = \frac{-2 \pm \sqrt{4+16}}{8} = \frac{-2 \pm 2\sqrt{5}}{8} = \frac{-1 \pm \sqrt{5}}{4}$

$\therefore \theta = 18^\circ$

$\therefore \sin \theta = \sin 18^\circ > 0$,

$\therefore \sin \theta = \sin 18^\circ = \frac{\sqrt{5}-1}{4}$

Value of $\cos 18^\circ$:

$$\cos^2 18^\circ = 1 - \sin^2 18^\circ = 1 - \left(\frac{\sqrt{5}-1}{4}\right)^2$$

$$= 1 - \frac{5+1-2\sqrt{5}}{16} = \frac{10+2\sqrt{5}}{16}$$

$$\Rightarrow \cos 18^\circ = \frac{1}{4} \sqrt{10+2\sqrt{5}} \quad [\because \cos 18^\circ > 0]$$

3. Value of $\cos 36^\circ$:

$$\cos 36^\circ = 1 - 2 \sin^2 18^\circ = 1 - 2 \left(\frac{\sqrt{5}-1}{4}\right)^2 = \frac{\sqrt{5}+1}{4}$$

Value of $\sin 36^\circ$:

$$\sin^2 36^\circ = 1 - \cos^2 36^\circ = 1 - \left(\frac{\sqrt{5}+1}{4}\right)^2$$

$$= 1 - \frac{6+2\sqrt{5}}{16} = \frac{16-6-2\sqrt{5}}{16} = \frac{10-2\sqrt{5}}{16}$$

$$\therefore \sin 36^\circ = \frac{1}{4}\sqrt{10-2\sqrt{5}} \quad [\because \sin 36^\circ > 0]$$

Note:

$$\bullet \sin 54^\circ = \sin(90^\circ - 36^\circ) = \cos 36^\circ = \frac{\sqrt{5}+1}{4}$$

$$\bullet \cos 54^\circ = \cos(90^\circ - 36^\circ) = \sin 36^\circ = \frac{1}{4}(\sqrt{10-2\sqrt{5}})$$

4. Value of $\tan 7\frac{1}{2}^\circ$:

$$\text{Let } \theta = 7\frac{1}{2}^\circ, \text{ then } 2\theta = 15^\circ$$

$$\tan \theta = \frac{1 - \cos 2\theta}{\sin 2\theta} \quad [\because 1 - \cos 2\theta = 2 \sin^2 \theta \text{ and } \sin 2\theta = 2 \sin \theta \cos \theta]$$

$$\begin{aligned} &= \frac{1 - \cos 15^\circ}{\sin 15^\circ} = \frac{1 - \frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}} \\ &= \frac{2\sqrt{2} - \sqrt{3} - 1}{\sqrt{3} - 1} \\ &= (\sqrt{3} - \sqrt{2})(\sqrt{2} - 1) \end{aligned}$$

Value of $\cot 82\frac{1}{2}^\circ$:

$$\begin{aligned} \cot 82\frac{1}{2}^\circ &= \cot \left(90^\circ - 7\frac{1}{2}^\circ \right) \\ &= \tan 7\frac{1}{2}^\circ = (\sqrt{3} - \sqrt{2})(\sqrt{2} - 1) \end{aligned}$$

Value of $\cot 7\frac{1}{2}^\circ$:

$$\text{Let } \theta = 7\frac{1}{2}^\circ, \text{ then } 2\theta = 15^\circ$$

$$\text{Now, } \cot \theta = \frac{1 + \cos 2\theta}{\sin 2\theta} = \frac{1 + \cos 15^\circ}{\sin 15^\circ}$$

$$\begin{aligned} &= \frac{1 + \frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}} \\ &= \frac{2\sqrt{2} + \sqrt{3} + 1}{\sqrt{3} - 1} \\ &= (\sqrt{3} + \sqrt{2})(\sqrt{2} + 1) \end{aligned}$$

Value of $\tan 82\frac{1}{2}^\circ$:

$$\begin{aligned} \tan 82\frac{1}{2}^\circ &= \tan \left(90^\circ - 7\frac{1}{2}^\circ \right) \\ &= \cot 7\frac{1}{2}^\circ = (\sqrt{3} + \sqrt{2})(\sqrt{2} + 1) \end{aligned}$$

5. Value of trigonometric functions for $\theta = 22.5^\circ$:

$$\text{We know that } \cos 2\theta = 2\cos^2 \theta - 1$$

$$\therefore \text{ for } \theta = 22.5^\circ, \text{ we have } \cos 45^\circ = 2\cos^2 22.5^\circ - 1$$

$$\therefore \cos 22.5^\circ = \sqrt{\frac{1 + \cos 45^\circ}{2}} = \sqrt{\frac{1 + \frac{1}{\sqrt{2}}}{2}}$$

$$= \sqrt{\frac{\sqrt{2} + 1}{2\sqrt{2}}} = \frac{\sqrt{2 + \sqrt{2}}}{2}$$

$$\sin 22.5^\circ = \sqrt{\frac{1 - \cos 45^\circ}{2}} = \frac{\sqrt{2 - \sqrt{2}}}{2}$$

$$\tan 22.5^\circ = \frac{\sqrt{2 - \sqrt{2}}}{\sqrt{2 + \sqrt{2}}} = \frac{2 - \sqrt{2}}{\sqrt{2}} = \sqrt{2} - 1$$

$$\cot 22.5^\circ = \frac{\sqrt{2 + \sqrt{2}}}{\sqrt{2 - \sqrt{2}}} = \sqrt{2} + 1$$

All above values are tabulated as follows:

	7.5°	15°	18°	22.5°	36°	67.5°	75°
sin	$\frac{\sqrt{8-2\sqrt{6}-2\sqrt{2}}}{4}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{2}-\sqrt{2}}{2}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$
cos	$\frac{\sqrt{8+2\sqrt{6}+2\sqrt{2}}}{4}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$
tan	$(\sqrt{3}-\sqrt{2})(\sqrt{2}-1)$	$2-\sqrt{3}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\sqrt{2}-1$	$\sqrt{5-2\sqrt{5}}$	$\sqrt{2}+1$	$2+\sqrt{3}$
cot	$(\sqrt{3}+\sqrt{2})(\sqrt{2}+1)$	$2+\sqrt{3}$	$\sqrt{(5+2\sqrt{5})}$	$\sqrt{2}+1$	$\sqrt{\left(1+\frac{2}{\sqrt{5}}\right)}$	$\sqrt{2}-1$	$2-\sqrt{3}$

ILLUSTRATION 3.66

Find the angle θ whose cosine is equal to its tangent.

Sol. Given,

$$\cos \theta = \tan \theta \quad \text{or} \quad \cos^2 \theta = \sin \theta$$

$$\text{or} \quad 1 - \sin^2 \theta = \sin \theta \quad \text{or} \quad \sin^2 \theta + \sin \theta - 1 = 0$$

$$\text{or} \quad \sin \theta = \frac{-1 \pm \sqrt{5}}{2} = 2 \times \frac{\sqrt{5}-1}{4} = 2 \sin 18^\circ$$

$$\text{or} \quad \theta = \sin^{-1}(2 \sin 18^\circ)$$

ILLUSTRATION 3.67

Find the value of $\cos 12^\circ + \cos 84^\circ + \cos 156^\circ + \cos 132^\circ$.

$$\begin{aligned} \text{Sol.} \quad & \cos 12^\circ + \cos 84^\circ + \cos 156^\circ + \cos 132^\circ \\ &= (\cos 12^\circ + \cos 132^\circ) + (\cos 84^\circ + \cos 156^\circ) \\ &= 2 \cos \left(\frac{12^\circ + 132^\circ}{2} \right) \cos \left(\frac{132^\circ - 12^\circ}{2} \right) \\ &\quad + 2 \cos \left(\frac{84^\circ + 156^\circ}{2} \right) \cos \left(\frac{156^\circ - 84^\circ}{2} \right) \\ &= 2 \cos 72^\circ \cos 60^\circ + 2 \cos 120^\circ \cos 36^\circ \\ &= 2 \sin 18^\circ \cos 60^\circ + 2 \cos 120^\circ \cos 36^\circ \\ &= 2 \left(\frac{\sqrt{5}-1}{4} \right) \frac{1}{2} + 2 \left(-\frac{1}{2} \right) \left(\frac{\sqrt{5}+1}{4} \right) = -\frac{1}{2} \end{aligned}$$

ILLUSTRATION 3.68

Prove that $\cos 36^\circ \cos 72^\circ \cos 108^\circ \cos 144^\circ = 1/16$.

$$\begin{aligned} \text{Sol.} \quad & \cos 36^\circ \cos 72^\circ \cos 108^\circ \cos 144^\circ \\ &= \cos 36^\circ \sin 18^\circ (-\sin 18^\circ) (-\cos 36^\circ) \\ &= \cos^2 36^\circ \sin^2 18^\circ \\ &= \left(\frac{\sqrt{5}+1}{4} \right)^2 \left(\frac{\sqrt{5}-1}{4} \right)^2 \\ &= \left[\left(\frac{\sqrt{5}+1}{4} \right) \left(\frac{\sqrt{5}-1}{4} \right) \right]^2 = \frac{1}{16} \end{aligned}$$

ILLUSTRATION 3.69

Show that $4 \sin 27^\circ = (5 + \sqrt{5})^{1/2} - (3 - \sqrt{5})^{1/2}$.

$$\begin{aligned} \text{Sol.} \quad & 16 \sin^2 27^\circ = 8(1 - \cos 54^\circ) \\ &= 8 \left(1 - \frac{\sqrt{10-2\sqrt{5}}}{4} \right) = 2 \left(4 - \sqrt{10-2\sqrt{5}} \right) \\ &= 8 - 2\sqrt{10-2\sqrt{5}} \\ &= (5 + \sqrt{5}) + (3 - \sqrt{5}) - 2\sqrt{(5 + \sqrt{5})(3 - \sqrt{5})} \\ &= \left\{ \sqrt{5 + \sqrt{5}} - \sqrt{3 - \sqrt{5}} \right\}^2 \\ \Rightarrow \quad & 4 \sin 27^\circ = \left(\sqrt{5 + \sqrt{5}} \right) - \left(\sqrt{3 - \sqrt{5}} \right) \end{aligned}$$

ILLUSTRATION 3.70

Prove that $\tan \frac{\pi}{16} = \sqrt{4 + 2\sqrt{2}} - (\sqrt{2} + 1)$

$$\text{Sol.} \quad \tan \frac{\pi}{16} = \tan 11.25^\circ$$

We know that $\tan 22.5^\circ = \sqrt{2} - 1$

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

$$\text{Put} \quad \theta = 11.5^\circ$$

$$\therefore \tan 22.5^\circ = \frac{2 \tan 11.25^\circ}{1 - \tan^2 11.25^\circ}$$

$$\therefore (\sqrt{2} - 1)x^2 + 2x - (\sqrt{2} - 1) = 0, \text{ where } x = \tan 11.25^\circ$$

$$\therefore x = \frac{-2 + \sqrt{4 + 4(\sqrt{2} - 1)^2}}{2(\sqrt{2} - 1)}$$

$$= \frac{-1}{\sqrt{2} - 1} + \frac{\sqrt{4 - 2\sqrt{2}}}{\sqrt{2} - 1}$$

$$= -(\sqrt{2} + 1) + \sqrt{4 - 2\sqrt{2}} \cdot (\sqrt{2} + 1)$$

$$= -(\sqrt{2} + 1) + \sqrt{(4 - 2\sqrt{2})(\sqrt{2} + 1)^2}$$

$$= -(\sqrt{2} + 1) + \sqrt{(4 - 2\sqrt{2})(3 + 2\sqrt{2})}$$

$$= -(\sqrt{2} + 1) + \sqrt{4 + 2\sqrt{2}}$$

ILLUSTRATION 3.71

Find the quadratic equation whose roots are $\tan \left(\frac{\pi}{8} \right)$ and $\tan \left(\frac{5\pi}{8} \right)$?

Sol. Sum of roots,

$$S = \tan \frac{\pi}{8} + \tan \frac{5\pi}{8}$$

$$= \tan \frac{\pi}{8} + \tan \left(\frac{\pi}{2} + \frac{\pi}{8} \right)$$

$$= \tan \frac{\pi}{8} - \cot \frac{\pi}{8}$$

$$= (\sqrt{2} - 1) - (\sqrt{2} + 1)$$

$$= -2$$

$$\text{Product of roots, } P = \left(\tan \frac{\pi}{8} \right) \left(\tan \frac{5\pi}{8} \right)$$

$$= \tan \frac{\pi}{8} \left(-\cot \frac{\pi}{8} \right)$$

$$= -1$$

Hence, the required equation is $x^2 + 2x - 1 = 0$.

CONCEPT APPLICATION EXERCISE 3.5

- Find the value of $(\cos^2 66^\circ - \sin^2 6^\circ)(\cos^2 48^\circ - \sin^2 12^\circ)$.
- Prove that $4(\sin 24^\circ + \cos 6^\circ) = \sqrt{3} + \sqrt{15}$.
- Prove that $\sin 47^\circ + \sin 61^\circ - \sin 11^\circ - \sin 25^\circ = \cos 7^\circ$.
- Find the values of the following:

$$(a) \frac{1 + \tan^2 7\frac{1}{2}^\circ}{\tan 7\frac{1}{2}^\circ} \quad (b) \frac{\tan^2 82\frac{1}{2}^\circ - 1}{\tan 82\frac{1}{2}^\circ}$$

$$(c) \frac{\tan^2 37\frac{1}{2}^\circ + 1}{\tan^2 37\frac{1}{2}^\circ - 1} \quad (d) \frac{3 \tan^2 5^\circ - 1}{3 \tan 5^\circ - \tan^3 5^\circ}$$

$$5. \text{ Find the value of } \cot\left(11\frac{1}{4}^\circ\right) + \tan\left(112\frac{1}{2}^\circ\right) - \cot\left(112\frac{1}{2}^\circ\right) - \tan\left(11\frac{1}{4}^\circ\right).$$

$$6. \text{ Find the value of } \frac{\tan 9^\circ + \cot 9^\circ}{\tan 27^\circ + \cot 27^\circ}.$$

ANSWERS

- $\frac{1}{16}$
- (a) $2(\sqrt{6} + \sqrt{2})$ (b) $2(2 + \sqrt{3})$
- (c) $-(\sqrt{6} + \sqrt{2})$ (d) $-(2 + \sqrt{3})$
- $2\sqrt{2}$
- $\frac{3 + \sqrt{5}}{2}$

SOME IMPORTANT RESULTS AND THEIR APPLICATIONS

Result 1. $\cos A \cos(60^\circ - A) \cos(60^\circ + A) = \frac{1}{4} \cos 3A$

Proof:

We have

$$\begin{aligned} \text{L.H.S.} &= \cos A \cos(60^\circ - A) \cos(60^\circ + A) \\ &= \cos A (\cos^2 60^\circ - \sin^2 A) \\ & \quad [\because \cos(A+B) \cos(A-B) = \cos^2 A - \sin^2 B] \\ &= \cos A \left(\frac{1}{4} - \sin^2 A\right) = \cos A \left(\frac{1}{4} - (1 - \cos^2 A)\right) \\ &= \cos A \left(-\frac{3}{4} + \cos^2 A\right) \\ &= \frac{1}{4} \cos A (-3 + 4 \cos^2 A) = \frac{1}{4} (4 \cos^3 A - 3 \cos A) \\ &= \frac{1}{4} \cos 3A = \text{R.H.S.} \end{aligned}$$

Result 2. $\sin A \sin(60^\circ - A) \sin(60^\circ + A) = \frac{1}{4} \sin 3A$

Proof:

We have

$$\begin{aligned} \text{L.H.S.} &= \sin A \sin(60^\circ - A) \sin(60^\circ + A) \\ &= \sin A (\sin^2 60^\circ - \sin^2 A) \\ & \quad [\because \sin(A+B) \sin(A-B) = \sin^2 A - \sin^2 B] \end{aligned}$$

$$\begin{aligned} &= \sin A \left(\frac{3}{4} - \sin^2 A\right) \\ &= \frac{1}{4} (3 \sin A - 4 \sin^3 A) \\ &= \frac{1}{4} \sin 3A = \text{R.H.S.} \end{aligned}$$

Result 3. $\tan A \tan(60^\circ - A) \tan(60^\circ + A) = \tan 3A$

Using the above two results, we can prove this result.

ILLUSTRATION 3.72

Prove that $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = 1/16$.

$$\begin{aligned} \text{Sol.} \quad \cos 20^\circ \cos 40^\circ \cos 80^\circ \cos 60^\circ &= \cos 20^\circ \cos(60^\circ - 20^\circ) \cos(60^\circ + 20^\circ) \cos 60^\circ \\ &= \frac{1}{4} \cos(3 \times 20^\circ) \cos 60^\circ \\ &= \frac{1}{4} \cos^2 60^\circ = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16} \end{aligned}$$

ILLUSTRATION 3.73

Prove that $\sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ = 1/16$.

$$\begin{aligned} \text{Sol.} \quad \sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ &= \sin 10^\circ \sin(60^\circ - 10^\circ) \sin(60^\circ + 10^\circ) \sin 30^\circ \\ &= \frac{1}{4} \sin(3 \times 10^\circ) \sin 30^\circ = \frac{1}{4} \sin^2 30^\circ = \frac{1}{16} \end{aligned}$$

ILLUSTRATION 3.74

Prove that $\tan 20^\circ \tan 40^\circ \tan 80^\circ = \tan 60^\circ$.

$$\begin{aligned} \text{Sol.} \quad \tan 20^\circ \tan 40^\circ \tan 80^\circ &= \tan 20^\circ \tan(60^\circ - 20^\circ) \tan(60^\circ + 20^\circ) \\ &= \tan(3 \times 20^\circ) = \tan 60^\circ \end{aligned}$$

Result 4. $\cos A \cos 2A \cos 2^2 A \cos 2^3 A \cdots \cos 2^{n-1} A = \frac{\sin 2^n A}{2^n \sin A}$

Proof: L.H.S. = $\cos A \cos 2A \cos 2^2 A \cos 2^3 A \cdots \cos 2^{n-1} A$

$$\begin{aligned} &= \frac{1}{2 \sin A} [(2 \sin A \cos A) \cos 2A \cos 2^2 A \cos 2^3 A \cdots \cos 2^{n-1} A] \\ &= \frac{1}{2 \sin A} [(\sin 2A \cos 2A \cos 2^2 A \cos 2^3 A \cdots \cos 2^{n-1} A)] \\ &= \frac{1}{2^2 \sin A} [(2 \sin 2A \cos 2A) \cos 2^2 A \cos 2^3 A \cdots \cos 2^{n-1} A] \\ &= \frac{1}{2^2 \sin A} [\sin 2(2A) \cos 2^2 A \cos 2^3 A \cdots \cos 2^{n-1} A] \\ &= \frac{1}{2^3 \sin A} [(2 \sin 2^2 A \cos 2^2 A) \cos 2^3 A \cdots \cos 2^{n-1} A] \\ &= \frac{1}{2^3 \sin A} [\sin(2 \times 2^2 A) \cos 2^3 A \cdots \cos 2^{n-1} A] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2^3 \sin A} [(\sin 2^3 A \cos 2^3 A \cos 2^4 A \cdots \cos 2^{n-1} A) \\
&\quad \cdots \\
&\quad \cdots] \\
&= \frac{1}{2^{n-1} \sin A} [\sin 2^{n-1} A \cos 2^{n-1} A] \\
&= \frac{1}{2^n \sin A} [2 \sin 2^{n-1} A \cos 2^{n-1} A] \\
&= \frac{1}{2^n \sin A} \sin(2 \times 2^{n-1} A) \\
&= \frac{1}{2^n \sin A} \sin 2^n A = \text{R.H.S.}
\end{aligned}$$

ILLUSTRATION 3.75

If $\theta = \frac{\pi}{2^n + 1}$, show that $\cos \theta \cos 2\theta \cos 2^2\theta \cdots \cos 2^{n-1}\theta$
 $= \frac{1}{2^n}$.

Sol. In the above result, put $\theta = \frac{\pi}{2^n + 1}$

$$\begin{aligned}
\Rightarrow \text{R.H.S.} &= \frac{\sin 2^n \theta}{2^n \sin \theta} = \frac{\sin \left(\frac{\pi}{2^n + 1} \right) 2^n}{2^n \sin \left(\frac{\pi}{2^n + 1} \right)} = \frac{\sin \left(\frac{2^n + 1 - 1}{2^n + 1} \right) \pi}{2^n \sin \left(\frac{\pi}{2^n + 1} \right)} \\
&= \frac{\sin \left(\pi - \frac{\pi}{2^n + 1} \right)}{2^n \sin \left(\frac{\pi}{2^n + 1} \right)} \\
&= \frac{\sin \left(\frac{\pi}{2^n + 1} \right)}{2^n \sin \left(\frac{\pi}{2^n + 1} \right)} \\
&= \frac{1}{2^n}
\end{aligned}$$

ILLUSTRATION 3.76

Prove that $\cos \frac{2\pi}{15} \cos \frac{4\pi}{15} \cos \frac{8\pi}{15} \cos \frac{14\pi}{15} = \frac{1}{16}$.

Sol. We have

$$\begin{aligned}
\text{L.H.S.} &= \cos \frac{2\pi}{15} \cos \frac{4\pi}{15} \cos \frac{8\pi}{15} \cos \left(\pi - \frac{\pi}{15} \right) \\
&= \left(\cos \frac{2\pi}{15} \cos \frac{4\pi}{15} \cos \frac{8\pi}{15} \right) \left(-\cos \frac{\pi}{15} \right) \\
&= -\cos \frac{\pi}{15} \cos \frac{2\pi}{15} \cos \frac{4\pi}{15} \cos \frac{8\pi}{15} \\
&= -\cos A \cos 2A \cos 2^2 A \cos 2^3 A, \text{ where } A = \pi/15
\end{aligned}$$

$$\begin{aligned}
&= -\left[\frac{\sin 2^4 A}{2^4 \sin A} \right] = -\frac{\sin 16A}{2^4 \sin A} \\
&= -\frac{\sin(15A + A)}{16 \sin A} = -\frac{\sin(\pi + A)}{16 \sin A} \quad [\because 15A = \pi] \\
&= \frac{\sin A}{16 \sin A} = \frac{1}{16} = \text{R.H.S.}
\end{aligned}$$

ILLUSTRATION 3.77

Prove that $\sin 6^\circ \sin 42^\circ \sin 66^\circ \sin 78^\circ = 1/16$.

$$\begin{aligned}
\text{Sol.} \quad \sin 6^\circ \sin 42^\circ \sin 66^\circ \sin 78^\circ &= \sin 6^\circ \cos 48^\circ \cos 24^\circ \cos 12^\circ \\
&= \sin 6^\circ \frac{2^3 \sin 12^\circ \cos 12^\circ \cos 24^\circ \cos 48^\circ}{2^3 \sin 12^\circ} \\
&= \sin 6^\circ \frac{\sin 96^\circ}{2^3 \sin 12^\circ} \\
&= \frac{2 \sin 6^\circ \cos 6^\circ}{2^4 \sin 12^\circ} \\
&= \frac{\sin 12^\circ}{2^4 \sin 12^\circ} \\
&= \frac{1}{16}
\end{aligned}$$

ILLUSTRATION 3.78

Find the value of $2 \cos^3 \frac{\pi}{7} - \cos^2 \frac{\pi}{7} - \cos \frac{\pi}{7}$.

Sol. Let $\frac{\pi}{7} = \theta$

$$\begin{aligned}
\text{Now } 2 \cos^3 \theta - \cos^2 \theta - \cos \theta &= \cos \theta [2 \cos^2 \theta - 1] - \cos^2 \theta \\
&= \cos \theta \cos 2\theta - \cos^2 \theta \\
&= \cos \theta [\cos 2\theta - \cos \theta] \\
&= -2 \cos \theta \cdot \sin \frac{3\theta}{2} \sin \frac{\theta}{2} \\
&= -2 \cos \frac{2\pi}{14} \sin \frac{3\pi}{14} \sin \frac{\pi}{14} \\
&= -2 \cos \frac{2\pi}{14} \cos \frac{4\pi}{14} \cos \frac{6\pi}{14} \\
&= -2 \cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{3\pi}{7} \\
&= 2 \cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{4\pi}{7} \\
&= \frac{\sin 8\pi/7}{4 \sin \pi/7} \\
&= \frac{\sin(\pi + (\pi/7))}{4 \sin(\pi/7)} = -\frac{1}{4}
\end{aligned}$$

CONCEPT APPLICATION EXERCISE 3.6

1. Prove that $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \frac{3}{16}$.
2. Prove that $\cos 10^\circ \cos 30^\circ \cos 50^\circ \cos 70^\circ = \frac{3}{16}$.
3. Prove that $\sin 12^\circ \sin 18^\circ \sin 42^\circ \sin 48^\circ \sin 72^\circ \sin 78^\circ = \frac{\cos 18^\circ}{32}$.
4. Find the value of $\sin \frac{9\pi}{14} \cdot \sin \frac{11\pi}{14} \sin \frac{13\pi}{14}$.
5. Find the value of $\sin \frac{\pi}{14} \sin \frac{3\pi}{14} \sin \frac{5\pi}{14} \sin \frac{7\pi}{14} \sin \frac{9\pi}{14} \sin \frac{11\pi}{14} \sin \frac{13\pi}{14}$.

ANSWERS

4. $1/8$ 5. $1/64$

SUM OF COSINES AND SINES WHEN ANGLES ARE IN A.P.

$$\begin{aligned} & \sin \alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \cdots + \sin(\alpha + \overline{n-1}\beta) \\ &= \frac{\sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}} \times \sin \left[\alpha + (n-1) \frac{\beta}{2} \right] \end{aligned}$$

Proof:

Let $S = \sin \alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \cdots + \sin(\alpha + \overline{n-1}\beta)$.
Here angles are in A.P. and common difference of angles is β .

Multiplying both sides by $2 \sin \frac{\beta}{2}$, we get

$$\begin{aligned} 2S \sin \frac{\beta}{2} &= 2 \sin \alpha \sin \frac{\beta}{2} + 2 \sin(\alpha + \beta) \sin \frac{\beta}{2} + \cdots \\ &\quad + 2 \sin(\alpha + \overline{n-1}\beta) \sin \frac{\beta}{2} \end{aligned} \quad (i)$$

$$\text{Now, } 2 \sin \alpha \sin \frac{\beta}{2} = \cos \left(\alpha - \frac{\beta}{2} \right) - \cos \left(\alpha + \frac{\beta}{2} \right)$$

$$2 \sin(\alpha + \beta) \sin \frac{\beta}{2} = \cos \left(\alpha + \frac{\beta}{2} \right) - \cos \left(\alpha + \frac{3\beta}{2} \right)$$

$$2 \sin(\alpha + 2\beta) \sin \frac{\beta}{2} = \cos \left(\alpha + \frac{3\beta}{2} \right) - \cos \left(\alpha + \frac{5\beta}{2} \right)$$

:

$$\begin{aligned} 2 \sin(\alpha + \overline{n-1}\beta) \sin \frac{\beta}{2} &= \cos \left[\alpha + (2n-3) \frac{\beta}{2} \right] \\ &\quad - \cos \left[\alpha + (2n-1) \frac{\beta}{2} \right] \end{aligned}$$

Adding, we get

$$2 \sin \frac{\beta}{2} S = \cos \left(\alpha - \frac{\beta}{2} \right) - \cos \left[\alpha + (2n-1) \frac{\beta}{2} \right]$$

$$\begin{aligned} \text{or } 2 \sin \frac{\beta}{2} S &= 2 \sin \left[\alpha + (n-1) \frac{\beta}{2} \right] \sin \frac{n\beta}{2} \\ \Rightarrow S &= \frac{\sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}} \sin \left[\alpha + (n-1) \frac{\beta}{2} \right] \end{aligned}$$

In the above result, replacing α by $\pi/2 + \alpha$, we get

$$\begin{aligned} & \cos \alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \cdots + \cos(\alpha + \overline{n-1}\beta) \\ &= \frac{\sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}} \cos \left[\alpha + (n-1) \frac{\beta}{2} \right] \end{aligned}$$

ILLUSTRATION 3.79

Find the value of $\cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7}$.

$$\begin{aligned} \text{Sol. } S &= \cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7} \\ &= \frac{\sin \left(3 \frac{\pi}{7} \right)}{\sin \left(\frac{\pi}{7} \right)} \cos \left(\frac{\pi}{7} + \frac{3\pi}{7} \right) = \frac{2 \sin \left(\frac{3\pi}{7} \right) \cos \left(\frac{4\pi}{7} \right)}{2 \sin \left(\frac{\pi}{7} \right)} \\ &= \frac{\sin \left(\frac{7\pi}{7} \right) - \sin \left(\frac{\pi}{7} \right)}{2 \sin \left(\frac{\pi}{7} \right)} = -\frac{1}{2} \end{aligned}$$

ILLUSTRATION 3.80

Prove that $\sin \theta + \sin 3\theta + \sin 5\theta + \cdots + \sin(2n-1)\theta = \frac{\sin^2 n\theta}{\sin \theta}$.

$$\begin{aligned} \text{Sol. } & \sin \theta + \sin 3\theta + \sin 5\theta + \cdots + \sin(2n-1)\theta \\ &= \frac{\sin \left[n \left(\frac{2\theta}{2} \right) \right]}{\sin \left(\frac{2\theta}{2} \right)} \sin \left(\frac{\theta + (2n-1)\theta}{2} \right) = \frac{\sin^2 n\theta}{\sin \theta} \end{aligned}$$

ILLUSTRATION 3.81

Prove that

$$\begin{aligned} & \frac{\cos 3x}{\sin 2x \sin 4x} + \frac{\cos 5x}{\sin 4x \sin 6x} + \frac{\cos 7x}{\sin 6x \sin 8x} + \frac{\cos 9x}{\sin 8x \sin 10x} \\ &= \frac{1}{2} (\operatorname{cosec} x) [\operatorname{cosec} 2x - \operatorname{cosec} 10x] \end{aligned}$$

Sol. Let

$$\begin{aligned} f(x) &= \frac{\cos 3x}{\sin 2x \sin 4x} + \frac{\cos 5x}{\sin 4x \sin 6x} + \frac{\cos 7x}{\sin 6x \sin 8x} \\ &\quad + \frac{\cos 9x}{\sin 8x \sin 10x} \end{aligned}$$

Multiply and divide by $(2 \sin x)$ in whole expression to get

$$\begin{aligned} f(x) &= \frac{\sin 4x - \sin 2x}{2 \sin x \sin 2x \sin 4x} + \frac{\sin 6x - \sin 4x}{2 \sin x \sin 4x \sin 6x} \\ &\quad + \frac{\sin 8x - \sin 6x}{2 \sin x \sin 6x \sin 8x} + \frac{\sin 10x - \sin 8x}{2 \sin x \sin 8x \sin 10x} \\ &= (1/2) \operatorname{cosec} x [\operatorname{cosec} 2x - \operatorname{cosec} 10x] \end{aligned}$$

ILLUSTRATION 3.82

Prove that $2 \sin 2^\circ + 4 \sin 4^\circ + 6 \sin 6^\circ + \dots + 180 \sin 180^\circ = 90 \cot 1^\circ$.

Sol. $x = 2 \sin 2^\circ + 4 \sin 4^\circ + 6 \sin 6^\circ + \dots$
 $\quad + 178 \sin 178^\circ + 180 \sin 180^\circ$
 $= (2 \sin 2^\circ + 178 \sin 178^\circ) + (4 \sin 4^\circ + 176 \sin 176^\circ)$
 $\quad + (6 \sin 6^\circ + 174 \sin 174^\circ)$
 $\quad \dots$
 $\quad \dots$
 $\quad + (88 \sin 88^\circ + 72 \sin 72^\circ)$
 $= (2 \sin 2^\circ + 178 \sin 2^\circ) + (4 \sin 4^\circ + 176 \sin 4^\circ)$
 $\quad + (6 \sin 6^\circ + 174 \sin 6^\circ)$
 $\quad \dots$
 $\quad \dots$
 $\quad + (88 \sin 88^\circ + 72 \sin 88^\circ) + 90 \sin 90^\circ$
 $= 180(\sin 2^\circ + \sin 4^\circ + \sin 6^\circ + \dots + \sin 88^\circ) + 90$
 $\therefore x = 180 \frac{\sin \frac{44 \times 2^\circ}{2}}{\sin 1^\circ} \sin \frac{2^\circ + 88^\circ}{2} + 90$
 $= 90 \left(\frac{2 \sin 44^\circ \sin 45^\circ}{\sin 1^\circ} + 1 \right) = 90 \left(\frac{\cos 1^\circ - \cos 89^\circ}{\sin 1^\circ} + 1 \right)$
 $= 90 \left(\cot 1^\circ - \frac{\sin 1^\circ}{\sin 1^\circ} + 1 \right) = 90 \cot 1^\circ$

CONCEPT APPLICATION EXERCISE 3.7

- Find the value of $\cos \frac{\pi}{11} + \cos \frac{3\pi}{11} + \cos \frac{5\pi}{11} + \cos \frac{7\pi}{11} + \cos \frac{9\pi}{11}$.
- Find the average value of $\sin 2^\circ, \sin 4^\circ, \sin 6^\circ, \dots, \sin 180^\circ$.
- Find the value of $\sum_{r=1}^{n-1} \sin^2 \frac{r\pi}{n}$.
- Sum the series : $\sqrt{1 + \cos \alpha} + \sqrt{1 + \cos 2\alpha} + \sqrt{1 + \cos 3\alpha} + \dots$ to n terms, where $0 < \alpha < \pi$
- Find the value of $(\cos^4 1^\circ + \cos^4 2^\circ + \cos^4 3^\circ + \dots + \cos^4 179^\circ) - (\sin^4 1^\circ + \sin^4 2^\circ + \sin^4 3^\circ + \dots + \sin^4 179^\circ)$.

ANSWERS

- 1/2
- $\frac{1}{90}(\cot 1^\circ)$
- $n/2$
- $\sqrt{2} \frac{\sin \frac{n\alpha}{4}}{\sin \frac{\alpha}{4}} \cos \left[(n+1) \frac{\alpha}{4} \right]$
- 1

CONDITIONAL IDENTITIES**SOME STANDARD IDENTITIES IN TRIANGLE**

1. $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

Proof:

$$\begin{aligned} &(\sin 2A + \sin 2B) + \sin 2C \\ &= 2 \sin(A+B) \cos(A-B) + \sin 2C \\ &= 2 \sin(\pi - C) \cos(A-B) + \sin 2C \\ &= 2 \sin C \cos(A-B) + 2 \sin C \cos C \\ &= 2 \sin C [\cos(A-B) + \cos C] \\ &= 2 \sin C [\cos(A-B) + \cos \{\pi - (A+B)\}] \\ &= 2 \sin C [\cos(A-B) - \cos(A+B)] \\ &= 2 \sin C \times 2 \sin A \sin B \\ &= 4 \sin A \sin B \sin C \end{aligned}$$

2. $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$

Proof:

$$\begin{aligned} &(\cos 2A + \cos 2B) + \cos 2C \\ &= 2 \cos(A+B) \cos(A-B) + 2 \cos^2 C - 1 \\ &= 2 \cos(\pi - C) \cos(A-B) + 2 \cos^2 C - 1 \\ &= -2 \cos C \cos(A-B) + 2 \cos^2 C - 1 \\ &= -2 \cos C [\cos(A-B) - \cos C] - 1 \\ &= -2 \cos C [\cos(A-B) - \cos \{\pi - (A+B)\}] \\ &= -2 \cos C [\cos(A-B) + \cos(A+B)] - 1 \\ &= -1 - 4 \cos A \cos B \cos C \end{aligned}$$

3. $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

Proof:

$$\begin{aligned} &(\cos A + \cos B) + \cos C - 1 \\ &= 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} + \cos C - 1 \\ &= 2 \cos \left(\frac{\pi}{2} - \frac{C}{2} \right) \cos \frac{A-B}{2} + \cos C - 1 \\ &= 2 \sin \frac{C}{2} \cos \frac{A-B}{2} + 1 - 2 \sin^2 \frac{C}{2} - 1 \\ &= 2 \sin \frac{C}{2} \cos \frac{A-B}{2} - 2 \sin^2 \frac{C}{2} \\ &= 2 \sin \frac{C}{2} \left[\cos \frac{A-B}{2} - \sin \frac{C}{2} \right] \\ &= 2 \cos \frac{C}{2} \left[\cos \frac{A-B}{2} - \sin \left(\frac{\pi}{2} - \frac{A+B}{2} \right) \right] \\ &= 2 \sin \frac{C}{2} \left[\cos \frac{A-B}{2} - \cos \frac{A+B}{2} \right] \\ &= 2 \sin \frac{C}{2} \left(2 \sin \frac{A}{2} \sin \frac{B}{2} \right) \\ &= 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \end{aligned}$$

$$4. \sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$$

Proof:

$$\begin{aligned} & (\sin A + \sin B) + \sin C \\ &= 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} + \sin C \\ &= 2 \sin \left(\frac{\pi}{2} - \frac{C}{2} \right) \cos \frac{A-B}{2} + \sin C \\ &= 2 \cos \frac{C}{2} \cos \frac{A-B}{2} + 2 \sin \frac{C}{2} \cos \frac{C}{2} \\ &= 2 \cos \frac{C}{2} \left[\cos \frac{A-B}{2} + \sin \frac{C}{2} \right] \\ &= 2 \cos \frac{C}{2} \left[\cos \frac{A-B}{2} + \sin \left(\frac{\pi}{2} - \frac{A+B}{2} \right) \right] \\ &= 2 \cos \frac{C}{2} \left[\cos \frac{A-B}{2} + \cos \frac{A+B}{2} \right] \\ &= 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \end{aligned}$$

ILLUSTRATION 3.83

If $A+B+C=180^\circ$, prove that $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C$.

$$\begin{aligned} \text{Sol. } \cos^2 A + \cos^2 B + \cos^2 C &= \frac{1+\cos 2A}{2} + \frac{1+\cos 2B}{2} + \frac{1+\cos 2C}{2} \\ &= \frac{1}{2} (\cos 2A + \cos 2B + \cos 2C) + \frac{3}{2} \\ &= \frac{1}{2} (-1 - 4 \cos A \cos B \cos C) + \frac{3}{2} \\ &= 1 - 2 \cos A \cos B \cos C \end{aligned}$$

ILLUSTRATION 3.84

Prove that in triangle ABC , $\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2 \sin A \sin B \cos C$.

$$\begin{aligned} \text{Sol. } \cos^2 A + \cos^2 B - \cos^2 C &= \cos^2 A + \sin^2 C - \sin^2 B \\ &= \cos^2 A + \sin(C+B)\sin(C-B) \\ &= 1 - \sin^2 A + \sin A \sin(C-B) \\ &= 1 - \sin A [\sin A - \sin(C-B)] \\ &= 1 - \sin A [\sin(B+C) - \sin(C-B)] \\ &= 1 - 2 \sin A \sin B \cos C \end{aligned}$$

ILLUSTRATION 3.85

In triangle ABC , prove that $\sin(B+C-A) + \sin(C+A-B) + \sin(A+B-C) = 4 \sin A \sin B \sin C$.

$$\begin{aligned} \text{Sol. } \sin(B+C-A) + \sin(C+A-B) + \sin(A+B-C) &= \sin(\pi - 2A) + \sin(\pi - 2B) + \sin(\pi - 2C) \\ &= \sin 2A + \sin 2B + \sin 2C \\ &= 4 \sin A \sin B \sin C \end{aligned}$$

ILLUSTRATION 3.86

If $A+B+C=\pi$, prove that

$$\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} - \sin^2 \frac{C}{2} = 1 - 2 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}.$$

$$\begin{aligned} \text{Sol. } \sin^2 \frac{A}{2} - \sin^2 \frac{C}{2} + \sin^2 \frac{B}{2} &= \sin \left(\frac{A+C}{2} \right) \sin \left(\frac{A-C}{2} \right) + 1 - \cos^2 \frac{B}{2} \\ &= \cos \left(\frac{B}{2} \right) \sin \left(\frac{A-C}{2} \right) - \cos^2 \frac{B}{2} + 1 \\ &= \cos \left(\frac{B}{2} \right) \left[\sin \left(\frac{A-C}{2} \right) - \cos \frac{B}{2} \right] + 1 \\ &= \cos \left(\frac{B}{2} \right) \left[\sin \left(\frac{A-C}{2} \right) - \sin \left(\frac{A+C}{2} \right) \right] + 1 \\ &= 1 - 2 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2} \end{aligned}$$

ILLUSTRATION 3.87

In triangle ABC , prove that $\sin^3 A \cos(B-C) + \sin^3 B \cos(C-A) + \sin^3 C \cos(A-B) = 3 \sin A \sin B \sin C$.

$$\begin{aligned} \text{Sol. } \text{LHS} &= \sin^2 A \sin(B+C) \cos(B-C) + \sin^2 B \sin(C+A) \cos(C-A) \\ &\quad + \sin^2 C \sin(A+B) \cos(A-B) \\ &= \frac{1}{2} \sin^2 A (\sin 2B + \sin 2C) + \frac{1}{2} \sin^2 B (\sin 2C + \sin 2A) \\ &\quad + \frac{1}{2} \sin^2 C (\sin 2A + \sin 2B) \\ &= \sin^2 A (\sin B \cos B + \sin C \cos C) \\ &\quad + \sin^2 B (\sin C \cos C + \sin A \cos A) \\ &\quad + \sin^2 C (\sin A \cos A + \sin B \cos B) \\ &= \sin A \sin B (\sin A \cos B + \cos A \sin B) \\ &\quad + \sin B \sin C (\sin B \cos C + \cos B \sin C) \\ &\quad + \sin C \sin A (\sin A \cos C + \cos A \sin C) \\ &= \sin A \sin B \sin(A+B) + \sin B \sin C \sin(B+C) \\ &\quad + \sin C \sin A \sin(A+C) \\ &= 3 \sin A \sin B \sin C = \text{RHS.} \end{aligned}$$

ILLUSTRATION 3.88

If $A+B+C=\pi$, prove that $\cot A + \cot B + \cot C = \text{cosec } A \cdot \text{cosec } B \cdot \text{cosec } C$.

$$\begin{aligned} \text{Sol. } \cot A + \cot B + \cot C - \text{cosec } A \cdot \text{cosec } B \cdot \text{cosec } C &= \frac{\cos A}{\sin A} + \frac{\cos B}{\sin B} + \frac{\cos C}{\sin C} - \frac{1}{\sin A \cdot \sin B \cdot \sin C} \\ &= \frac{(\cos A \sin B \sin C + \cos B \sin C \sin A + \cos C \sin A \sin B - 1)}{\sin A \sin B \sin C} \end{aligned}$$

$$\begin{aligned}
& \frac{\{\sin C(\cos A \sin B + \cos B \sin A) + \cos C \sin A \sin B - 1\}}{\sin A \sin B \sin C} \\
&= \frac{\sin C \sin(A+B) + \cos C \sin A \sin B - 1}{\sin A \sin B \sin C} \\
&= \frac{\sin^2 C + \cos C \sin A \sin B - 1}{\sin A \sin B \sin C} \\
&\quad \{\because \sin(A+B) = \sin(\pi - C) = \sin C\} \\
&= \frac{\cos C \cdot \sin A \sin B - \cos^2 C}{\sin A \sin B \sin C} \\
&= \frac{\cos C \{\sin A \sin B - \cos(\pi - (A+B))\}}{\sin A \sin B \sin C} \\
&= \frac{\cos C \cdot \{\sin A \sin B + \cos(A+B)\}}{\sin A \sin B \sin C} \\
&= \frac{\cos C \cdot \cos A \cdot \cos B}{\sin A \sin B \sin C} \\
&= \cot A \cdot \cot B \cdot \cot C
\end{aligned}$$

ILLUSTRATION 3.89

If $A + B + C = \pi$, prove that

$$\frac{\tan A}{\tan B \cdot \tan C} + \frac{\tan B}{\tan A \cdot \tan C} + \frac{\tan C}{\tan A \cdot \tan B} = \tan A + \tan B + \tan C - 2\cot A - 2\cot B - 2\cot C$$

Sol.

$$\begin{aligned}
& \frac{\tan A}{\tan B \cdot \tan C} + \frac{\tan B}{\tan A \cdot \tan C} + \frac{\tan C}{\tan A \cdot \tan B} \\
&= \frac{\tan^2 A + \tan^2 B + \tan^2 C}{\tan A \cdot \tan B \cdot \tan C} \\
&= \frac{(\tan A + \tan B + \tan C)^2 - 2(\tan A \cdot \tan B + \tan B \cdot \tan C + \tan C \cdot \tan A)}{\tan A \cdot \tan B \cdot \tan C} \\
&= \frac{(\tan A + \tan B + \tan C)^2}{\tan A + \tan B + \tan C} - \frac{2(\tan A \cdot \tan B + \tan B \cdot \tan C + \tan C \cdot \tan A)}{\tan A \cdot \tan B \cdot \tan C} \\
&\quad (\because \tan A + \tan B + \tan C = \tan A \tan B \tan C) \\
&= \tan A + \tan B + \tan C - 2\cot A - 2\cot B - 2\cot C
\end{aligned}$$

ILLUSTRATION 3.90

In triangle ABC , if $\cot A \cdot \cot C = \frac{1}{2}$ and $\cot B \cdot \cot C = \frac{1}{18}$, then find the value of $\tan C$.

Sol. Let $x = \tan C$,

$$\text{so } \tan A = \frac{2}{x} \text{ and } \tan B = \frac{18}{x}$$

As, in $\triangle ABC$, $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$$\Rightarrow \frac{2}{x} + \frac{18}{x} + x = \frac{2}{x} \cdot \frac{18}{x} \cdot x$$

$$\Rightarrow 20 + x^2 = 36$$

$$\Rightarrow x = 4 = \tan C \quad (\because \cot A \cot C = \frac{1}{2}, \text{ so } \tan C = 4)$$

ILLUSTRATION 3.91

If $\cos(A+B+C) = \cos A \cos B \cos C$, then find the value of $\frac{8 \sin(B+C) \sin(C+A) \sin(A+B)}{\sin 2A \sin 2B \sin 2C}$.

Sol. We have to prove that

$$\begin{aligned}
& \cos A \cos(B+C) - \sin A \sin(B+C) = \cos A \cos B \cos C - \sin A \sin B \sin C \\
\Rightarrow & \cos A (\cos(B+C) - \cos B \cos C) = \sin A (\sin(B+C) - \sin B \sin C) \\
\Rightarrow & \sin(B+C) = -\frac{\cos A \sin B \sin C}{\sin A}
\end{aligned}$$

Similarly

$$\begin{aligned}
\sin(C+A) &= -\frac{\cos B \sin C \sin A}{\sin B} \\
\text{and } \sin(A+B) &= -\frac{\cos C \sin A \sin B}{\sin C} \\
\therefore \sin(A+B) \sin(B+C) \sin(C+A) \\
&= -\frac{\cos A \sin B \sin C}{\sin A} \cdot \frac{\cos B \sin C \sin A}{\sin B} \cdot \frac{\cos C \sin A \sin B}{\sin C} \\
&= -\frac{1}{8} \sin 2A \sin 2B \sin 2C \\
\Rightarrow \frac{8 \sin(B+C) \sin(C+A) \sin(A+B)}{\sin 2A \sin 2B \sin 2C} &= -1
\end{aligned}$$

ILLUSTRATION 3.92

If $x + y + z = \frac{\pi}{2}$, then prove that

$$\begin{vmatrix} \sin x & \sin y & \sin z \\ \cos x & \cos y & \cos z \\ \cos^3 x & \cos^3 y & \cos^3 z \end{vmatrix} = 0.$$

Sol.

$$D = \begin{vmatrix} \sin x & \sin y & \sin z \\ \cos x & \cos y & \cos z \\ \cos^3 x & \cos^3 y & \cos^3 z \end{vmatrix}$$

Expanding along R_3 , we get

$$D = \Sigma \cos^3 x \sin(y-z)$$

$$\text{Given } x + y + z = \frac{\pi}{2}$$

$$\begin{aligned}
\therefore D &= \Sigma \sin^3(y+z) \sin(y-z) \\
&= \Sigma \sin^2(y+z) \sin(y+z) \sin(y-z) \\
&= \Sigma (1 - \sin^2 x) (\sin^2 y - \sin^2 z) \\
&= 0
\end{aligned}$$

ILLUSTRATION 3.93

The product of the sines of the angles of a triangle is p and the product of their cosines is q . Show that the tangents of the angles are the roots of the equation $qx^3 - px^2 + (1+q)x - p = 0$.

Sol. From the question,

$$\sin A \sin B \sin C = p \text{ and } \cos A \cos B \cos C = q$$

$$\tan A \tan B \tan C = p/q \quad (i)$$

$$\therefore \tan A + \tan B + \tan C = \tan A \tan B \tan C = p/q \quad (ii)$$

$$\text{Also, } \tan A \tan B + \tan B \tan C + \tan C \tan A$$

$$\text{Now, } \frac{\sin A \sin B \cos C + \sin B \sin C \cos A + \sin C \sin A \cos B}{\cos A \cos B \cos C}$$

$$= \frac{1}{2q} [(\sin^2 A + \sin^2 B - \sin^2 C) + (\sin^2 B + \sin^2 C - \sin^2 A) + (\sin^2 C + \sin^2 A - \sin^2 B)]$$

$$[\because A + B + C = \pi \text{ and } 2 \sin A \sin B \cos C$$

$$= \sin^2 A + \sin^2 B - \sin^2 C]$$

$$= \frac{1}{2q} [\sin^2 A + \sin^2 B + \sin^2 C]$$

$$= \frac{1}{4q} [3 - (\cos 2A + \cos 2B + \cos 2C)]$$

$$= \frac{1}{q} [1 + \cos A \cos B \cos C] = \frac{1}{q} (1 + q)$$

The equation whose roots are $\tan A$, $\tan B$ and $\tan C$ will be given by $x^3 - (\tan A + \tan B + \tan C)x^2 + (\tan A \tan B + \tan B \tan C + \tan C \tan A)x - \tan A \tan B \tan C = 0$

$$\text{or } x^3 - \frac{p}{q}x^2 + \frac{1+q}{q}x - \frac{p}{q} = 0$$

$$\text{or } qx^3 - px^2 + (1+q)x - p = 0$$

CONCEPT APPLICATION EXERCISE 3.8

1. In triangle ABC , prove that

$$(a) \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} - \cos^2 \frac{C}{2} = 2 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}$$

$$(b) \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} = 2 + 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

2. If $A + B + C = \pi/2$, show that

$$(a) \sin^2 A + \sin^2 B + \sin^2 C = 1 - 2 \sin A \sin B \sin C$$

$$(b) \cos^2 A + \cos^2 B + \cos^2 C = 2 + 2 \sin A \sin B \sin C$$

3. (a) If $A + B = C$, prove that $\cos^2 A + \cos^2 B + \cos^2 C$

$$= 1 + 2 \cos A \cos B \cos C.$$

$$(b) \text{ If } \alpha + \beta = 60^\circ, \text{ prove that } \cos^2 \alpha + \cos^2 \beta - \cos \alpha \cos \beta = 3/4.$$

4. Prove that $\cos^2(\beta - \gamma) + \cos^2(\gamma - \alpha) + \cos^2(\alpha - \beta)$

$$= 1 + 2 \cos(\beta - \alpha) \cos(\gamma - \alpha) \cos(\alpha - \beta).$$

5. If $A + B + C = \pi/2$, show that

$$(a) \cot A + \cot B + \cot C = \cot A \cot B \cot C$$

$$(b) \tan A \tan B + \tan B \tan C + \tan C \tan A = 1$$

6. If $A + B + C = \pi$, prove that

$$(a) \tan 3A + \tan 3B + \tan 3C = \tan 3A \tan 3B \tan 3C$$

$$(b) \cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$$

7. If $A + B + C = \pi$, prove that

$$\frac{\cos A}{\sin B \sin C} + \frac{\cos B}{\sin C \sin A} + \frac{\cos C}{\sin A \sin B} = 2.$$

8. In a triangle ABC , $\cos 3A + \cos 3B + \cos 3C = 1$ and $\angle A + \angle B < \angle C$, then find possible measure of $\angle C$.

9. In $\triangle ABC$ if $2 \sin^2 C = 2 + \cos 2A + \cos 2B$, then prove that triangle is right angled.

ANSWERS

8. 120°

FINDING RANGE OF EXPRESSION USING TRIGONOMETRIC SUBSTITUTION

Range of different trigonometric functions helps us to use these functions as substitutions in many algebraic expressions.

This simplifies the given expression to another expression, the range of which can be easily determined.

ILLUSTRATION 3.94

If $x^2 + y^2 = 4$, then find the maximum value of $\frac{x^3 + y^3}{x + y}$.

Sol. $x^2 + y^2 = 4$

Let $x = 2 \cos \theta$,

$$y = 2 \sin \theta$$

Now, $\frac{x^3 + y^3}{x + y} = x^2 + y^2 - xy$

$$= 4 \cos^2 \theta + 4 \sin^2 \theta - (2 \sin \theta)(2 \cos \theta)$$

$$= 4 - 2 \sin 2\theta$$

$$\Rightarrow \left(\frac{x^3 + y^3}{x + y} \right)_{\max} = 4 - 2(-1) = 6$$

ILLUSTRATION 3.95

If $\frac{x^2}{4} + \frac{y^2}{9} = 1$, then find the range of $2x + y$.

Sol. We have $\left(\frac{x}{2} \right)^2 + \left(\frac{y}{3} \right)^2 = 1$

Put $x = 2 \cos \theta$ and $y = 3 \sin \theta$

$$\therefore 2x + y = 4 \cos \theta + 3 \sin \theta$$

$$4 \cos \theta + 3 \sin \theta \in \left[-\sqrt{4^2 + 3^2}, \sqrt{4^2 + 3^2} \right] \text{ or } [-5, 5]$$

$$\therefore 2x + y \in [-5, 5]$$

ILLUSTRATION 3.96

If $x^2 + y^2 = x^2 y^2$ then find the range of $\frac{5x+12y+7xy}{xy}$.

Sol. We have $x^2 + y^2 = x^2 y^2$ or $\left(\frac{1}{x}\right)^2 + \left(\frac{1}{y}\right)^2 = 1$

Let $x = \sec \theta$ and $y = \operatorname{cosec} \theta$

$$\Rightarrow \frac{5x+12y+7xy}{xy} = 5 \sin \theta + 12 \cos \theta + 7 \in [-6, 20]$$

$$\therefore \frac{5x+12y+7xy}{xy} \in [-6, 20]$$

ILLUSTRATION 3.97

For all, $x, y \in R$, find the range of $\frac{(x+y)(1-xy)}{(1+x^2)(1+y^2)}$.

Sol. Let $x = \tan \alpha$, $y = \tan \beta$

$$\frac{(x+y)(1-xy)}{(1+x^2)(1+y^2)}$$

$$= \frac{(\tan \alpha + \tan \beta)(1 - \tan \alpha \tan \beta)}{\sec^2 \alpha \sec^2 \beta}$$

$$= \sin(\alpha + \beta) \cos(\alpha + \beta)$$

$$= \frac{1}{2} \sin(2\alpha + 2\beta)$$

$$\therefore \text{Thus, required range is } \left[-\frac{1}{2}, \frac{1}{2}\right].$$

ILLUSTRATION 3.98

If $x, y \in R$ and $x^2 + y^2 + xy = 1$, then find the minimum value of $x^3 y + xy^3 + 4$.

Sol. $x^2 + y^2 + xy = 1$

$$\text{Let } x = r \cos \theta, y = r \sin \theta$$

$$\therefore r^2 \cos^2 \theta + r^2 \sin^2 \theta + r^2 (\cos \theta)(\sin \theta) = 1$$

$$\therefore r^2 \sin \theta \cos \theta = 1 - r^2$$

$$\therefore r^2 = \frac{2}{2 + \sin 2\theta}$$

$$\text{Now } -1 \leq \sin 2\theta \leq 1$$

$$\therefore 1 \leq 2 + \sin 2\theta \leq 3$$

$$\therefore \frac{1}{3} \leq \frac{1}{2 + \sin 2\theta} \leq 1$$

$$\therefore \frac{2}{3} \leq \frac{2}{2 + \sin 2\theta} \leq 2$$

$$\text{or } \frac{2}{3} \leq r^2 \leq 2$$

$$\text{Now } E = x^3 y + xy^3 + 4$$

$$= xy(x^2 + y^2) + 4$$

$$= r^2 \sin \theta \cos \theta (r^2 \cos^2 \theta + r^2 \sin^2 \theta) + 4$$

$$= r^2 (1 - r^2) + 4$$

$$= \frac{17}{4} - \left(r^2 - \frac{1}{2}\right)^2$$

$$\therefore E_{\min} = \frac{17}{4} - \left(2 - \frac{1}{2}\right)^2 = 2$$

CONCEPT APPLICATION EXERCISE 3.9

1. Let $x, y \in R$, then find the maximum and minimum values

$$\text{of expression } \frac{x^2 + y^2}{x^2 + xy + 4y^2}.$$

2. Let $a^2 + b^2 = \alpha^2 + \beta^2 = 2$. Then show that the maximum value of $S = (1-a)(a-b) + (1-\alpha)(1-\beta)$ is 8.

3. If $x^2 + 2y^2 + 2xy = 1$, then find the maximum value of $x^2 + y^2$.

4. If $\frac{x^2}{144} - \frac{y^2}{25} = 1$, then find the range of $\frac{144}{x} + \frac{25}{y}$.

5. If $x^2 + y^2 + 6x - 4y - 12 = 0$, then find the range of $2x + y$.

ANSWERS

$$1. \text{ Max. value} = \frac{2}{5 - \sqrt{10}}; \text{ Min. value} = \frac{2}{5 + \sqrt{10}}$$

$$3. \frac{2}{3 - \sqrt{5}} \quad 4. [-13, 13] \quad 5. [-5\sqrt{5} - 4, 5\sqrt{5} - 4]$$

INEQUALITIES IN TRIANGLE**SOME STANDARD INEQUALITIES IN TRIANGLE**

1. In acute angled triangle ABC , $\tan A + \tan B + \tan C \geq 3\sqrt{3}$

Proof:

Method I: Using A.M. $\geq$ G.M.

We know that in a triangle,

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C$$

Since triangle is acute angled, $\tan A$, $\tan B$ and $\tan C$ are positive.

So, using A.M. $\geq$ G.M., we get

$$\frac{\tan A + \tan B + \tan C}{3} \geq \sqrt[3]{\tan A \tan B \tan C}$$

$$\Rightarrow \tan A \tan B \tan C \geq 3 \sqrt[3]{\tan A \tan B \tan C}$$

$$\Rightarrow \tan^3 A \tan^3 B \tan^3 C \geq 27 \tan A \tan B \tan C$$

$$\Rightarrow \tan^2 A \tan^2 B \tan^2 C \geq 27$$

$$\Rightarrow \tan A \tan B \tan C \geq 3\sqrt{3}$$

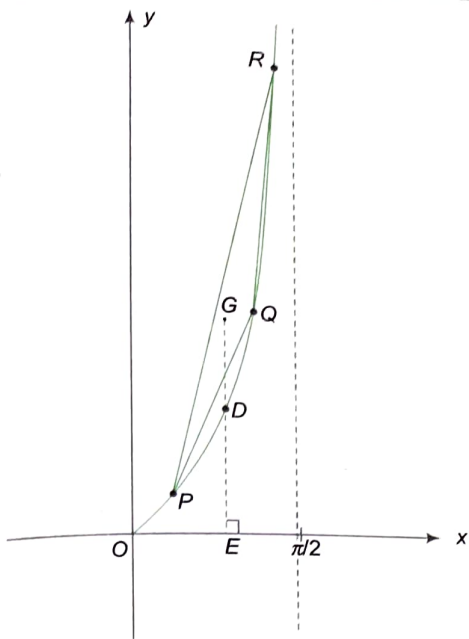
$$\Rightarrow \tan A + \tan B + \tan C \geq 3\sqrt{3}$$

Method II:

Since triangle ABC is acute angled, $0 < A, B, C < \frac{\pi}{2}$

So, consider the graph of function $y = \tan x$ for $x \in (0, \frac{\pi}{2})$

The graph of function is concave upward as shown in the figure.



Now, consider three points on the graph $P(A, \tan A)$, $Q(B, \tan B)$ and $R(C, \tan C)$.

Join these points to get triangle PQR .

Centroid of the triangle PQR is

$$G\left(\frac{A+B+C}{3}, \frac{\tan A + \tan B + \tan C}{3}\right).$$

Through point G , draw a line parallel to y -axis meeting graph at point D and x -axis at point E .

The abscissa of point E is $\frac{A+B+C}{3}$.

So, coordinates of point D are

$$\left(\frac{A+B+C}{3}, \tan\left(\frac{A+B+C}{3} \right) \right).$$

Clearly, $GE \geq DE$

$$\therefore \frac{\tan A + \tan B + \tan C}{3} \geq \tan\left(\frac{A+B+C}{3}\right)$$

$$\Rightarrow \frac{\tan A + \tan B + \tan C}{3} \geq \tan \frac{\pi}{3}$$

$$\Rightarrow \tan A + \tan B + \tan C \geq 3\sqrt{3}$$

2. In $\triangle ABC$, $\cos A + \cos B + \cos C \leq 3/2$

Proof:

$$\cos A + \cos B + \cos C$$

$$= 2 \cos\left(\frac{A+B}{2}\right) \cdot \cos\left(\frac{A-B}{2}\right) + 1 - 2 \sin^2 \frac{C}{2}$$

$$= 2\sin\frac{C}{2}.\cos\left(\frac{A-B}{2}\right)+1-2\sin^2\frac{C}{2}$$

$$\leq 2\sin\frac{C}{2}+1-2\sin^2\frac{C}{2} \quad \left(\because \cos\frac{A-B}{2}\leq 1\right)$$

$$= 1 - 2 \left(\sin^2 \frac{C}{2} - \sin \frac{C}{2} \right)$$

$$= 1 - 2 \left(\left(\sin \frac{C}{2} - \frac{1}{2} \right)^2 - \frac{1}{4} \right)$$

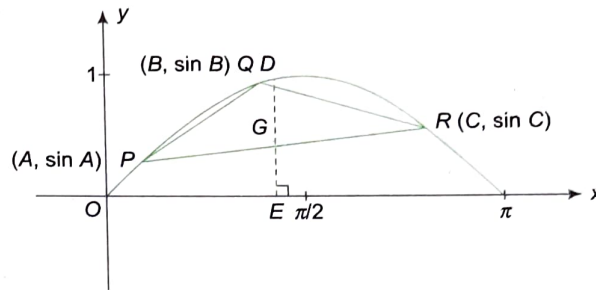
$$= \frac{3}{2} - \left(\sin \frac{C}{2} - \frac{1}{2} \right)^2 \leq \frac{3}{2}$$

3. In ΔABC , $\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}$

Proof:

In triangle ABC , $0 < A, B, C < \pi$

So, consider the graph of function $y = \sin x$ for $x \in (0, \pi)$.



The graph of function is concave downward.

Now, consider three points on the graph $P(A, \sin A)$, $Q(B, \sin B)$ and $R(C, \sin C)$.

Join these points to get triangle PQR .

Centroid of the triangle PQR is

$$G\left(\frac{A+B+C}{3}, \frac{\sin A + \sin B + \sin C}{3}\right).$$

Through point G draw a line parallel to y -axis meeting graph at point D and x -axis at point E .

The abscissa of point D is $\frac{A+B+C}{3}$.

So, coordinates of point D are

$$\left(\frac{A+B+C}{3}, \sin\left(\frac{A+B+C}{3} \right) \right).$$

Clearly, $DE \geq GE$

$$\therefore \sin\left(\frac{A+B+C}{3}\right) \geq \frac{\sin A + \sin B + \sin C}{3}$$

$$\Rightarrow \sin \frac{\pi}{3} \geq \frac{\sin A + \sin B + \sin C}{3}$$

$$\Rightarrow \sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}$$

ILLUSTRATION 3.99

Prove that in a $\triangle ABC$, $\sin^2 A + \sin^2 B + \sin^2 C \leq \frac{9}{4}$.

Sol. $\sin^2 A + \sin^2 B + \sin^2 C$

$$= 1 - \cos^2 A + \sin^2 B + \sin^2 C$$

$$= 1 - (\cos^2 A - \sin^2 B) + \sin^2 C$$

$$= 1 - \cos(A - B) \cos(A + B) + 1 - \cos^2 C$$

$$= 2 + \cos(A - B) \cos C - \cos^2 C$$

$$\leq 2 + \cos C - \cos^2 C \quad [\text{As } \cos(A - B) \leq 1]$$

$$\begin{aligned}
 &= 2 - ((\cos C - 1/2)^2 - 1/4) \\
 &= 9/4 - (\cos C - 1/2)^2 \leq 9/4
 \end{aligned}$$

ILLUSTRATION 3.100

Prove that in $\triangle ABC$, $\cos A \cos B \cos C \leq \frac{1}{8}$.

Sol. $2 \cos A \cos B \cos C$

$$\begin{aligned}
 &= [\cos(A+B) + \cos(A-B)] \cos C \\
 &= [-\cos C + \cos(A-B)] \cos C \\
 &= \cos(A-B) \cos C - \cos^2 C \\
 &\leq \cos C - \cos^2 C \quad [\text{As } \cos(A-B) \leq 1] \\
 &= 1/4 - (\cos C - 1/2)^2 \leq 1/4
 \end{aligned}$$

So, $\cos A \cos B \cos C \leq 1/8$

ILLUSTRATION 3.101

In $\triangle ABC$, prove that $\cos^2 A + \cos^2 B + \cos^2 C \geq \frac{3}{4}$.

Sol. $\cos^2 A + \cos^2 B + \cos^2 C$

$$\begin{aligned}
 &= 1 - \sin^2 A + \cos^2 B + \cos^2 C \\
 &= 1 + \cos(A-B) \cos(A+B) + \cos^2 C \\
 &= 1 - \cos(A-B) \cos C + \cos^2 C \\
 &\geq 1 - \cos C + \cos^2 C \quad [\text{As } \cos(A-B) \leq 1] \\
 &= (\cos C - 1/2)^2 + 3/4 \geq 3/4
 \end{aligned}$$

ILLUSTRATION 3.102

In $\triangle ABC$, prove that $\sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} \leq \frac{3}{2}$.

Sol. $\sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2}$

$$\begin{aligned}
 &= 2 \sin \frac{A+B}{4} \cos \frac{A-B}{4} + \sin \frac{C}{2} \\
 &= 2 \sin \frac{\pi-C}{4} \cos \frac{A-B}{4} + \cos \frac{\pi-C}{2} \\
 &= 2 \sin \frac{\pi-C}{4} \cos \frac{A-B}{4} + 1 - 2 \sin^2 \frac{\pi-C}{4} \\
 &\leq 1 + 2 \sin \frac{\pi-C}{4} - 2 \sin^2 \frac{\pi-C}{4} \quad \left(\because \cos \frac{A-B}{4} \leq 1 \right) \\
 &= 1 - 2 \left[\left(\sin \frac{\pi-C}{4} - \frac{1}{2} \right)^2 - \frac{1}{4} \right] \\
 &= \frac{3}{2} - 2 \left(\sin \frac{\pi-C}{4} - \frac{1}{2} \right)^2 \leq \frac{3}{2}
 \end{aligned}$$

ILLUSTRATION 3.103

In acute angled triangle ABC , prove that $\sec A + \sec B + \sec C \geq 6$.

Sol.

Method I:

In acute angled triangle, $\sec A$, $\sec B$ and $\sec C$ are positive.

Now, A.M. $\geq$ H.M.

$$\Rightarrow \frac{\sec A + \sec B + \sec C}{3} \geq \frac{3}{\cos A + \cos B + \cos C}$$

But in $\triangle ABC$,

$$\cos A + \cos B + \cos C \leq 3/2$$

$$\Rightarrow \frac{\sec A + \sec B + \sec C}{3} \geq 2$$

$$\Rightarrow \sec A + \sec B + \sec C \geq 6$$

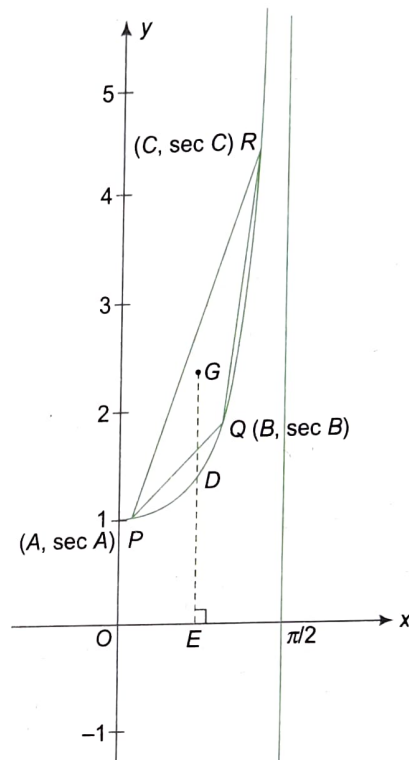
Method II:

Triangle ABC is acute angled.

$$\therefore 0 < A, B, C < \pi/2$$

So, consider the graph of function

$$y = \sec x \text{ for } x \in (0, \pi/2).$$



The graph of function is concave upward.

Now, consider three points on the graph

$$P(A, \sec A), Q(B, \sec B)$$

and $R(C, \sec C)$.

Join these points to get triangle PQR .

Centroid of the triangle PQR is

$$G\left(\frac{A+B+C}{3}, \frac{\sec A + \sec B + \sec C}{3}\right).$$

Through point G , draw a line parallel to y -axis meeting graph at point D and x -axis at point E .

The abscissa of point D is $\frac{A+B+C}{3}$.

So, coordinates of point D are $\left(\frac{A+B+C}{3}, \sec\left(\frac{A+B+C}{3}\right)\right)$.

Clearly, $GE \geq DE$

$$\therefore \frac{\sec A + \sec B + \sec C}{3} \geq \sec\left(\frac{A+B+C}{3}\right)$$

$$\Rightarrow \frac{\sec A + \sec B + \sec C}{3} \geq \sec \frac{\pi}{3}$$

$$\Rightarrow \sec A + \sec B + \sec C \geq 6$$

ILLUSTRATION 3.104

If $A + B + C = \pi$, prove that $\tan^2 \frac{A}{2} + \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2} \geq 1$

Sol.

$$\left(\tan \frac{A}{2} - \tan \frac{B}{2} \right)^2 + \left(\tan \frac{B}{2} - \tan \frac{C}{2} \right)^2 + \left(\tan \frac{C}{2} - \tan \frac{A}{2} \right)^2 \geq 0$$

$$\therefore 2 \left(\tan^2 \frac{A}{2} + \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2} \right) - 2 \left(\tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{A}{2} \tan \frac{C}{2} \right) \geq 0$$

We know that in $\triangle ABC$,

$$\tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{A}{2} \tan \frac{C}{2} = 1$$

$$\therefore 2 \left(\tan^2 \frac{A}{2} + \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2} \right) - 2(1) \geq 0$$

$$\therefore \tan^2 \frac{A}{2} + \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2} \geq 1$$

CONCEPT APPLICATION EXERCISE 3.10

1. In $\triangle ABC$,

(a) Prove that $\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} \leq \frac{9}{4}$

(b) If $\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} = y \left(x^2 + \frac{1}{x^2} \right)$ then find the maximum value of y .

2. Let $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma = \frac{\pi}{2}$. Then prove that $\sqrt{\tan \alpha \tan \beta} + \sqrt{\tan \beta \tan \gamma} + \sqrt{\tan \alpha \tan \gamma} \leq \sqrt{3}$

3. In acute angled $\triangle ABC$ prove that $\tan^2 A + \tan^2 B + \tan^2 C \geq 9$.

4. In $\triangle ABC$, prove that

(a) $\sin A \sin B \sin C \leq \frac{3\sqrt{3}}{8}$

(b) $\sin 2A + \sin 2B + \sin 2C \leq \frac{3\sqrt{3}}{2}$

5. In triangle ABC , prove that $\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \leq \frac{1}{8}$ and

hence, prove that $\operatorname{cosec} \frac{A}{2} + \operatorname{cosec} \frac{B}{2} + \operatorname{cosec} \frac{C}{2} \geq 6$.

ANSWERS

1. (b) $9/8$

Solved Examples**EXAMPLE 3.1**

In a $\triangle ABC$, if $\tan A/2, \tan B/2, \tan C/2$ are in A.P., then show that $\cos A, \cos B, \cos C$ are in A.P.

Sol. Given $\tan A/2, \tan B/2, \tan C/2$ are in A.P.

$$\therefore \tan A/2 - \tan B/2 = \tan B/2 - \tan C/2$$

$$\therefore \frac{\sin A/2}{\cos A/2} - \frac{\sin B/2}{\cos B/2} = \frac{\sin B/2}{\cos B/2} - \frac{\sin C/2}{\cos C/2}$$

$$\Rightarrow \frac{\sin A/2 \cdot \cos B/2 - \sin B/2 \cdot \cos A/2}{\cos A/2 \cdot \cos B/2}$$

$$= \frac{\sin B/2 \cdot \cos C/2 - \sin C/2 \cdot \cos B/2}{\cos B/2 \cdot \cos C/2}$$

$$\Rightarrow \frac{\sin \left(\frac{A-B}{2} \right)}{\cos A/2} = \frac{\sin \left(\frac{B-C}{2} \right)}{\cos C/2} \quad \dots(i)$$

$$\Rightarrow \sin \left(\frac{A-B}{2} \right) \cdot \sin \left(\frac{A+B}{2} \right) = \sin \left(\frac{B-C}{2} \right) \cdot \sin \left(\frac{B+C}{2} \right)$$

$$\Rightarrow \cos B - \cos A = \cos C - \cos B$$

Hence, $\cos A, \cos B, \cos C$ are in A.P.

EXAMPLE 3.2

If ABC is a triangle and $\tan \frac{A}{2}, \tan \frac{B}{2}, \tan \frac{C}{2}$ are in H.P., then find the minimum value of $\cot B/2$.

Sol. $A + B + C = \pi$

or $\frac{A}{2} + \frac{B}{2} = \frac{\pi}{2} - \frac{C}{2}$

or $\cot \left(\frac{A}{2} + \frac{B}{2} \right) = \cot \left(\frac{\pi}{2} - \frac{C}{2} \right)$

or $\frac{\cot \frac{A}{2} \cot \frac{B}{2} - 1}{\cot \frac{A}{2} + \cot \frac{B}{2}} = \tan \frac{C}{2} = \frac{1}{\cot \frac{C}{2}}$

or $\cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2} = \cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2}$

But $\tan \frac{A}{2}, \tan \frac{B}{2}, \tan \frac{C}{2}$ are in H.P. ...(i)

$\therefore \cot \frac{A}{2}, \cot \frac{B}{2}, \cot \frac{C}{2}$ are in A.P.

So, $\cot \frac{A}{2} + \cot \frac{C}{2} = 2 \cot \frac{B}{2}$

Hence, Eq. (i) becomes

$$\cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2} = 3 \cot \frac{B}{2}$$

$$\Rightarrow \cot \frac{A}{2} \cot \frac{C}{2} = 3$$

Thus, G.M. of $\cot \frac{A}{2}$ and $\cot \frac{C}{2}$ is

$$\sqrt{\cot \frac{A}{2} \cot \frac{C}{2}} = \sqrt{3}$$

and A.M. of $\cot \frac{A}{2}$ and $\cot \frac{C}{2}$ is

$$\frac{\cot \frac{A}{2} + \cot \frac{C}{2}}{2} = \cot \frac{B}{2}$$

But A.M. $\geq$ G.M. Thus,

$$\cot \frac{B}{2} \geq \sqrt{3}$$

Therefore, the minimum value of $\cot B/2$ is $\sqrt{3}$.

EXAMPLE 3.3

In ΔABC , if $\sin^3 \theta = \sin(A - \theta) \sin(B - \theta) \sin(C - \theta)$, then prove that $\cot \theta = \cot A + \cot B + \cot C$.

Sol. Let $\cot \theta = \cot A + \cot B + \cot C$

$$\text{or } \cot \theta - \cot A = \cot B + \cot C$$

$$\text{or } \frac{\sin(A - \theta)}{\sin A \sin \theta} = \frac{\sin(B + C)}{\sin B \sin C}$$

$$\text{or } \sin(A - \theta) = \frac{\sin^2 A \sin \theta}{\sin B \sin C} \quad \dots(i)$$

Similarly,

$$\sin(B - \theta) = \frac{\sin^2 B \sin \theta}{\sin A \sin C} \quad \dots(ii)$$

$$\text{and } \sin(C - \theta) = \frac{\sin^2 C \sin \theta}{\sin A \sin B} \quad \dots(iii)$$

By multiplying corresponding sides of Eqs. (i), (ii), and (iii), we have

$$\sin^3 \theta = \sin(A - \theta) \sin(B - \theta) \sin(C - \theta)$$

EXAMPLE 3.4

Find the sum of the series $\operatorname{cosec} \theta + \operatorname{cosec} 2\theta + \operatorname{cosec} 4\theta + \dots$ to n terms.

$$\begin{aligned} \text{Sol. } \operatorname{cosec} \theta &= \frac{1}{\sin \theta} = \frac{1}{\sin \theta} \frac{\sin \theta/2}{\sin \theta/2} \\ &= \frac{\sin\left(\theta - \frac{\theta}{2}\right)}{\sin \theta \sin\left(\frac{\theta}{2}\right)} = \frac{\sin \theta \cos \frac{\theta}{2} - \cos \theta \sin \frac{\theta}{2}}{\sin \theta \sin \frac{\theta}{2}} \end{aligned}$$

$$\therefore \operatorname{cosec} \theta = \cot \frac{\theta}{2} - \cot \theta$$

Similarly, $\operatorname{cosec} 2\theta = \cot \theta - \cot 2\theta$

$$\operatorname{cosec} 4\theta = \cot 2\theta - \cot 4\theta$$

...

...

$$\operatorname{cosec} 2^{n-1} \theta = \cot 2^{n-2} \theta - \cot 2^{n-1} \theta$$

$$\therefore \text{Sum} = \cot \frac{\theta}{2} - \cot 2^{n-1} \theta$$

EXAMPLE 3.5

If $\frac{x}{\tan(\theta + \alpha)} = \frac{y}{\tan(\theta + \beta)} = \frac{z}{\tan(\theta + \gamma)}$, then show that

$$\sum \frac{x+y}{x-y} \sin^2(\alpha - \beta) = 0.$$

Sol. Here

$$\frac{x}{y} = \frac{\tan(\theta + \alpha)}{\tan(\theta + \beta)}$$

By componendo and dividendo, we get

$$\frac{x+y}{x-y} = \frac{\tan(\theta + \alpha) + \tan(\theta + \beta)}{\tan(\theta + \alpha) - \tan(\theta + \beta)} = \frac{\sin(2\theta + \alpha + \beta)}{\sin(\alpha - \beta)}$$

$$\therefore \frac{x+y}{x-y} \sin^2(\alpha - \beta) = \sin(2\theta + \alpha + \beta) \sin(\alpha - \beta)$$

$$= \frac{1}{2} [\cos 2(\theta + \beta) - \cos 2(\theta + \alpha)]$$

Similarly,

$$\frac{y+z}{y-z} \sin^2(\beta - \gamma) = \frac{1}{2} [\cos 2(\theta + \gamma) - \cos 2(\theta + \beta)] \quad \dots(ii)$$

$$\text{and } \frac{z+x}{z-x} \sin^2(\gamma - \alpha) = \frac{1}{2} [\cos 2(\theta + \alpha) - \cos 2(\theta + \gamma)] \quad \dots(iii)$$

Adding Eqs. (i), (ii), and (iii), we get L.H.S. = 0.

EXAMPLE 3.6

If $\tan 6\theta = p/q$, find the value of $\frac{1}{2}(p \operatorname{cosec} 2\theta - q \sec 2\theta)$ in terms of p and q .

Sol. Here, we have $\tan 6\theta = p/q$

$$\text{or } \frac{\sin 6\theta}{\cos 6\theta} = \frac{p}{q}$$

$$\text{or } \frac{p}{\sin 6\theta} = \frac{q}{\cos 6\theta} = \frac{\sqrt{p^2 + q^2}}{\sqrt{1}} = \sqrt{p^2 + q^2} = k \text{ (say)}$$

$$\text{Now } y = \frac{1}{2}(p \operatorname{cosec} 2\theta - q \sec 2\theta)$$

$$= \frac{1}{2} \left(\frac{p}{\sin 2\theta} - \frac{q}{\cos 2\theta} \right)$$

$$= \frac{1}{2} \left[\frac{p \cos 2\theta - q \sin 2\theta}{\sin 2\theta \cos 2\theta} \right]$$

$$= \left[\frac{2k \sin 6\theta \cos 2\theta - 2k \cos 6\theta \sin 2\theta}{4 \sin 2\theta \cos 2\theta} \right]$$

$$= k \frac{\sin(6\theta - 2\theta)}{\sin 4\theta}$$

$$= k = \sqrt{p^2 + q^2}$$

EXAMPLE 3.7

If $0 < \alpha < \pi/2$ and $\sin \alpha + \cos \alpha + \tan \alpha + \cot \alpha + \sec \alpha + \csc \alpha = 7$, then prove that $\sin 2\alpha$ is a root of the equation $x^2 - 44x + 36 = 0$.

Sol. $\sin \alpha + \cos \alpha + (\tan \alpha + \cot \alpha) + (\sec \alpha + \csc \alpha) = 7$

$$\text{or } (\sin \alpha + \cos \alpha) + \frac{1}{\sin \alpha \cos \alpha} + \frac{\sin \alpha + \cos \alpha}{\sin \alpha \cos \alpha} = 7$$

$$\text{or } (\sin \alpha + \cos \alpha) \left(1 + \frac{1}{\sin \alpha \cos \alpha} \right)$$

$$= 7 - \frac{1}{\sin \alpha \cos \alpha}$$

Squaring both sides, we get

$$\text{or } (1 + \sin 2\alpha) \left(1 + \frac{4}{\sin 2\alpha} + \frac{4}{\sin^2 2\alpha} \right)$$

$$= 49 - \frac{28}{\sin 2\alpha} + \frac{4}{\sin^2 2\alpha}$$

Let $\sin 2\alpha = x$. Then,

$$(1+x) \left(1 + \frac{4}{x} + \frac{4}{x^2} \right) = 49 - \frac{28}{x} + \frac{4}{x^2}$$

$$\text{or } (1+x)(x^2 + 4x + 4) = 49x^2 - 28x + 4$$

$$\text{or } x^3 - 44x^2 + 36x = 0$$

$$\text{or } x^2 - 44x + 36 = 0 \quad (\text{as } x = \sin 2\alpha \neq 0)$$

EXAMPLE 3.8

Let A, B, C be three angles such that $A = \pi/4$ and $\tan B \tan C = p$. Find all possible values of p such that A, B, C are the angles of a triangle.

Sol. $A + B + C = \pi$

$$\Rightarrow B + C = \frac{3\pi}{4} \Rightarrow 0 < B, C < \frac{3\pi}{4}$$

Also $\tan B \tan C = p$

$$\text{or } \frac{\sin B \sin C}{\cos B \cos C} = \frac{p}{1}$$

$$\text{or } \frac{\cos B \cos C - \sin B \sin C}{\cos B \cos C + \sin B \sin C} = \frac{1-p}{1+p}$$

$$\text{or } \frac{\cos(B+C)}{\cos(B-C)} = \frac{1-p}{1+p}$$

$$\text{or } \frac{1+p}{\sqrt{2}(p-1)} = \cos(B-C) \quad \dots(i)$$

Since B or C can vary from 0 to $3\pi/4$, we get

$$-\frac{3\pi}{4} < B-C < \frac{3\pi}{4} \text{ or } -\frac{1}{\sqrt{2}} < \cos(B-C) \leq 1.$$

$$\text{Equation (i) will now lead to } -\frac{1}{\sqrt{2}} < \frac{p+1}{\sqrt{2}(p-1)} \leq 1$$

$$\text{For } 0 < 1 + \frac{p+1}{p-1}$$

$$\text{or } \frac{2p}{(p-1)} > 0$$

$$\therefore p < 0 \text{ or } p > 1 \quad \dots(ii)$$

$$\text{Also } \frac{p+1-\sqrt{2}(p-1)}{\sqrt{2}(p-1)} \leq 0$$

$$\text{or } \frac{(p-(\sqrt{2}+1)^2)}{(p-1)} \geq 0$$

$$\therefore p < 1 \text{ or } p \geq (\sqrt{2}+1)^2 \quad \dots(iii)$$

Combining Eqs. (ii) and (iii), we get $p < 0$ or $p \geq (\sqrt{2}+1)^2$.

EXAMPLE 3.9

Show that $\frac{\sin x}{\cos 3x} + \frac{\sin 3x}{\cos 9x} + \frac{\cos 9x}{\cos 27x} = \frac{1}{2} [\tan 27x - \tan x]$.

Sol.

$$\begin{aligned} \text{R.H.S.} &= \frac{1}{2} [\tan 27x - \tan x] \\ &= \frac{1}{2} [(\tan 27x - \tan 9x) + (\tan 9x - \tan 3x) + (\tan 3x - \tan x)] \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2} \left[\left(\frac{\sin 27x}{\cos 27x} - \frac{\sin 9x}{\cos 9x} \right) + \left(\frac{\sin 9x}{\cos 9x} - \frac{\sin 3x}{\cos 3x} \right) + \left(\frac{\sin 3x}{\cos 3x} - \frac{\sin x}{\cos x} \right) \right] \end{aligned}$$

$$= \frac{1}{2} \left[\frac{\sin(27x-9x)}{\cos 27x \cos 9x} + \frac{\sin(9x-3x)}{\cos 9x \cos 3x} + \frac{\sin(3x-x)}{\cos 3x \cos x} \right]$$

$$= \frac{1}{2} \left[\frac{\sin 18x}{\cos 27x \cos 9x} + \frac{\sin 6x}{\cos 9x \cos 3x} + \frac{\sin 2x}{\cos 3x \cos x} \right]$$

$$= \frac{1}{2} \left[\frac{2 \sin 9x \cos 9x}{\cos 27x \cos 9x} + \frac{2 \sin 3x \cos 3x}{\cos 9x \cos 3x} + \frac{2 \sin x \cos x}{\cos 3x \cos x} \right]$$

$$= \frac{\sin 9x}{\cos 27x} + \frac{\sin 3x}{\cos 9x} + \frac{\sin x}{\cos 3x}$$

$$= \frac{\sin x}{\cos 3x} + \frac{\sin 3x}{\cos 9x} + \frac{\sin 9x}{\cos 27x} = \text{L.H.S.}$$

EXAMPLE 3.10

Prove that $\frac{2 \cos 2^n \theta + 1}{2 \cos \theta + 1} = (2 \cos \theta - 1)(2 \cos 2\theta - 1) \times (\cos 2^2 \theta - 1) \dots (2 \cos 2^{n-1} \theta - 1)$.

Sol.

We have to prove that

$$\frac{2 \cos 2^n \theta + 1}{2 \cos \theta + 1} = (2 \cos \theta - 1)(2 \cos 2\theta - 1)(2 \cos 2^2 \theta - 1) \dots$$

$$\text{or } 2 \cos 2^n \theta + 1 = [(2 \cos \theta + 1)(2 \cos \theta - 1)](2 \cos 2\theta - 1) \\ (2 \cos 2^2 \theta - 1) \dots (2 \cos 2^{n-1} \theta - 1)$$

$$\begin{aligned} \text{Now } & [(2 \cos \theta + 1)(2 \cos \theta - 1)](2 \cos 2\theta - 1) \\ & (2 \cos 2^2 \theta - 1) \dots (2 \cos 2^{n-1} \theta - 1) \\ & = (4 \cos^2 \theta - 1)(2 \cos 2\theta - 1)(2 \cos 2^2 \theta - 1) \dots \\ & (2 \cos 2^{n-1} \theta - 1) \\ & = (2 \cos 2\theta + 1)(2 \cos 2\theta - 1)(2 \cos 2^2 \theta - 1) \dots \\ & (2 \cos 2^{n-1} \theta - 1) \quad [\text{using } \cos 2\theta = 2 \cos^2 \theta - 1] \\ & = (4 \cos^2 2\theta - 1)(2 \cos 2^2 \theta - 1) \dots (2 \cos 2^{n-1} \theta - 1) \\ & = (2 \cos 2^2 \theta + 1)(2 \cos 2^2 \theta - 1) \dots (2 \cos 2^{n-1} \theta - 1) \\ & = (4 \cos^2 2^2 \theta - 1)(2 \cos 2^3 \theta - 1) \dots (2 \cos 2^{n-1} \theta - 1) \\ & \dots \\ & \dots \\ & \dots \\ & = (2 \cos 2^{n-1} \theta + 1)(2 \cos 2^{n-1} \theta - 1) \\ & = 4 \cos^2 2^{n-1} \theta - 1 \\ & = 2 \cos 2^n \theta + 1 \end{aligned}$$

EXAMPLE 3.11

If $\tan\left(\frac{\pi}{4} + \frac{y}{2}\right) = \tan^3\left(\frac{\pi}{4} + \frac{x}{2}\right)$, prove that $\frac{\sin y}{\sin x} = \frac{3 + \sin^2 x}{1 + 3 \sin^2 x}$.

Sol.

We have,

$$\tan\left(\frac{\pi}{4} + \frac{y}{2}\right) = \tan^3\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

$$\Rightarrow \frac{\tan \pi/4 + \tan y/2}{1 - \tan \pi/4 \tan y/2} = \left[\frac{\tan \pi/4 + \tan x/2}{1 - \tan \pi/4 \tan x/2} \right]^3$$

$$\Rightarrow \frac{1 + \tan y/2}{1 - \tan y/2} = \left(\frac{1 + \tan x/2}{1 - \tan x/2} \right)^3$$

$$\Rightarrow \frac{\cos y/2 + \sin y/2}{\cos y/2 - \sin y/2} = \left(\frac{\cos x/2 + \sin x/2}{\cos x/2 - \sin x/2} \right)^3$$

$$\Rightarrow \left(\frac{\cos y/2 + \sin y/2}{\cos y/2 - \sin y/2} \right)^2 = \left[\left(\frac{\cos x/2 + \sin x/2}{\cos x/2 - \sin x/2} \right)^2 \right]^3$$

$$\Rightarrow \frac{1 + 2 \sin y/2 \cos y/2}{1 - 2 \sin y/2 \cos y/2} = \left(\frac{1 + 2 \sin x/2 \cos x/2}{1 - 2 \sin x/2 \cos x/2} \right)^3$$

$$\Rightarrow \frac{1 + \sin y}{1 - \sin y} = \left(\frac{1 + \sin x}{1 - \sin x} \right)^3$$

Applying componendo and dividendo

$$\Rightarrow \frac{(1 + \sin y) - (1 - \sin y)}{(1 + \sin y) + (1 - \sin y)} = \frac{(1 + \sin x)^3 - (1 - \sin x)^3}{(1 + \sin x)^3 + (1 - \sin x)^3}$$

$$\frac{2 \sin y}{2} = \frac{6 \sin x + 2 \sin^3 x}{2 + 6 \sin^2 x}$$

$$\Rightarrow \frac{\sin y}{\sin x} = \frac{3 + \sin^2 x}{1 + 3 \sin^2 x}$$

EXAMPLE 3.12

If $(1 + \sin t)(1 + \cot t) = \frac{5}{4}$ then find the value of $(1 - \sin t)(1 - \cos t)$.

Sol. If $(1 + \sin t)(1 + \cot t) = \frac{5}{4}$

$$\therefore 1 + \sin t \cos t + (\sin t + \cos t) = 5/4$$

$$\therefore \sin t + \cos t = \frac{1 - 2 \sin 2t}{4}$$

Squaring both sides, we get

$$1 + \sin 2t = \frac{1 + 4 \sin^2 2t - 4 \sin 2t}{16}$$

$$\therefore 16 + 16 \sin 2t = 1 + 4 \sin^2 2t - 4 \sin 2t$$

$$\therefore 4 \sin^2 2t - 20 \sin 2t - 15 = 0$$

$$\therefore \sin 2t = \frac{20 \pm \sqrt{640}}{8}$$

$$\therefore \sin 2t = \frac{5 - 2\sqrt{10}}{2}$$

Now, $E = (1 - \sin t)(1 - \cos t)$

$$= 1 - (\sin t + \cos t) + \frac{\sin 2t}{2}$$

$$= 1 - \frac{(1 - 2 \sin 2t)}{4} + \frac{\sin 2t}{2}$$

$$= \frac{3 + 4 \sin 2t}{4}$$

$$= \frac{3 + 2(5 - 2\sqrt{10})}{4}$$

$$= \frac{13}{4} - \sqrt{10}$$

EXAMPLE 3.13

For all θ in $[0, \pi/2]$ show that $\cos(\sin \theta) \geq \sin(\cos \theta)$.

Sol. We have,

$$\begin{aligned} \cos \theta + \sin \theta &= \sqrt{2} \left[\frac{1}{\sqrt{2}} \cos \theta + \frac{1}{\sqrt{2}} \sin \theta \right] \\ &= \sqrt{2} \sin(\pi/4 + \theta) \end{aligned}$$

$$\therefore \cos \theta + \sin \theta \leq \sqrt{2} < \pi/2$$

$$\therefore \cos \theta + \sin \theta < \pi/2$$

$$\text{or } \cos \theta < \pi/2 - \sin \theta$$

As $\theta \in [0, \pi/2]$ in which $\sin \theta$ increases.

Taking $\sin$ on both the sides of Eq. (i), we get

$$\sin(\cos \theta) < \sin(\pi/2 - \sin \theta)$$

$$\text{or } \sin(\cos \theta) < \cos(\sin \theta)$$

$$\text{or } \cos(\sin \theta) > \sin(\cos \theta)$$

EXAMPLE 3.14

Show that the value of $\frac{\tan x}{\tan 3x}$, wherever defined, never lies between $\frac{1}{3}$ and 3.

Sol. Let

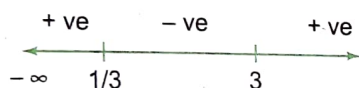
$$\begin{aligned} y &= \frac{\tan x}{\tan 3x} \\ &= \frac{\tan x (1 - 3 \tan^2 x)}{3 \tan x - \tan^3 x} \\ &= \frac{1 - 3 \tan^2 x}{3 - \tan^2 x} \end{aligned}$$

or $3y - (\tan^2 x)y = 1 - 3 \tan^2 x$

or $(y - 3) \tan^2 x = 3y - 1$

or $\tan^2 x = \frac{3y - 1}{y - 3}$

or $\frac{3y - 1}{y - 3} \geq 0$ (L.H.S. is a perfect square)



From the sign scheme $y \leq \frac{1}{3}$ or $y > 3$

Thus, y never lies between $1/3$ and 3.

EXAMPLE 3.15

Prove that $\sum_{k=1}^{n-1} (n-k) \cos \frac{2k\pi}{n} = -\frac{n}{2}$, where $n \geq 3$ is an integer.

Sol.
$$\begin{aligned} S &= \sum_{k=1}^{n-1} (n-k) \cos \frac{2k\pi}{n} \\ &= (n-1) \cos \frac{2\pi}{n} + (n-2) \cos 2 \frac{2\pi}{n} + \dots \\ &\quad + 1 \cos (n-1) \frac{2\pi}{n} \quad \dots(i) \end{aligned}$$

We know that $\cos \theta = \cos(2\pi - \theta)$. Replacing each angle θ by $2\pi - \theta$ in Eq. (i), we get

$$\begin{aligned} S &= (n-1) \cos (n-1) \frac{2\pi}{n} + (n-2) \cos (n-2) \frac{2\pi}{n} + \dots \\ &\quad + 1 \cos \frac{2\pi}{n} \quad [\text{using Eq. (i)}] \quad \dots(ii) \end{aligned}$$

Adding terms having the same angle and taking n common, we have

$$\begin{aligned} 2S &= n \left[\cos \frac{2\pi}{n} + \cos \frac{4\pi}{n} + \cos \frac{6\pi}{n} + \dots + \cos (n-1) \frac{2\pi}{n} \right] \\ &= n \left[\frac{\sin(n-1) \frac{\pi}{n} \cos \frac{2\pi}{n} + (n-1) \frac{2\pi}{n}}{\sin \frac{\pi}{n}} \right] \\ &= n \left[\frac{\sin \left(\pi - \frac{\pi}{n} \right)}{\sin \frac{\pi}{n}} \cdot \cos \pi \right] = -n \end{aligned}$$

$\therefore S = -n/2$

Exercises

Single Correct Answer Type

- If $\cos(A - B) = 3/5$ and $\tan A \tan B = 2$, then
 - $\cos A \cos B = 1/5$
 - $\sin A \sin B = -2/5$
 - $\cos A \cos B = -1/5$
 - $\sin A \sin B = -1/5$
- If $A = \sin 45^\circ + \cos 45^\circ$ and $B = \sin 44^\circ + \cos 44^\circ$, then
 - $A > B$
 - $A < B$
 - $A = B$
 - none of these
- $\tan 100^\circ + \tan 125^\circ + \tan 100^\circ \tan 125^\circ$ is equal to
 - 0
 - 1/2
 - 1
 - 1
- If $\cot(\alpha + \beta) = 0$, then $\sin(\alpha + 2\beta)$ can be
 - $-\sin \alpha$
 - $\sin \beta$
 - $\cos \alpha$
 - $\cos \beta$
- In triangle ABC , if $\sin A \cos B = \frac{1}{4}$ and $3 \tan A = \tan B$, then $\cot^2 A$ is equal to
 - 2
 - 3
 - 4
 - 5
- Let $\frac{\sin(\theta - \alpha)}{\sin(\theta - \beta)} = \frac{a}{b}$ and $\frac{\cos(\theta - \alpha)}{\cos(\theta - \beta)} = \frac{c}{d}$ then $\frac{ac + bd}{ad + bc} =$
 - $\cos(\alpha - \beta)$
 - $\sin(\alpha - \beta)$
 - $\sin(\alpha + \beta)$
 - none of these
- If A, B, C are angles of a triangle, then $2 \sin \frac{A}{2} \operatorname{cosec} \frac{B}{2}$
 $\sin \frac{C}{2} - \sin A \cot \frac{B}{2} - \cos A$ is
 - independent of A, B, C
 - function of A, B
 - function of C
 - none of these
- If $a \leq 3 \cos x + 5 \sin(x - \pi/6) \leq b$ for all x , then (a, b) is
 - $(-\sqrt{19}, \sqrt{19})$
 - $(-17, 17)$
 - $(-\sqrt{21}, \sqrt{21})$
 - none of these
- If $\frac{x}{\cos \theta} = \frac{y}{\cos(\theta - \frac{2\pi}{3})} = \frac{z}{\cos(\theta + \frac{2\pi}{3})}$, then $x + y + z$ is
 equal to
 - 1
 - 0
 - 1
 - none of these
- Let $x = \sin 1^\circ$, then the value of the expression
 $\frac{1}{\cos 0^\circ \cdot \cos 1^\circ} + \frac{1}{\cos 1^\circ \cdot \cos 2^\circ} + \frac{1}{\cos 2^\circ \cdot \cos 3^\circ} + \dots$
 $+ \frac{1}{\cos 44^\circ \cdot \cos 45^\circ}$ is equal to
 - x
 - $1/x$
 - $\sqrt{2}/x$
 - $x/\sqrt{2}$
- If θ is eliminated from the equations $x = a \cos(\theta - \alpha)$ and
 $y = b \cos(\theta - \beta)$, then $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} \cos(\alpha - \beta)$ is equal
 to
 - $\sec^2(\alpha - \beta)$
 - $\operatorname{cosec}^2(\alpha - \beta)$
 - $\cos^2(-\beta)$
 - $\sin^2(\alpha - \beta)$
- The minimum vertical distance between the graphs
 $y = 2 + \sin x$ and $y = \cos x$ is
 - 2
 - 1
 - $\sqrt{2}$
 - $2 - \sqrt{2}$
- If $\tan^2 \frac{\pi - A}{4} + \tan^2 \frac{\pi - B}{4} + \tan^2 \frac{\pi - C}{4} = 1$, then $\triangle ABC$ is
 - equilateral
 - isosceles
 - scalene
 - none of these
- If $(1 + \tan \alpha)(1 + \tan 4\alpha) = 2$, $\alpha \in (0, \pi/16)$, then α is equal
 to
 - $\frac{\pi}{20}$
 - $\frac{\pi}{30}$
 - $\frac{\pi}{40}$
 - $\frac{\pi}{60}$
- If $\cos 28^\circ + \sin 28^\circ = k^3$, then $\cos 17^\circ$ is equal to
 - $\frac{k^3}{\sqrt{2}}$
 - $-\frac{k^3}{\sqrt{2}}$
 - $\pm \frac{k^3}{\sqrt{2}}$
 - none of these
- Let $f(\theta) = \frac{\cot \theta}{1 + \cot \theta}$ and $\alpha + \beta = \frac{5\pi}{4}$, then the value of
 $f(\alpha)f(\beta)$ is
 - $\frac{1}{2}$
 - $-\frac{1}{2}$
 - 2
 - none of these
- If $y = (1 + \tan A)(1 - \tan B)$, where $A - B = \frac{\pi}{4}$, then
 $(y + 1)^{y+1}$ is equal to
 - 9
 - 4
 - 27
 - 81
- If $\frac{\sin x}{\sin y} = \frac{1}{2}$, $\frac{\cos x}{\cos y} = \frac{3}{2}$, where $x, y \in (0, \frac{\pi}{2})$, then the
 value of $\tan(x + y)$ is equal to
 - $\sqrt{13}$
 - $\sqrt{14}$
 - $\sqrt{17}$
 - $\sqrt{15}$
- If $\cot^2 x = \cot(x - y) \cot(x - z)$, then $\cot 2x$ is equal to (where
 $x \neq \pm \pi/4$)
 - $\frac{1}{2}(\tan y + \tan z)$
 - $\frac{1}{2}(\cot y + \cot z)$
 - $\frac{1}{2}(\sin y + \sin z)$
 - none of these
- In a $\triangle ABC$, if $\tan A : \tan B : \tan C = 3 : 4 : 5$, then the value
 of $\sin A \cdot \sin B \cdot \sin C$ is equal to
 - $\frac{2}{\sqrt{5}}$
 - $\frac{2\sqrt{5}}{7}$
 - $\frac{2\sqrt{5}}{9}$
 - $\frac{2}{3\sqrt{5}}$

21. Which of the following is not the value of $\sin 27^\circ - \cos 27^\circ$?
- (1) $-\frac{\sqrt{3}-\sqrt{5}}{2}$ (2) $-\frac{\sqrt{5}-\sqrt{3}}{2}$
 (3) $-\frac{\sqrt{5}-1}{2\sqrt{2}}$ (4) none of these
22. If $\cos \theta_1 = 2 \cos \theta_2$, then $\tan \frac{\theta_1 - \theta_2}{2} \tan \frac{\theta_1 + \theta_2}{2}$ is equal to
- (1) $\frac{1}{3}$ (2) $-\frac{1}{3}$ (3) 1 (4) -1
23. Let α and β be such that $\pi < \alpha - \beta < 3\pi$. If $\sin \alpha + \sin \beta = -\frac{21}{65}$ and $\cos \alpha + \cos \beta = -\frac{27}{65}$, then the value of $\cos \frac{\alpha - \beta}{2}$ is
- (1) $-\frac{3}{\sqrt{130}}$ (2) $\frac{3}{\sqrt{130}}$
 (3) $\frac{6}{65}$ (4) $-\frac{6}{65}$
24. If $\alpha = \frac{\pi}{14}$, then the value of $(\tan \alpha \tan 2\alpha + \tan 2\alpha \tan 4\alpha + \tan 4\alpha \tan \alpha)$ is
- (1) 1 (2) $1/2$ (3) 2 (4) $1/3$
25. $\frac{\sin 3\theta + \sin 5\theta + \sin 7\theta + \sin 9\theta}{\cos 3\theta + \cos 5\theta + \cos 7\theta + \cos 9\theta}$ is equal to
- (1) $\tan 3\theta$ (2) $\cot 3\theta$ (3) $\tan 6\theta$ (4) $\cot 6\theta$
26. If x, y, z are in A.P., then $\frac{\sin x - \sin z}{\cos z - \cos x}$ is equal to
- (1) $\tan y$ (2) $\cot y$ (3) $\sin y$ (4) $\cos y$
27. If $\frac{\cos x}{a} = \frac{\cos(x+\theta)}{b} = \frac{\cos(x+2\theta)}{c} = \frac{\cos(x+3\theta)}{d}$ then $\frac{a+c}{b+d}$ is equal to
- (1) $\frac{a}{d}$ (2) $\frac{c}{b}$ (3) $\frac{b}{c}$ (4) $\frac{d}{a}$
28. If $\cos \alpha + \cos \beta = 0 = \sin \alpha + \sin \beta$, then $\cos 2\alpha + \cos 2\beta$ is equal to
- (1) $-2 \sin(\alpha + \beta)$ (2) $-2 \cos(\alpha + \beta)$
 (3) $2 \sin(\alpha + \beta)$ (4) $2 \cos(\alpha + \beta)$
29. Value of $\frac{3 + \cot 80^\circ \cot 20^\circ}{\cot 80^\circ + \cot 20^\circ}$ is equal to
- (1) $\cot 20^\circ$ (2) $\tan 50^\circ$
 (3) $\cot 50^\circ$ (4) $\cot \sqrt{20^\circ}$
30. If $\tan \alpha$ is equal to the integral solution of the inequality $4x^2 - 16x + 15 < 0$ and $\cos \beta$ is equal to the slope of the bisector of the first quadrant, then $\sin(\alpha + \beta) \sin(\alpha - \beta)$ is equal to
- (1) $\frac{3}{5}$ (2) $\frac{3}{5}$ (3) $\frac{2}{\sqrt{5}}$ (4) $\frac{4}{5}$
31. Let $f(n) = 2 \cos nx \forall n \in N$, then $f(1) f(n+1) - f(n)$ is equal to
- (1) $f(n+3)$ (2) $f(n+2)$
 (3) $f(n+1)f(2)$ (4) $f(n+2)f(2)$
32. If $\sin \theta_1 - \sin \theta_2 = a$ and $\cos \theta_1 + \cos \theta_2 = b$, then
- (1) $a^2 + b^2 \geq 4$ (2) $a^2 + b^2 \leq 4$
 (3) $a^2 + b^2 \geq 3$ (4) $a^2 + b^2 \leq 2$
33. $\frac{\sqrt{2} - \sin \alpha - \cos \alpha}{\sin \alpha - \cos \alpha}$ is equal to
- (1) $\sec\left(\frac{\alpha}{2} - \frac{\pi}{8}\right)$ (2) $\cos\left(\frac{\pi}{8} - \frac{\alpha}{2}\right)$
 (3) $\tan\left(\frac{\alpha}{2} - \frac{\pi}{8}\right)$ (4) $\cot\left(\frac{\alpha}{2} - \frac{\pi}{2}\right)$
34. If x_1 and x_2 are two distinct roots of the equation $a \cos x + b \sin x = c$, then $\tan \frac{x_1 + x_2}{2}$ is equal to
- (1) $\frac{a}{b}$ (2) $\frac{b}{a}$ (3) $\frac{c}{a}$ (4) $\frac{a}{c}$
35. If $\sin(y+z-x)$, $\sin(z+x-y)$, $\sin(x+y-z)$ are in A.P., then $\tan x$, $\tan y$, $\tan z$ are in
- (1) A.P. (2) G.P.
 (3) H.P. (4) none of these
36. If $\frac{\tan(\alpha + \beta - \gamma)}{\tan(\alpha - \beta + \gamma)} = \frac{\tan \gamma}{\tan \beta}$, ($\beta \neq \gamma$) then $\sin 2\alpha + \sin 2\beta + \sin 2\gamma =$
- (1) 0 (2) 1 (3) 2 (4) $1/2$
37. If $\sin \theta_1 \sin \theta_2 - \cos \theta_1 \cos \theta_2 + 1 = 0$, then the value of $\tan(\theta_1/2) \cot(\theta_2/2)$ is equal to
- (1) -1 (2) 1 (3) 2 (4) -2
38. The value of expression $\frac{2(\sin 1^\circ + \sin 2^\circ + \sin 3^\circ + \dots + \sin 89^\circ)}{2(\cos 1^\circ + \cos 2^\circ + \dots + \cos 44^\circ) + 1}$ equals
- (1) $\sqrt{2}$ (2) $1/\sqrt{2}$ (3) $1/2$ (4) 0
39. If A, B, C are the angles of a triangle such that $\cot \frac{A}{2} = 3 \tan \frac{C}{2}$, then $\sin A, \sin B, \sin C$ are in
- (1) A.P. (2) G.P.
 (3) H.P. (4) none of these
40. If $2 \sec 2\theta = \tan \phi + \cot \phi$, then one of the values of $\theta + \phi$ is
- (1) $\pi/2$ (2) $\pi/4$
 (3) $\pi/3$ (4) none of these
41. The roots of the equation $4x^2 - 2\sqrt{5}x + 1 = 0$ are
- (1) $\sin 36^\circ, \sin 18^\circ$ (2) $\sin 18^\circ, \cos 36^\circ$
 (3) $\sin 36^\circ, \cos 18^\circ$ (4) $\cos 18^\circ, \cos 36^\circ$
42. If A and B are acute positive angles satisfying the equations $3 \sin^2 A + 2 \sin^2 B = 1$ and $3 \sin 2A - 2 \sin 2B = 0$, then $A + 2B$ is equal to
- (1) π (2) $\frac{\pi}{2}$ (3) $\frac{\pi}{4}$ (4) $\frac{\pi}{6}$

43. If $\cos 25^\circ + \sin 25^\circ = p$, then $\cos 50^\circ$ is

- (1) $\sqrt{2-p^2}$ (2) $-\sqrt{2-p^2}$
 (3) $p\sqrt{2-p^2}$ (4) $-p\sqrt{2-p^2}$

44. The value of $\cot \frac{7\pi}{16} + 2 \cot \frac{3\pi}{8} + \cot \frac{15\pi}{16}$ is

- (1) 4 (2) 2 (3) -2 (4) -4

45. If $\alpha, \beta, \gamma, \delta$ are the smallest positive angles in ascending order of magnitude which have their sines equal to the positive quantity k , then the value of $4 \sin \frac{\alpha}{2} + 3 \sin \frac{\beta}{2} + 2 \sin \frac{\gamma}{2} + \sin \frac{\delta}{2}$ is equal to

- (1) $2\sqrt{1-k}$ (2) $2\sqrt{1+k}$
 (3) $\frac{\sqrt{1+k}}{2}$ (4) none of these

46. $\frac{\sin^2 A - \sin^2 B}{\sin A \cos A - \sin B \cos B}$ is equal to

- (1) $\tan(A-B)$ (2) $\tan(A+B)$
 (3) $\cot(A-B)$ (4) $\cot(A+B)$

47. If $\cos(\alpha-\beta) = 3 \sin(\alpha+\beta)$, then $\frac{1}{1-3 \sin 2\alpha} + \frac{1}{1-3 \sin 2\beta} =$

- (1) $\frac{1}{2}$ (2) $-\frac{1}{2}$ (3) $\frac{1}{4}$ (4) $-\frac{1}{4}$

48. The value of $\cos^2 10^\circ - \cos 10^\circ \cos 50^\circ + \cos^2 50^\circ$ is equal to

- (1) $\frac{4}{3}$ (2) $\frac{1}{3}$ (3) $\frac{3}{4}$ (4) 3

49. If $\tan^2 \theta = 2 \tan^2 \phi + 1$, then $\cos 2\theta + \sin^2 \phi$ equals

- (1) -1 (2) 0
 (3) 1 (4) none of these

50. If $\tan A \cdot \tan B = \frac{1}{2}$, then $(5 - 3 \cos 2A)(5 - 3 \cos 2B) =$

- (1) 2 (2) 8 (3) 12 (4) 16

51. If $\sin x + \operatorname{cosec} x + \tan y + \cot y = 4$ where x and $y \in \left[0, \frac{\pi}{2}\right]$,

then $\tan \frac{y}{2}$ is a root of the equation

- (1) $\alpha^2 + 2\alpha + 1 = 0$ (2) $\alpha^2 + 2\alpha - 1 = 0$
 (3) $2\alpha^2 - 2\alpha - 1 = 0$ (4) $\alpha^2 - \alpha - 1 = 0$

52. If $2|\sin 2\alpha| = |\tan \beta + \cot \beta|$, $\alpha, \beta \in \left(\frac{\pi}{2}, \pi\right)$, then the value of $\alpha + \beta$ is

- (1) $\frac{3\pi}{4}$ (2) π (3) $\frac{3\pi}{2}$ (4) $\frac{5\pi}{4}$

53. The value of $\sin^3 10^\circ + \sin^3 50^\circ - \sin^3 70^\circ$ is equal to

- (1) $-\frac{3}{2}$ (2) $\frac{3}{4}$ (3) $-\frac{3}{4}$ (4) $-\frac{3}{8}$

54. Let $P(x) = \left(\frac{1 - \cos 2x + \sin 2x}{1 + \cos 2x + \sin 2x}\right)^2 + \left(\frac{1 + \cot x + \cot^2 x}{1 + \tan x + \tan^2 x}\right)^2$

then the minimum value of $P(x)$ equals

- (1) 1 (2) 2 (3) 4 (4) 16

55. If $\frac{3 - \tan^2 \frac{\pi}{7}}{1 - \tan^2 \frac{\pi}{7}} = k \cos \frac{\pi}{7}$ then the value of k is

- (1) 1 (2) 2 (3) 3 (4) 4

56. $\operatorname{cosec} \frac{360^\circ}{7} + \operatorname{cosec} \frac{540^\circ}{7} =$

- (1) $\operatorname{cosec} \frac{180^\circ}{7}$ (2) $\operatorname{cosec} \frac{90^\circ}{7}$
 (3) $\sec \frac{180^\circ}{7}$ (4) $\sec \frac{90^\circ}{7}$

57. If θ_1 and θ_2 are two values lying in $[0, 2\pi]$ for which $\tan \theta = \lambda$, then $\tan \frac{\theta_1}{2} \tan \frac{\theta_2}{2}$ is equal to

- (1) 0 (2) -1 (3) 2 (4) 1

58. If $\tan \theta = \sqrt{n}$, where $n \in \mathbb{N}$, $n \geq 2$, then $\sec 2\theta$ is always

- (1) a rational number (2) an irrational number
 (3) a positive integer (4) a negative integer

59. If $\sin 2\theta = \cos 3\theta$ and θ is an acute angle, then $\sin \theta$ equals

- (1) $\frac{\sqrt{5}-1}{4}$ (2) $-\left(\frac{\sqrt{5}-1}{4}\right)$
 (3) $\frac{\sqrt{5}+1}{4}$ (4) $\frac{-\sqrt{5}-1}{4}$

60. If $\cos x = \tan y$, $\cos y = \tan z$, $\cos z = \tan x$, then the value of $\sin x$ is

- (1) $2 \cos 18^\circ$ (2) $\cos 18^\circ$
 (3) $\sin 18^\circ$ (4) $2 \sin 18^\circ$

61. The value of $\cot 70^\circ + 4 \cos 70^\circ$ is

- (1) $\frac{1}{\sqrt{3}}$ (2) $\sqrt{3}$ (3) $2\sqrt{3}$ (4) $\frac{1}{2}$

62. If $\sin x + \cos x = \frac{\sqrt{7}}{2}$, where $x \in \left(0, \frac{\pi}{4}\right)$, then $\tan \frac{x}{2}$ is equal to

- (1) $\frac{3-\sqrt{7}}{3}$ (2) $\frac{\sqrt{7}-2}{3}$
 (3) $\frac{4-\sqrt{7}}{4}$ (4) none of these

63. If $\frac{\tan 3A}{\tan A} = k (k \neq 1)$ then which of the following is true?

- (1) $\frac{\cos A}{\cos 3A} = \frac{k-1}{2}$ (2) $\frac{\sin 3A}{\sin A} = \frac{2k}{k-1}$
 (3) $\frac{\cot 3A}{\cot A} = \frac{1}{k}$ (4) none of these

64. If $x \in \left(\pi, \frac{3\pi}{2}\right)$, then $4 \cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right) + \sqrt{4 \sin^4 x + \sin^2 2x}$ is always equal to

- (1) 1
(2) 2
(3) -2
(4) none of these

65. If $\cos x = \frac{2 \cos y - 1}{2 - \cos y}$, where $x, y \in (0, \pi)$, then $\tan \frac{x}{2} \cot \frac{y}{2}$ is equal to

- (1) $\sqrt{2}$
(2) $\sqrt{3}$
(3) $\frac{1}{\sqrt{2}}$
(4) $\frac{1}{\sqrt{3}}$

66. $\cot 16^\circ \cot 44^\circ + \cot 44^\circ \cot 76^\circ - \cot 76^\circ \cot 16^\circ =$

- (1) 1
(2) 2
(3) 3
(4) 4

67. If $\tan x = b/a$, then $\sqrt{(a+b)/(a-b)} + \sqrt{(a-b)/(a+b)}$ is equal to (where $x \in \left(0, \frac{\pi}{2}\right)$, $a > b > 0$)

- (1) $2 \sin x / \sqrt{\sin 2x}$
(2) $2 \cos x / \sqrt{\cos 2x}$
(3) $2 \cos x / \sqrt{\sin 2x}$
(4) $2 \sin x / \sqrt{\cos 2x}$

68. Given that $(1 + \sqrt{1+x}) \tan y = 1 + \sqrt{1-x}$. Then $\sin 4y$ is equal to

- (1) $4x$
(2) $2x$
(3) x
(4) none of these

69. If $\cos 2B = \frac{\cos(A+C)}{\cos(A-C)}$, then $\tan A, \tan B, \tan C$ are in

- (1) A.P.
(2) G.P.
(3) H.P.
(4) none of these

70. If $\frac{\cos(x-y)}{\cos(x+y)} + \frac{\cos(z+t)}{\cos(z-t)} = 0$, then the value of expression $\tan x \tan y \tan z \tan t$ is equal to

- (1) 1
(2) -1
(3) 2
(4) -2

71. If $\tan \beta = 2 \sin \alpha \sin \gamma \operatorname{cosec}(\alpha + \gamma)$, then $\cot \alpha, \cot \beta, \cot \gamma$ are in

- (1) A.P.
(2) G.P.
(3) H.P.
(4) none of these

72. The value of $\tan 9^\circ - \tan 27^\circ - \tan 63^\circ + \tan 81^\circ$ is

- (1) 2
(2) 3
(3) 4
(4) none of these

73. $\cos^3 x \sin 2x = \sum_{r=0}^n a_r \sin(rx) \quad \forall x \in R$, then

- (1) $n = 5, a_1 = 1/2$
(2) $n = 5, a_1 = 1/4$
(3) $n = 5, a_2 = 1/8$
(4) $n = 5, a_2 = 1/4$

74. $\tan^6 \frac{\pi}{9} - 33 \tan^4 \frac{\pi}{9} + 27 \tan^2 \frac{\pi}{9}$ is equal to

- (1) 0
(2) $\sqrt{3}$
(3) 3
(4) 9

75. Given that a, b, c are the sides of a $\triangle ABC$ which is right angled at C , then the minimum value of $\left(\frac{c}{a} + \frac{c}{b}\right)^2$ is

- (1) 0
(2) 4
(3) 6
(4) 8

76. If $\theta = 3\alpha$ and $\sin \theta = \frac{a}{\sqrt{a^2 + b^2}}$, the value of the expression $a \operatorname{cosec} \alpha - b \sec \alpha$ is

- (1) $\frac{a}{\sqrt{a^2 + b^2}}$
(2) $2\sqrt{a^2 + b^2}$
(3) $a + b$
(4) none of these

77. The value of $\tan 6^\circ \tan 42^\circ \tan 66^\circ \tan 78^\circ$ is

- (1) 1
(2) $1/2$
(3) $1/4$
(4) $1/8$

78. In triangle ABC , if angle C is 90° and the area of triangle is 30 sq. units, then the minimum possible value of the hypotenuse c is equal to

- (1) $30\sqrt{2}$
(2) $60\sqrt{2}$
(3) $120\sqrt{2}$
(4) $2\sqrt{30}$

79. If $\sqrt{2} \cos A = \cos B + \cos^3 B$, and $\sqrt{2} \sin A = \sin B - \sin^3 B$ then $\sin(A - B) =$

- (1) ± 1
(2) $\pm \frac{1}{2}$
(3) $\pm \frac{1}{3}$
(4) $\pm \frac{1}{4}$

80. In a right angled triangle the hypotenuse is $2\sqrt{2}$ times the perpendicular drawn from the opposite vertex. Then the other acute angles of the triangle are

- (1) $\frac{\pi}{3}$ and $\frac{\pi}{6}$
(2) $\frac{\pi}{8}$ and $\frac{3\pi}{8}$
(3) $\frac{\pi}{4}$ and $\frac{\pi}{4}$
(4) $\frac{\pi}{5}$ and $\frac{3\pi}{10}$

81. A circular ring of radius 3 cm hangs horizontally from a point 4 cm vertically above the centre by 4 strings attached at equal intervals to its circumference. If the angle between two consecutive strings be θ , then $\cos \theta$ is equal to

- (1) $\frac{4}{5}$
(2) $\frac{4}{25}$
(3) $\frac{16}{25}$
(4) none of these

82. The distance between the two parallel lines is 1 unit. A point 'A' is chosen to lie between the lines at a distance 'd' from one of them. Triangle ABC is equilateral with B on one line and C on the other parallel line. The length of the side of the equilateral triangle is

- (1) $\frac{2}{3}\sqrt{d^2 + d + 1}$
(2) $2\sqrt{\frac{d^2 - d + 1}{3}}$
(3) $2\sqrt{d^2 - d + 1}$
(4) $\sqrt{d^2 - d + 1}$

83. If $\sin^{-1} a + \sin^{-1} b + \sin^{-1} c = \pi$, then $a\sqrt{1-a^2} + b\sqrt{1-b^2} + c\sqrt{1-c^2}$ is equal to

- (1) $a + b + c$
(2) $a^2 b^2 c^2$
(3) $2abc$
(4) $4abc$

84. If $A + B + C = 3\pi/2$, then $\cos 2A + \cos 2B + \cos 2C$ is equal to

- (1) $1 - 4 \cos A \cos B \cos C$
(2) $4 \sin A \sin B \sin C$
(3) $1 + 2 \cos A \cos B \cos C$
(4) $1 - 4 \sin A \sin B \sin C$

85. If $\tan(\alpha - \beta) = \frac{\sin 2\beta}{3 - \cos 2\beta}$, then
 (1) $\tan \alpha = 2 \tan \beta$ (2) $\tan \beta = 2 \tan \alpha$
 (3) $2 \tan \alpha = 3 \tan \beta$ (4) $3 \tan \alpha = 2 \tan \beta$
86. In any triangle ABC , $\sin^2 A - \sin^2 B + \sin^2 C$ is always equal to
 (1) $2 \sin A \sin B \cos C$ (2) $2 \sin A \cos B \sin C$
 (3) $2 \sin A \cos B \cos C$ (4) $2 \sin A \sin B \sin C$
87. The value of $\sum_{r=0}^{10} \cos^3 \frac{r\pi}{3}$ is equal to
 (1) $1/4$ (2) $1/8$ (3) $-1/4$ (4) $-1/8$
88. In triangle ABC , $\frac{\sin A + \sin B + \sin C}{\sin A + \sin B - \sin C}$ is equal to
 (1) $\tan \frac{A}{2} \cot \frac{B}{2}$ (2) $\cot \frac{A}{2} \tan \frac{B}{2}$
 (3) $\cot \frac{A}{2} \cot \frac{B}{2}$ (4) $\tan \frac{A}{2} \tan \frac{B}{2}$
89. $\frac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C}$ is equal to
 (1) $8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$ (2) $8 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$
 (3) $8 \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}$ (4) $8 \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$
90. If $\cos^2 A + \cos^2 B + \cos^2 C = 1$, then ΔABC is
 (1) equilateral (2) isosceles
 (3) right angled (4) none of these
91. In triangle ABC , $\tan A + \tan B + \tan C = 6$ and $\tan A \tan B = 2$, then the values of $\tan A$, $\tan B$, $\tan C$ are, respectively
 (1) $1, 2, 3$ (2) $3, 2/3, 7/3$
 (3) $4, 1/2, 3/2$ (4) none of these
92. If $\cos x + \cos y - \cos(x+y) = \frac{3}{2}$, then
 (1) $x+y=0$ (2) $x=2y$
 (3) $x=y$ (4) $2x=y$
93. If $a \sin x + b \cos(x+\theta) + b \cos(x-\theta) = d$, then the minimum value of $|\cos \theta|$ is equal to
 (1) $\frac{1}{2|b|} \sqrt{d^2 - a^2}$ (2) $\frac{1}{2|a|} \sqrt{d^2 - a^2}$
 (3) $\frac{1}{2|d|} \sqrt{d^2 - a^2}$ (4) none of these
94. If $u = \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} + \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}$, then the difference between the maximum and minimum values of u^2 is given by
 (1) $2(a^2 + b^2)$ (2) $2\sqrt{a^2 + b^2}$
 (3) $(a+b)^2$ (4) $(a-b)^2$
95. If $\tan x = n \tan y$, $n \in R^+$, then the maximum value of $\sec^2 x$ is equal to
 (1) $\frac{(n+1)^2}{2n}$ (2) $\frac{(n+1)^2}{n}$
 (3) $\frac{(n+1)^2}{2}$ (4) $\frac{(n+1)^2}{4n}$
96. If $0 \leq x \leq \frac{\pi}{3}$ then range of $f(x) = \sec\left(\frac{\pi}{6} - x\right) + \sec\left(\frac{\pi}{6} + x\right)$ is
 (1) $\left(\frac{4}{\sqrt{3}}, \infty\right)$ (2) $\left[\frac{4}{\sqrt{3}}, \infty\right)$
 (3) $\left[0, \frac{4}{\sqrt{3}}\right]$ (4) $\left(0, \frac{4}{\sqrt{3}}\right)$
97. The maximum value of $\cos x \left\{ \sin x + \sqrt{\sin^2 x + \sin^2 \frac{\pi}{6}} \right\}$
 (1) $\frac{\sqrt{5}}{3}$ (2) $\sqrt{\frac{3}{2}}$ (3) $\sqrt{\frac{5}{2}}$ (4) $\frac{\sqrt{5}}{2}$
98. If α, β, γ are acute angles and $\cos \theta = \sin \beta / \sin \alpha$, $\cos \phi = \sin \gamma / \sin \alpha$ and $\cos(\theta - \phi) = \sin \beta \sin \gamma$, then the value $\tan^2 \alpha - \tan^2 \beta - \tan^2 \gamma$ is equal to
 (1) -1 (2) 0 (3) 1 (4) 2
99. The value of $\sum_{n=1}^{\infty} \frac{\tan\left(\frac{\theta}{2^n}\right)}{2^{n-1} \cos \frac{\theta}{2^{n-1}}}$
 (1) $\frac{2}{\sin 2\theta} - \frac{1}{\theta}$ (2) $\frac{2}{\sin 2\theta} + \frac{1}{\theta}$
 (3) $\frac{1}{\sin 2\theta} - \frac{1}{\theta}$ (4) $\frac{1}{\sin \theta} - \frac{1}{\theta}$
100. If $x \sin a + y \sin 2a + z \sin 3a = \sin 4a$,
 $x \sin b + y \sin 2b + z \sin 3b = \sin 4b$,
 $x \sin c + y \sin 2c + z \sin 3c = \sin 4c$,
 then the roots of the equation
 $t^3 - \left(\frac{z}{2}\right)t^2 - \left(\frac{y+2}{4}\right)t + \left(\frac{z-x}{8}\right) = 0$,
 $a, b, c \neq n\pi$, are
 (1) $\sin a, \sin b, \sin c$
 (2) $\cos a, \cos b, \cos c$
 (3) $\sin 2a, \sin 2b, \sin 2c$
 (4) $\cos 2a, \cos 2b, \cos 2c$

Multiple Correct Answers Type

1. If $\cos \beta$ is the geometric mean between $\sin \alpha$ and $\cos \alpha$, where $0 < \alpha, \beta < \pi/2$, then $\cos 2\beta$ is equal to

(1) $-2 \sin^2\left(\frac{\pi}{4} - \alpha\right)$ (2) $-2 \cos^2\left(\frac{\pi}{4} + \alpha\right)$

(3) $2 \sin^2\left(\frac{\pi}{4} + \alpha\right)$ (4) $2 \cos^2\left(\frac{\pi}{4} - \alpha\right)$

2. Which of the following statements are always correct (where Q denotes the set of rationals)?

- (1) $\cos 2\theta \in Q$ and $\sin 2\theta \in Q \Rightarrow \tan \theta \in Q$ (if defined)
 (2) $\tan \theta \in Q \Rightarrow \sin 2\theta, \cos 2\theta$ and $\tan 2\theta \in Q$ (if defined)
 (3) if $\sin \theta \in Q$ and $\cos \theta \in Q \Rightarrow \tan 3\theta \in Q$ (if defined)
 (4) if $\sin \theta \in Q \Rightarrow \cos 3\theta \in Q$

3. Which of the following quantities are rational?

(1) $\sin\left(\frac{11\pi}{12}\right) \sin\left(\frac{5\pi}{12}\right)$

(2) $\operatorname{cosec}\left(\frac{9\pi}{10}\right) \sec\left(\frac{4\pi}{5}\right)$

(3) $\sin^4\left(\frac{\pi}{8}\right) + \cos^4\left(\frac{\pi}{8}\right)$

(4) $\left(1 + \cos \frac{2\pi}{9}\right)\left(1 + \cos \frac{4\pi}{9}\right)\left(1 + \cos \frac{8\pi}{9}\right)$

4. In which of the following sets the inequality $\sin^6 x + \cos^6 x > 5/8$ holds good?

- (1) $(-\pi/8, \pi/8)$ (2) $(3\pi/8, 5\pi/8)$
 (3) $(\pi/4, 3\pi/4)$ (4) $(7\pi/8, 9\pi/8)$

5. Let $f(x) = x^2 - 2\sqrt{(\sin \sqrt{3} - \sin \sqrt{2})}x - (\cos \sqrt{3} - \cos \sqrt{2})$ then

- (1) $f(x)$ is positive $\forall x \in R$
 (2) $f(x)$ assumes both positive and negative values
 (3) $f(x) = 0$ has no real roots
 (4) $y = f(x)$ touches the line $y = 0$.

6. For $\alpha = \pi/7$ which of the following hold(s) good?

- (1) $\tan \alpha \tan 2\alpha \tan 3\alpha = \tan 3\alpha - \tan 2\alpha - \tan \alpha$
 (2) $\operatorname{cosec} \alpha = \operatorname{cosec} 2\alpha + \operatorname{cosec} 4\alpha$
 (3) $\cos \alpha - \cos 2\alpha + \cos 3\alpha = 1/2$
 (4) $8 \cos \alpha \cos 2\alpha \cos 4\alpha = 1$

7. Which of the following identities, wherever defined, hold(s) good?

- (1) $\cot \alpha - \tan \alpha = 2 \cot 2\alpha$
 (2) $\tan(45^\circ + \alpha) - \tan(45^\circ - \alpha) = 2 \operatorname{cosec} 2\alpha$
 (3) $\tan(45^\circ + \alpha) + \tan(45^\circ - \alpha) = 2 \sec 2\alpha$
 (4) $\tan \alpha + \cot \alpha = 2 \tan 2\alpha$

8. The expression $(\tan^4 x + 2 \tan^2 x + 1) \cos^2 x$, when $x = \pi/12$, can be equal to

- (1) $4(2 - \sqrt{3})$ (2) $4(\sqrt{2} + 1)$
 (3) $16 \cos^2 \pi/12$ (4) $16 \sin^2 \pi/12$

9. Let α, β , and γ be some angles in the first quadrant satisfying $\tan(\alpha + \beta) = 15/8$ and $\operatorname{cosec} \gamma = 17/8$, then which of the following hold(s) good?

- (1) $\alpha + \beta + \gamma = \pi$
 (2) $\cot \alpha \cot \beta \cot \gamma = \cot \alpha + \cot \beta + \cot \gamma$
 (3) $\tan \alpha + \tan \beta + \tan \gamma = \tan \alpha \tan \beta \tan \gamma$
 (4) $\tan \alpha \tan \beta + \tan \beta \tan \gamma + \tan \gamma \tan \alpha = 1$

10. Let $f_n(\theta) = \frac{\cos \frac{\theta}{2} + \cos 2\theta + \cos \frac{7\theta}{2} + \dots + \cos(3n-2)\frac{\theta}{2}}{\sin \frac{\theta}{2} + \sin 2\theta + \sin \frac{7\theta}{2} + \dots + \sin(3n-2)\frac{\theta}{2}}$, then

(1) $f_3\left(\frac{3\pi}{16}\right) = \sqrt{2} - 1$ (2) $f_5\left(\frac{\pi}{28}\right) = \sqrt{2} + 1$

(3) $f_7\left(\frac{\pi}{60}\right) = (2 + \sqrt{3})$ (4) none of these

11. If $\sin(x + 20^\circ) = 2 \sin x \cos 40^\circ$, where $x \in (0, \pi/2)$, then which of the following hold(s) good?

- (1) $\cos 2x = 1/2$ (2) $\operatorname{cosec} 4x = 2$
 (3) $\sec \frac{x}{2} = \sqrt{6} - \sqrt{2}$ (4) $\tan \frac{x}{2} = (2 - \sqrt{3})$

12. The expression $\cos^2(\alpha + \beta) + \cos^2(\alpha - \beta) - \cos 2\alpha \cdot \cos 2\beta$, is

- (1) independent of α
 (2) independent of β
 (3) independent of α and β
 (4) dependent on α and β .

13. If $\cot^3 \alpha + \cot^2 \alpha + \cot \alpha = 1$ then

- (1) $\cos 2\alpha \cdot \tan \alpha = -1$ (2) $\cos 2\alpha \cdot \tan \alpha = 1$
 (3) $\cos 2\alpha - \tan 2\alpha = 1$ (4) $\cos 2\alpha - \tan 2\alpha = -1$

14. If $p = \sin(A - B) \sin(C - D)$, $q = \sin(B - C) \sin(A - D)$, $r = \sin(C - A) \sin(B - D)$ then

- (1) $p + q - r = 0$ (2) $p + q + r = 0$
 (3) $p - q + r = 0$ (4) $p^3 + q^3 + r^3 = 3pqr$

15. If $\cos x - \sin \alpha \cot \beta \sin x = \cos \alpha$, then the value of $\tan(x/2)$ is

- (1) $-\tan(\alpha/2) \cot(\beta/2)$ (2) $\tan(\alpha/2) \tan(\beta/2)$
 (3) $-\cot(\alpha/2) \tan(\beta/2)$ (4) $\cot(\alpha/2) \cot(\beta/2)$

16. Let $f(x) = ab \sin x + b\sqrt{1-a^2} \cos x + c$, where $|a| < 1$, $b > 0$ then

- (1) maximum value of $f(x)$ is b if $c = 0$
 (2) difference of maximum and minimum values of $f(x)$ is $2b$
 (3) $f(x) = c$ if $x = -\cos^{-1} a$
 (4) $f(x) = c$ if $x = \cos^{-1} a$

17. Let $P(k) = \left(1 + \cos \frac{\pi}{4k}\right) \left(1 + \cos \frac{(2k-1)\pi}{4k}\right)$

$\left(1 + \cos \frac{(2k+1)\pi}{4k}\right) \left(1 + \cos \frac{(4k-1)\pi}{4k}\right)$. Then

(1) $P(3) = \frac{1}{16}$

(2) $P(4) = \frac{2-\sqrt{2}}{16}$

(3) $P(5) = \frac{3-\sqrt{5}}{32}$

(4) $P(6) = \frac{2-\sqrt{3}}{16}$

18. If $3 \sin \beta = \sin(2\alpha + \beta)$, then $\tan(\alpha + \beta) - 2 \tan \alpha$ is

- (1) independent of α
 (2) independent of β
 (3) dependent of both α and β
 (4) independent of both α and β

19. $x = \sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha} + \sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}$

then $x^2 = a^2 + b^2 + 2\sqrt{p(a^2 + b^2) - p^2}$, where p is equal to

- (1) $a^2 \cos^2 \alpha + b^2 \sin^2 \alpha$
 (2) $a^2 \sin^2 \alpha + b^2 \cos^2 \alpha$
 (3) $\frac{1}{2} [a^2 + b^2 + (a^2 - b^2) \cos 2\alpha]$
 (4) $\frac{1}{2} [a^2 + b^2 - (a^2 - b^2) \cos 2\alpha]$

20. If $(x-a) \cos \theta + y \sin \theta = (x-a) \cos \phi + y \sin \phi = a$ and $\tan(\theta/2) - \tan(\phi/2) = 2b$, then

- (1) $y^2 = 2ax - (1-b^2)x^2$ (2) $\tan \frac{\theta}{2} = \frac{1}{x} (y+bx)$
 (3) $y^2 = 2bx - (1-a^2)x^2$ (4) $\tan \frac{\phi}{2} = \frac{1}{x} (y-bx)$

21. If $\cos(x-y)$, $\cos x$ and $\cos(x+y)$ are in H.P., then

$$\cos x \sec\left(\frac{y}{2}\right) =$$

- (1) $-\sqrt{3}$ (2) $-\sqrt{2}$ (3) $\sqrt{2}$ (4) $\sqrt{3}$

22. Difference between maximum and minimum values of $(60 \sin \alpha + p \cos \alpha)$ is 122 then p can be

- (1) 61 (2) 11 (3) -61 (4) -11

Linked Comprehension Type**For Problems 1-3**If $\sin \alpha = A \sin(\alpha + \beta)$, $A \neq 0$, then1. The value of $\tan \alpha$ is

- (1) $\frac{A \sin \beta}{1 - A \cos \beta}$ (2) $\frac{A \sin \beta}{1 + A \cos \beta}$
 (3) $\frac{A \cos \beta}{1 - A \sin \beta}$ (4) $\frac{A \sin \beta}{1 + A \cos \beta}$

2. The value of $\tan \beta$ is

- (1) $\frac{\sin \alpha(1 + A \cos \beta)}{A \cos \alpha \cos \beta}$ (2) $\frac{\sin \alpha(1 - A \cos \beta)}{A \cos \alpha \cos \beta}$
 (3) $\frac{\cos \alpha(1 - A \sin \beta)}{A \cos \alpha \cos \beta}$ (4) $\frac{\cos \alpha(1 + A \sin \beta)}{A \cos \alpha \cos \beta}$

3. Which of the following is not the value of $\tan(\alpha + \beta)$?

- (1) $\frac{\sin \beta}{\cos \beta - A}$ (2) $\frac{\sin \alpha \cos \alpha}{A \cos \beta - \sin^2 \alpha}$
 (3) $\frac{\sin \alpha \cos \alpha}{A \cos \beta + \sin^2 \alpha}$ (4) none of these

For Problems 4-6If $\alpha, \beta, \gamma, \delta$ are the solutions of the equation $\tan\left(\theta + \frac{\pi}{4}\right) = 3 \tan \theta$, no two of which have equal tangents.4. The value of $\tan \alpha + \tan \beta + \tan \gamma + \tan \delta$ is

- (1) 1/3 (2) 8/3 (3) -8/3 (4) 0

5. The value of $\tan \alpha \tan \beta \tan \gamma \tan \delta$ is

- (1) -1/3 (2) -2
 (3) 0 (4) none of these

6. The value of $\frac{1}{\tan \alpha} + \frac{1}{\tan \beta} + \frac{1}{\tan \gamma} + \frac{1}{\tan \delta}$ is

- (1) -8 (2) 8 (3) 2/3 (4) 1/3

For Problems 7-9 $\sin \alpha + \sin \beta = \frac{1}{4}$ and $\cos \alpha + \cos \beta = \frac{1}{3}$ 7. The value of $\sin(\alpha + \beta)$ is

- (1) $\frac{24}{25}$ (2) $\frac{13}{25}$
 (3) $\frac{12}{13}$ (4) none of these

8. The value of $\cos(\alpha + \beta)$ is

- (1) $\frac{12}{25}$ (2) $\frac{7}{25}$
 (3) $\frac{12}{13}$ (4) none of these

9. The value of $\tan(\alpha + \beta)$ is

- (1) $\frac{25}{7}$ (2) $\frac{25}{12}$ (3) $\frac{25}{13}$ (4) $\frac{24}{7}$

For Problems 10-12To find the sum $\sin^2 \frac{2\pi}{7} + \sin^2 \frac{4\pi}{7} + \sin^2 \frac{8\pi}{7}$, we follow the following method.Put $7\theta = 2n\pi$, where n is any integer. Then

$$\sin 4\theta = \sin(2n\pi - 3\theta) = -\sin 3\theta$$

This means that $\sin \theta$ takes the values 0, $\pm \sin(2\pi/7)$, $\pm \sin(4\pi/7)$ and $\pm \sin(8\pi/7)$.

From Eq. (i), we now get

$$2 \sin 2\theta \cos 2\theta = 4 \sin^3 \theta - 3 \sin \theta$$

$$\text{or } 4 \sin \theta \cos \theta (1 - 2 \sin^2 \theta) = \sin \theta (4 \sin^2 \theta - 3)$$

Rejecting the value $\sin \theta = 0$, we get

$$4 \cos \theta (1 - 2 \sin^2 \theta) = 4 \sin^2 \theta - 3$$

$$\text{or } 16 \cos^2 \theta (1 - 2 \sin^2 \theta)^2 = (4 \sin^2 \theta - 3)^2$$

$$\text{or } 16 (1 - \sin^2 \theta) (1 - 4 \sin^2 \theta + 4 \sin^4 \theta) = 16 \sin^4 \theta - 24 \sin^2 \theta + 9$$

$$64 \sin^6 \theta - 112 \sin^4 \theta - 56 \sin^2 \theta - 7 = 0$$

or This is cubic in $\sin^2 \theta$ with the roots $\sin^2(2\pi/7)$, $\sin^2(4\pi/7)$, and $\sin^2(8\pi/7)$.

The sum of these roots is

$$\sin^2 \frac{2\pi}{7} + \sin^2 \frac{4\pi}{7} + \sin^2 \frac{8\pi}{7} = \frac{112}{64} = \frac{7}{4}.$$

Now answer the following questions:

10. The value of $\left(\tan^2 \frac{\pi}{7} + \tan^2 \frac{2\pi}{7} + \tan^2 \frac{3\pi}{7} \right) \times$

$$\left(\cot^2 \frac{\pi}{7} + \cot^2 \frac{2\pi}{7} + \cot^2 \frac{3\pi}{7} \right) \text{ is}$$

- (1) 105 (2) 35
(3) 210 (4) none of these

11. The value of $\frac{\tan^2 \frac{\pi}{7} + \tan^2 \frac{2\pi}{7} + \tan^2 \frac{3\pi}{7}}{\cot^2 \frac{\pi}{7} + \cot^2 \frac{2\pi}{7} + \cot^2 \frac{3\pi}{7}}$ is

- (1) 7 (2) 35/3
(3) 21/5 (4) none of these

12. The value of $\tan^2 \frac{\pi}{7} \tan^2 \frac{2\pi}{7} \tan^2 \frac{3\pi}{7}$ is

- (1) -3 (2) 7
(3) -5 (4) none of these

For Problems 13–15

In $\triangle ABC$, if $\cos A \cos B \cos C = \frac{\sqrt{3}-1}{8}$ and $\sin A \sin B \sin C = \frac{3+\sqrt{3}}{8}$, then

13. The value of $\tan A + \tan B + \tan C$ is

- (1) $\frac{3+\sqrt{3}}{\sqrt{3}-1}$ (2) $\frac{\sqrt{3}+4}{\sqrt{3}-1}$ (3) $\frac{6-\sqrt{3}}{\sqrt{3}-1}$ (4) $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-1}$

14. The value of $\tan A \tan B + \tan B \tan C + \tan C \tan A$ is

- (1) $5-4\sqrt{3}$ (2) $5+4\sqrt{3}$
(3) $6+\sqrt{3}$ (4) $6-\sqrt{3}$

15. The respective values of $\tan A$, $\tan B$ and $\tan C$ are

- (1) $1, \sqrt{3}, \sqrt{2}$ (2) $1, \sqrt{3}, 2$
(3) $1, 2, \sqrt{3}$ (4) $1, \sqrt{3}, 2+\sqrt{3}$

For Problems 16 and 17

If the angles α, β, γ of a triangle satisfy the relation,

$$\sin\left(\frac{\alpha-\beta}{2}\right) + \sin\left(\frac{\alpha-\gamma}{2}\right) + \sin\left(\frac{3\alpha}{2}\right) = \frac{3}{2}, \text{ then}$$

16. The measure of the smallest angle of the triangle is

- (1) 30° (2) 40° (3) 45° (4) 50°

17. Triangle is

- (1) acute angled
(2) right angled but not isosceles
(3) isosceles
(4) isosceles right angled

For Problems 18–20

A line OA of length r starts from its initial position OX and traces an angle $AOB = \alpha$ in the anticlockwise direction. It then traces back in the clockwise direction an angle $BOC = 3\theta$ (where $\alpha > 3\theta$). L is the foot of the perpendicular from C on OA . Also,

$$\frac{\sin^3 \theta}{CL} = \frac{\cos^3 \theta}{OL} = 1.$$

18. $\frac{1-r \cos \alpha}{r \sin \alpha}$ is equal to

- (1) $\tan 2\theta$ (2) $\cot 2\theta$ (3) $\sin 2\theta$ (4) $\cos 2\theta$

19. $\frac{2r \sin \alpha}{1+2r \cos \alpha}$ is equal to

- (1) $\tan^2 \theta$ (2) $\cot^2 \theta$ (3) $\cot 2\theta$ (4) $\tan 2\theta$

20. $\frac{2r^2-1}{r}$ is equal to

- (1) $\sin \alpha$ (2) $\cos \alpha$ (3) $\sin \theta$ (4) $\cos \theta$

Matrix Match Type

1. If $\cos \theta - \sin \theta = \frac{1}{5}$, where $0 < \theta < \frac{\pi}{4}$, then

List I	List II
a. $(\cos \theta + \sin \theta)/2$	p. $\frac{4}{5}$
b. $\sin 2\theta$	q. $\frac{7}{10}$
c. $\cos 2\theta$	r. $\frac{24}{25}$
d. $\cos \theta$	s. $\frac{7}{25}$

2. If $\cos \alpha + \cos \beta = 1/2$ and $\sin \alpha + \sin \beta = 1/3$, then

List I	List II
a. $\cos\left(\frac{\alpha+\beta}{2}\right)$	p. $\pm \frac{\sqrt{13}}{12}$
b. $\cos\left(\frac{\alpha-\beta}{2}\right)$	q. $\frac{2}{3}$
c. $\tan\left(\frac{\alpha+\beta}{2}\right)$	r. $\pm \frac{3}{\sqrt{13}}$
d. $\tan\left(\frac{\alpha-\beta}{2}\right)$	s. $\pm \sqrt{\frac{131}{13}}$

3.

List I	List II
a. The maximum value of $\{\cos(2A+\theta) + \cos(2B+\theta)\}$, where A, B are constants, is	p. $2 \sin(A+B)$
b. The maximum value of $\{\cos 2A + \cos 2B\}$, where $(A+B)$ is constant and $A, B \in (0, \pi/2)$, is	q. $2 \sec(A+B)$

c. The minimum value of $\{\sec 2A + \sec 2B\}$, where $(A+B)$ is constant and $A, B \in (0, \pi/4)$, is	r. $2 \cos(A+B)$
d. The minimum value of $\sqrt{\{\tan \theta + \cot \theta - 2 \cos 2(A+B)\}}$, where A, B are constants and $\theta \in (0, \pi/2)$, is	s. $2 \cos(A-B)$

4.

List I	List II
a. $\cos 20^\circ + \cos 80^\circ - \sqrt{3} \cos 50^\circ$	p. -1
b. $\cos 0^\circ + \cos \frac{\pi}{7} + \cos \frac{2\pi}{7} + \cos \frac{3\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{5\pi}{7} + \cos \frac{6\pi}{7}$	q. $-\frac{3}{4}$
c. $\cos 20^\circ + \cos 40^\circ + \cos 60^\circ - 4 \cos 10^\circ \cos 20^\circ \cos 30^\circ$	r. 1
d. $\cos 20^\circ \cos 100^\circ + \cos 100^\circ \cos 140^\circ - \cos 140^\circ \cos 200^\circ$	s. 0

5.

List I	List II
a. If $x^2 + y^2 = 1$ and $P = (3x - 4x^3)^2 + (3y - 4y^3)^2$, then P is equal to	p. 1
b. If $a + b = 3 - \cos 4\theta$ and $a - b = 4 \sin 2\theta$, then the maximum value of ab is	q. 4
c. The least positive integral value of x for which $3 \cos \theta = x^2 - 8x + 19$ holds good is	r. 5
d. If $x = \frac{4\lambda}{1+\lambda^2}$ and $y = \frac{2-2\lambda^2}{1+\lambda^2}$, where λ is a real parameter, such that $x^2 - xy + y^2$ lies between $[a, b]$ then $(a+b)$ is	s. 8

6.

List I	List II
a. In triangle ABC , $3 \sin A + 4 \cos B = 6$ and $3 \cos A + 4 \sin B = 1$, then $\angle C$ can be	p. 60°
b. In any triangle, if $(\sin A + \sin B + \sin C)(\sin A + \sin B - \sin C) = 3 \sin A \sin B$, then the angle $\angle C$ is	q. 30°
c. If $8 \sin x \cos^5 x - 8 \sin^5 x \cos x = 1$, then x is	r. 165°
d. 'O' is the center of the inscribed circle in a $30^\circ-60^\circ-90^\circ$ triangle ABC with right angled at C . If the circle is tangent to AB at D , then the angle $\angle COD$ is	s. 7.5°

7.

List I	List II
a. The value of $\left(\frac{4 + \sec 20^\circ}{\operatorname{cosec} 20^\circ}\right)^2$, is	p. 1
b. The minimum value of $\frac{1 + \cos 2x + 8 \sin^2 x}{2 \sin 2x}$, $x \in (0, \pi/2)$, is	q. 2

c. The value of $\frac{8 \sin 40^\circ \cdot \sin 50^\circ \cdot \tan 10^\circ}{\cos 80^\circ}$ is	r. 3
d. If $\frac{\cos 5A}{\cos A} + \frac{\sin 5A}{\sin A} \equiv a + b \cos 4A$, then a^2/b is	s. 4

8.

List I	List II
a. The value of $\frac{(1 + \tan 8^\circ)(1 + \tan 37^\circ)}{(1 + \tan 22^\circ)(1 + \tan 23^\circ)}$ is	p. 5
b. In a triangle ABC , $2 \sum \left(\frac{\cot A + \cot B}{\tan A + \tan B} \right)$ equals to	q. 2
c. The minimum value of $ \cos(\cos 3x + 3 \cos x) $ is	r. 0
d. The value of $16 \sin \frac{\pi}{5} \sin \frac{2\pi}{5} \sin \frac{3\pi}{5} \sin \frac{4\pi}{5}$ is	s. 1

Codes

	a	b	c	d
(1)	s	q	r	p
(2)	r	s	q	p
(3)	p	r	s	q
(4)	q	r	s	p

9.

List I	List II
a. In $\triangle ABC$, if $\cos 2A + \cos 2B + \cos 2C = -1$ then we can conclude that triangle is	p. Equilateral triangle
b. In $\triangle ABC$ if $\tan A > 0$, $\tan B > 0$ and $\tan A \tan B < 1$, then triangle is	q. Right angled triangle
c. In $\triangle ABC$ if $\cos^3 A + \cos^3 B + \cos^3 C = 3 \cos A \cos B \cos C$ then triangle is	r. Acute angled triangle
d. In $\triangle ABC$ if $\cot A > 0$, $\cot B > 0$ and $\cot A \cot B < 1$, then triangle is	s. Obtuse angled triangle

Codes

	a	b	c	d
(1)	s	q	r	p
(2)	r	s	q	p
(3)	q	s	p	r
(4)	q	3	s	p

Numerical Value Type

- If $f(\theta) = \frac{1 - \sin 2\theta + \cos 2\theta}{2 \cos 2\theta}$, then value of $f(11^\circ) \cdot f(12^\circ)$ is _____.
- If $f(x) = 2(7 \cos x + 24 \sin x)(7 \sin x - 24 \cos x)$, for $x \in R$, then maximum value of $f(x)$ is _____.

3. In a triangle ABC , $\angle C = \frac{\pi}{2}$. If $\tan\left(\frac{A}{2}\right)$ and $\tan\left(\frac{B}{2}\right)$ are the roots of the equation $ax^2 + bx + c = 0$ ($a \neq 0$), then the value of $\frac{a+b}{c}$ (where, a, b, c are sides of Δ opposite to angles A, B, C , respectively) is _____.
4. If $x, y \in R$ satisfies $(x+5)^2 + (y-12)^2 = (14)^2$, then the minimum value of $\sqrt{x^2 + y^2}$ is _____.
5. Suppose x and y are real numbers such that $\tan x + \tan y = 42$ and $\cot x + \cot y = 49$. Then the value of $\tan(x+y)$ is _____.
6. Let $0 \leq a, b, c, d \leq \pi$, where b and c are not complementary, such that $2 \cos a + 6 \cos b + 7 \cos c + 9 \cos d = 0$ and $2 \sin a - 6 \sin b + 7 \sin c - 9 \sin d = 0$, then the value of $\frac{\cos(a+d)}{\cos(b+c)}$ is _____.
7. Suppose A and B are two angles such that $A, B \in (0, \pi)$, and satisfy $\sin A + \sin B = 1$ and $\cos A + \cos B = 0$. Then the value of $12 \cos 2A + 4 \cos 2B$ is _____.
8. α and β are the positive acute angles and satisfying equations $5 \sin 2\beta = 3 \sin 2\alpha$ and $\tan \beta = 3 \tan \alpha$ simultaneously. Then the value of $\tan \alpha + \tan \beta$ is _____.
9. The absolute value of the expression $\tan \frac{\pi}{16} + \tan \frac{5\pi}{16} + \tan \frac{9\pi}{16} + \tan \frac{13\pi}{16}$ is _____.
10. The greatest integer less than or equal to $\frac{1}{\sqrt{3} \sin 250^\circ} + \frac{1}{\cos 290^\circ}$ is _____.
11. The maximum value of $y = \frac{1}{\sin^6 x + \cos^6 x}$ is _____.
12. The maximum value of $\cos^2(45^\circ + x) + (\sin x - \cos x)^2$ is _____.
13. The value of $\operatorname{cosec} 10^\circ + \operatorname{cosec} 50^\circ - \operatorname{cosec} 70^\circ$ is _____.
14. Number of triangles ABC if $\tan A = x$, $\tan B = x + 1$, and $\tan C = 1 - x$ is _____.
15. If $\log_{10} \sin x + \log_{10} \cos x = -1$ and $\log_{10}(\sin x + \cos x) = \frac{(\log_{10} n) - 1}{2}$, then the value of n is _____.
16. The value of $\frac{\sin 1^\circ + \sin 3^\circ + \sin 5^\circ + \sin 7^\circ}{\cos 1^\circ \cdot \cos 2^\circ \cdot \sin 4^\circ}$ is _____.
17. In a triangle ABC , if $A - B = 120^\circ$ and $\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = \frac{1}{32}$, then the value of $8 \cos C$ is _____.
18. If $\frac{\tan x}{2} = \frac{\tan y}{3} = \frac{\tan z}{5}$, $x + y + z = \pi$ and $\tan^2 x + \tan^2 y + \tan^2 z = \frac{38}{K}$ then $K =$ _____.
19. If $\sin^3 x \cos 3x + \cos^3 x \sin 3x = 3/8$, then the value of $\sin 4x$ is _____.
20. The value of $\operatorname{cosec} \frac{\pi}{18} - 4 \sin \frac{7\pi}{18}$ is _____.
21. If $\tan x + \tan 2x + \tan 3x = \tan x \tan 2x \tan 3x$ then value of $|\sin 3x + \cos 3x|$ is _____.
22. $16 \left(\cos \theta - \cos \frac{\pi}{8} \right) \left(\cos \theta - \cos \frac{3\pi}{8} \right) \left(\cos \theta - \cos \frac{5\pi}{8} \right) \left(\cos \theta - \cos \frac{7\pi}{8} \right) = \lambda \cos 4\theta$, then the value of λ is _____.
23. If $\frac{\tan(\ln 6) \cdot \tan(\ln 2) \cdot \tan(\ln 3)}{\tan(\ln 6) - \tan(\ln 2) - \tan(\ln 3)} = k$, then the value of k is _____.
24. If $\cot(\theta - \alpha)$, $3 \cot \theta$, $\cot(\theta + \alpha)$ are in A.P. and θ is not an integral multiple of $\frac{\pi}{2}$, then the value of $\frac{4 \sin^2 \theta}{3 \sin^2 \alpha} =$ _____.
25. The value of $\frac{2 \sin x}{\sin 3x} + \frac{\tan x}{\tan 3x}$ is _____.
26. If $\cot^2 A \cot^2 B = 3$, then the value of $(2 - \cos 2A)(2 - \cos 2B)$ is _____.
27. The value of $f(x) = x^4 + 4x^3 + 2x^2 - 4x + 7$, when $x = \cot \frac{11\pi}{8}$ is _____.
28. The value of $\sin^2 12^\circ + \sin^2 21^\circ + \sin^2 39^\circ + \sin^2 48^\circ - \sin^2 9^\circ - \sin^2 18^\circ$ is _____.
29. Given that, $f(n\theta) = \frac{2 \sin 2\theta}{\cos 2\theta - \cos 4n\theta}$, and $f(\theta) + f(2\theta) + f(3\theta) + \dots + f(n\theta) = \frac{\sin \lambda\theta}{\sin \theta \sin \mu\theta}$, then the value of $\mu - \lambda$ is _____.
30. Suppose $\sin^3 x \sin 3x = \sum_{m=0}^n C_m \cos mx$ is an identity in x , where $C_0, C_1, \dots, C_n$ are constants, and $C_n \neq 0$, then the value of n is _____.
31. If $\sec \alpha$ is the average of $\sec(\alpha - 2\beta)$ and $\sec(\alpha + 2\beta)$ then the value of $(2 \sin^2 \beta - \sin^2 \alpha)$ is _____.
32. If A, B and C are three values lying in $[0, 2\pi]$ for which $\tan \theta = K$ then $\tan \frac{A}{3} \tan \frac{B}{3} + \tan \frac{B}{3} \tan \frac{C}{3} + \tan \frac{C}{3} \tan \frac{A}{3}$ is equal to _____.
33. The value of $\left[\left(\sin \frac{\pi}{9} \right) \left(4 + \sec \frac{\pi}{9} \right) \right]^2$ is _____.
34. $\left(\frac{\sin 33^\circ}{\sin 11^\circ \sin 49^\circ \sin 71^\circ} \right)^2 + \left(\frac{\cos 33^\circ}{\cos 11^\circ \cos 49^\circ \cos 71^\circ} \right)^2$ is equal to _____.

35. If $f(\theta) = \sin^3 \theta + \sin^3 \left(\theta + \frac{2\pi}{3} \right) + \sin^3 \left(\theta + \frac{4\pi}{3} \right)$ then the value of $f\left(\frac{\pi}{18}\right) + f\left(\frac{7\pi}{18}\right)$ is _____.
36. The value of $\frac{1 + \sin 22^\circ \sin 33^\circ \sin 35^\circ}{\cos^2 22^\circ + \cos^2 33^\circ + \cos^2 35^\circ}$ is _____.

37. If $A > 0, B > 0, A + B = \pi/3$, and maximum value of $\tan A \tan B$ is M then the value of $1/M$ is _____.
38. If $\frac{\sin^3 \theta}{\sin(2\theta + \alpha)} = \frac{\cos^3 \theta}{\cos(2\theta + \alpha)}$ and $\tan 2\theta = \lambda \tan(3\theta + \alpha)$ then the value of λ is _____.

Archives

JEE MAIN

Single Correct Answer Type

- Let A and B denote the statements
 $A : \cos \alpha + \cos \beta + \cos \gamma = 0$
 $B : \sin \alpha + \sin \beta + \sin \gamma = 0$
 If $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -\frac{3}{2}$, then
 (1) A is true and B is false.
 (2) A is false and B is true.
 (3) Both A and B are true.
 (4) Both A and B are false. (AIEEE 2009)
- Let $\cos(\alpha + \beta) = 4/5$ and let $\sin(\alpha - \beta) = 5/13$, where $0 \leq \alpha, \beta \leq \pi/4$. Then $\tan 2\alpha$ is equal to
 (1) $\frac{20}{7}$ (2) $\frac{25}{16}$ (3) $\frac{56}{33}$ (4) $\frac{19}{12}$ (AIEEE 2010)
- If $A = \sin^2 x + \cos^4 x$, then for all real x
 (1) $\frac{3}{4} \leq A \leq \frac{13}{16}$ (2) $\frac{3}{4} \leq A \leq 1$
 (3) $\frac{13}{16} \leq A \leq 1$ (4) $1 \leq A \leq 2$ (AIEEE 2011)
- In a ΔPQR , if $3 \sin P + 4 \cos Q = 6$ and $4 \sin Q + 3 \cos P = 1$, then the angle R is equal to:
 (1) $\frac{5\pi}{6}$ (2) $\frac{\pi}{6}$
 (3) $\frac{\pi}{4}$ (4) $\frac{3\pi}{4}$ (AIEEE 2012)
- If $5(\tan^2 x - \cos^2 x) = 2 \cos 2x + 9$, then the value of $\cos 4x$ is
 (1) $-\frac{7}{9}$ (2) $-\frac{3}{5}$
 (3) $\frac{1}{3}$ (4) $\frac{2}{9}$ (JEE Main 2017)

JEE ADVANCED

Single Correct Answer Type

- The value of $\sum_{k=1}^{13} \frac{1}{\sin\left(\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right) \sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)}$ equal to
 (1) $3 - \sqrt{3}$ (2) $2(3 - \sqrt{3})$
 (3) $2(\sqrt{3} - 1)$ (4) $2(2 + \sqrt{3})$ (JEE Advanced 2010)

Multiple Correct Answers Type

- Let $f : (-1, 1) \rightarrow R$ be such that $f(\cos 4\theta) = \frac{2}{2 - \sec^2 \theta}$ for $\theta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. Then the value(s) of $f\left(\frac{1}{3}\right)$ (are)
 (1) $1 - \sqrt{\frac{3}{2}}$ (2) $1 + \sqrt{\frac{3}{2}}$
 (3) $1 - \sqrt{\frac{2}{3}}$ (4) $1 + \sqrt{\frac{2}{3}}$ (IIT-JEE 2010)
- Let α and β be non-zero real numbers such that $2(\cos \beta - \cos \alpha) + \cos \alpha \cos \beta = 1$. Then which of the following is/are true?
 (1) $\tan\left(\frac{\alpha}{2}\right) + \sqrt{3} \tan\left(\frac{\beta}{2}\right) = 0$
 (2) $\sqrt{3} \tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right) = 0$
 (3) $\tan\left(\frac{\alpha}{2}\right) - \sqrt{3} \tan\left(\frac{\beta}{2}\right) = 0$
 (4) $\sqrt{3} \tan\left(\frac{\alpha}{2}\right) - \tan\left(\frac{\beta}{2}\right) = 0$ (JEE Advanced 2011)

Matrix Match Type

1. Match List I with List II and select the correct answer using the codes given below the lists :

List I	List II
(P) $\left(\frac{1}{y^2} \left(\frac{\cos(\tan^{-1} y) + y \sin(\tan^{-1} y)}{\cot(\sin^{-1} y) + \tan(\sin^{-1} y)} \right)^2 + y^4 \right)^{1/2}$ takes value	(I) $\frac{1}{2} \sqrt{\frac{5}{3}}$
(Q) If $\cos x + \cos y + \cos z = 0 = \sin x + \sin y + \sin z$ then possible value of $\cos \frac{x-y}{2}$ is	(II) $\sqrt{2}$
(R) If $\cos \left(\frac{\pi}{4} - x \right) \cos 2x + \sin x \sin 2x \sec x$ $= \cos x \sin 2x \sec x + \cos \left(\frac{\pi}{4} + x \right) \cos 2x$ then possible value of $\sec x$ is	(III) $1/2$
(S) If $\cot \left(\sin^{-1} \sqrt{1-x^2} \right) = \sin \left(\tan^{-1} (x\sqrt{6}) \right)$, $x \neq 0$, then possible value of x is	(IV) 1

Codes:

- (P) (Q) (R) (S)
 (1) IV III I II
 (2) IV III II I
 (3) III IV II I
 (4) III IV I II

(IIT-JEE 2013)

Numerical Value Type

1. The maximum value of the expression

$$\frac{1}{\sin^2 \theta + 3 \sin \theta \cos \theta + 5 \cos^2 \theta} \text{ is } \underline{\hspace{2cm}}.$$

(IIT-JEE 2010)

2. The positive integer value of $n > 3$ satisfying the equation

$$\frac{1}{\sin \left(\frac{\pi}{n} \right)} = \frac{1}{\sin \left(\frac{2\pi}{n} \right)} + \frac{1}{\sin \left(\frac{3\pi}{n} \right)} \text{ is } \underline{\hspace{2cm}}.$$

(IIT-JEE 2011)

Answers Key

EXERCISES

Single Correct Answer Type

- | | | | | |
|---------|---------|---------|---------|----------|
| 1. (1) | 2. (1) | 3. (4) | 4. (4) | 5. (2) |
| 6. (1) | 7. (1) | 8. (1) | 9. (2) | 10. (2) |
| 11. (4) | 12. (4) | 13. (1) | 14. (1) | 15. (1) |
| 16. (1) | 17. (3) | 18. (4) | 19. (2) | 20. (2) |
| 21. (2) | 22. (2) | 23. (1) | 24. (1) | 25. (3) |
| 26. (2) | 27. (3) | 28. (2) | 29. (2) | 30. (4) |
| 31. (2) | 32. (2) | 33. (3) | 34. (2) | 35. (1) |
| 36. (1) | 37. (1) | 38. (1) | 39. (1) | 40. (2) |
| 41. (2) | 42. (2) | 43. (3) | 44. (4) | 45. (2) |
| 46. (2) | 47. (4) | 48. (3) | 49. (2) | 50. (4) |
| 51. (2) | 52. (3) | 53. (4) | 54. (2) | 55. (4) |
| 56. (1) | 57. (2) | 58. (1) | 59. (1) | 60. (4) |
| 61. (2) | 62. (2) | 63. (4) | 64. (2) | 65. (2) |
| 66. (3) | 67. (2) | 68. (3) | 69. (2) | 70. (2) |
| 71. (1) | 72. (3) | 73. (2) | 74. (3) | 75. (4) |
| 76. (2) | 77. (1) | 78. (4) | 79. (3) | 80. (2) |
| 81. (3) | 82. (2) | 83. (3) | 84. (4) | 85. (1) |
| 86. (2) | 87. (4) | 88. (3) | 89. (1) | 90. (3) |
| 91. (1) | 92. (3) | 93. (1) | 94. (4) | 95. (4) |
| 96. (2) | 97. (4) | 98. (2) | 99. (1) | 100. (2) |

Multiple Correct Answers Type

- | | |
|-----------------------|------------------|
| 1. (1), (2) | 2. (1), (2), (3) |
| 3. (1), (2), (3), (4) | 4. (1), (2), (4) |
| 5. (1), (3) | 6. (1), (2), (3) |

7. (1), (3)

9. (2), (4)

11. (1), (3), (4)

13. (1), (3)

15. (1), (2)

17. (1), (2), (3), (4)

19. (1), (2), (3), (4)

21. (2), (3)

8. (1), (4)

10. (1), (2), (3)

12. (1), (2), (3)

14. (2), (4)

16. (1), (2), (3)

18. (1), (2), (4)

20. (1), (2), (4)

22. (2), (4)

Linked Comprehension Type

- | | | | | |
|---------|---------|---------|---------|---------|
| 1. (1) | 2. (2) | 3. (3) | 4. (4) | 5. (1) |
| 6. (2) | 7. (1) | 8. (2) | 9. (4) | 10. (1) |
| 11. (3) | 12. (2) | 13. (1) | 14. (2) | 15. (4) |
| 16. (2) | 17. (3) | 18. (1) | 19. (4) | 20. (2) |

Matrix Match Type

- $a \rightarrow q; b \rightarrow r; c \rightarrow s; d \rightarrow p$
- $a \rightarrow r; b \rightarrow p; c \rightarrow q; d \rightarrow s$
- $a \rightarrow s; b \rightarrow r; c \rightarrow q; d \rightarrow p$
- $a \rightarrow s; b \rightarrow r; c \rightarrow p; d \rightarrow q$
- $a \rightarrow p; b \rightarrow p; c \rightarrow q; d \rightarrow s$
- $a \rightarrow q; b \rightarrow p; c \rightarrow s; d \rightarrow r$
- $a \rightarrow r; b \rightarrow q; c \rightarrow s; d \rightarrow p$
- (1)
- (3)

Numerical Value Type

- | | | | | |
|----------|----------|--------|--------|----------|
| 1. (0.5) | 2. (625) | 3. (1) | 4. (1) | 5. (294) |
| 6. (7) | 7. (8) | 8. (4) | 9. (4) | 10. (2) |

3.48 Trigonometry

- | | | | | |
|-----------|----------|---------|-----------|----------|
| 11. (4) | 12. (3) | 13. (6) | 14. (0) | 15. (12) |
| 16. (4) | 17. (7) | 18. (3) | 19. (0.5) | 20. (2) |
| 21. (1) | 22. (2) | 23. (1) | 24. (2) | 25. (1) |
| 26. (3) | 27. (6) | 28. (1) | 29. (1) | 30. (6) |
| 31. (1) | 32. (-3) | 33. (3) | 34. (32) | 35. (0) |
| 36. (0.5) | 37. (3) | 38. (2) | | |

ARCHIVES

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Single Correct Answer Type

- | | | | | |
|--------|--------|--------|--------|--------|
| 1. (3) | 2. (3) | 3. (2) | 4. (2) | 5. (1) |
|--------|--------|--------|--------|--------|

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Single Correct Answer Type

1. (3)

Multiple Correct Answers Type

1. (1), (2)

2. (1), (3)

Matrix Match Type

1. (2)

Numerical Value Type

1. (2) 2. (7)

4

Trigonometric Equations

TRIGONOMETRIC EQUATIONS

A trigonometric equation is any equation that contains trigonometric functions. We have already learned about trigonometric identities, which are satisfied for every value of the involved angles whereas, trigonometric equation is satisfied only for some values (finite or infinite in number) of the angles. A value of the unknown angle which satisfies the given trigonometric equation is called a solution or a root of the equation.

For example, equation $2 \sin x = 1$ is satisfied by $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$ so, $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ are solutions of the equation between 0 and 2π . The solutions of a trigonometric equation lying in the interval $[0, 2\pi)$ are called its principal solutions.

We know that trigonometric ratios are periodic functions. Functions $\sin x$, $\cos x$, $\sec x$ and $\operatorname{cosec} x$ are periodic with period 2π and functions $\tan x$ and $\cot x$ are periodic functions with period π . Therefore, solutions of trigonometric equations can be generalized with the help of period of trigonometric functions. The solution consisting of all possible solutions of a trigonometric equation is called its general solution.

For $2 \sin x = 1$, general solutions are given by $x = \frac{\pi}{6} + 2n\pi$ and $x = \frac{5\pi}{6} + 2n\pi$, where $n \in \mathbb{Z}$.

We have following trigonometric equations whose solutions are quadrantile angles.

Equation	Solution
$\sin \theta = 0$ or $\tan \theta = 0$	$\theta = n\pi, n \in \mathbb{Z}$
$\cos \theta = 0$ or $\cot \theta = 0$	$\theta = (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$
$\sin \theta = 1$	$\theta = (4n+1)\frac{\pi}{2}, n \in \mathbb{Z}$
$\sin \theta = -1$	$\theta = (4n-1)\frac{\pi}{2}, n \in \mathbb{Z}$
$\cos \theta = 1$	$\theta = 2n\pi, n \in \mathbb{Z}$
$\cos \theta = -1$	$\theta = (2n+1)\pi, n \in \mathbb{Z}$

ILLUSTRATION 4.1

Find general value of θ which satisfies both $\sin \theta = -1/2$ and $\tan \theta = 1/\sqrt{3}$ simultaneously.

Sol. Here $\sin \theta < 0$ and $\tan \theta > 0$, then θ lies in the third quadrant. Now

$$\sin \theta = -\frac{1}{2}$$

$$\Rightarrow \theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$$

Generalizing, we have $\theta = 2n\pi + \frac{7\pi}{6}, n \in \mathbb{Z}$.

ILLUSTRATION 4.2

Find the values of θ which satisfy $r \sin \theta = 3$ and $r = 4(1 + \sin \theta)$, $0 \leq \theta \leq 2\pi$.

Sol. We have

$$r \sin \theta = 3 \text{ and } r = 4(1 + \sin \theta)$$

Eliminating r , we get

$$4 \sin^2 \theta + 4 \sin \theta - 3 = 0$$

$$\therefore (2 \sin \theta - 1)(2 \sin \theta + 3) = 0$$

$$\Rightarrow \sin \theta = \frac{1}{2}, -\frac{3}{2} \text{ (not possible)}$$

$$\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

Thus, we have two solutions.

ILLUSTRATION 4.3

If $\sin A = \sin B$ and $\cos A = \cos B$, then find the value of A in terms of B .

Sol. $\sin A - \sin B = 0$ and $\cos A - \cos B = 0$

$$\Rightarrow 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} = 0 \text{ and } 2 \sin \frac{A+B}{2} \sin \frac{B-A}{2} = 0$$

We observe that the common factor gives $\sin \frac{A+B}{2} = 0$. Thus,

$$\frac{A+B}{2} = n\pi, n \in \mathbb{Z}$$

$$\text{or } A+B = 2n\pi, n \in \mathbb{Z}$$

$$\text{or } A = 2n\pi + B, n \in \mathbb{Z}$$

ILLUSTRATION 4.4

Find the number of roots of the equation $16 \sec^3 \theta - 12 \tan^2 \theta - 4 \sec \theta = 9$ in interval $(-\pi, \pi)$.

Sol. We have $16 \sec^3 \theta - 12 (\sec^2 \theta - 1) - 4 \sec \theta - 9 = 0$

$$\Rightarrow 16 \sec^3 \theta - 12 \sec^2 \theta - 4 \sec \theta + 3 = 0$$

4.2 Trigonometry

Let $\sec \theta = t$

$$\Rightarrow 16t^3 - 12t^2 - 4t + 3 = 0$$

$$\Rightarrow (2t-1)(2t+1)(4t-3) = 0$$

$$\therefore t = \frac{1}{2}, \frac{-1}{2}, \frac{3}{4}$$

$$\Rightarrow \cos \theta = 2, -2, \frac{4}{3} \quad (\text{Not possible})$$

$\therefore$ Thus, equation has no real roots.

ILLUSTRATION 4.5

Find the number of solutions of $\sin^2 x - \sin x - 1 = 0$ in $[-2\pi, 2\pi]$.

Sol. $\sin^2 x - \sin x - 1 = 0$

$$\Rightarrow \sin x = \frac{1 \pm \sqrt{5}}{2}$$

$$= \frac{1 - \sqrt{5}}{2} \quad [\sin x = \frac{1 + \sqrt{5}}{2} > 1 \text{ not possible}]$$

Hence, x can attain two values in $[0, 2\pi]$ and two more values in $[-2\pi, 0)$. Thus, there are four solutions.

ILLUSTRATION 4.6

If $x \in (0, 2\pi)$ and $y \in (0, 2\pi)$, then find the number of distinct ordered pairs (x, y) satisfying the equation $9 \cos^2 x + \sec^2 y - 6 \cos x - 4 \sec y + 5 = 0$.

Sol. $9 \cos^2 x + \sec^2 y - 6 \cos x - 4 \sec y + 5 = 0$

$$\Rightarrow (3 \cos x - 1)^2 + (\sec y - 2)^2 = 0$$

$$\Rightarrow \cos x = \frac{1}{3} \text{ and } \sec y = 2 \text{ (simultaneously)}$$

$\therefore$ Number of distinct x in $(0, 2\pi)$ is 2 and number of distinct y in $(0, 2\pi)$ is 2.

$\therefore$ Number of distinct ordered pairs $(x, y) = 2 \times 2 = 4$

ILLUSTRATION 4.7

If $2 \tan^2 x - 5 \sec x = 1$ for exactly seven distinct values of $x \in [0, n\pi/2]$, $n \in \mathbb{N}$ then find the greatest value of n .

Sol. $2 \tan^2 x - 5 \sec x = 1$

$$\text{or } 2 \sec^2 x - 2 - 5 \sec x = 1$$

$$\text{or } 2 \sec^2 x - 5 \sec x - 3 = 0$$

$$\text{or } \sec x = 3 \quad (\because \sec x = -\frac{1}{2} \text{ not possible})$$

This gives two values of x in each of $[0, 2\pi]$, $(2\pi, 4\pi]$, $(4\pi, 6\pi]$ and one value in $\left(6\pi, 6\pi + \frac{3\pi}{2}\right]$ for n to be greatest.

$\therefore$ Greatest value of $n = 15$

ILLUSTRATION 4.8

Solve $16^{\sin^2 x} + 16^{\cos^2 x} = 10$, $0 \leq x < 2\pi$.

Sol. $16^{\sin^2 x} + 16^{1-\sin^2 x} = 10$

$$\text{If } 16^{\sin^2 x} = t, \text{ then } t + \frac{16}{t} = 10$$

Then Eq. (i) becomes

$$t^2 - 10t + 16 = 0$$

or $t = 2, 8$

$$\Rightarrow 16^{\sin^2 x} = 16^{1/4} \text{ or } 16^{3/4}$$

$$\Rightarrow \sin x = \pm \frac{1}{2}, \pm \frac{\sqrt{3}}{2}$$

$$\text{Now } \sin x = \frac{1}{2}, \text{ then } x = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\sin x = -\frac{1}{2}, \text{ then } x = \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\sin x = \frac{\sqrt{3}}{2}, \text{ then } x = \frac{\pi}{3}, \frac{2\pi}{3}$$

$$\sin x = -\frac{\sqrt{3}}{2}, \text{ then } x = \frac{4\pi}{3}, \frac{5\pi}{3}$$

Hence, there will be eight solutions in all.

ILLUSTRATION 4.9

Find the number of solutions of the equation $e^{\sin x} - e^{-\sin x} - 4 = 0$.

Sol. Put $e^{\sin x} = t$

$$\Rightarrow t^2 - 4t - 1 = 0$$

$$\Rightarrow t = e^{\sin x} = 2 \pm \sqrt{5}$$

Now $\sin x \in [-1, 1]$. Thus,

$$e^{\sin x} \in [e^{-1}, e^1] \text{ and } 2 \pm \sqrt{5} \notin [e^{-1}, e^1]$$

Hence, there does not exist any solution.

ILLUSTRATION 4.10

Find the number of solutions of $[\cos x] + |\sin x| = 1$ in $\pi \leq x \leq 3\pi$ (where $[.]$ denotes the greatest integer function).

Sol. We have $[\cos x] + |\sin x| = 1$

Now, $-1 \leq \cos x \leq 1$

$\therefore$ Possible values of $[\cos x]$ are 0, 1, -1

Case I:

$$[\cos x] = -1 \text{ or } \cos x \in [-1, 0)$$

$$\therefore -1 + |\sin x| = 1$$

$$\Rightarrow |\sin x| = 2$$

(not possible)

Case II:

$$[\cos x] = 0 \text{ or } \cos x \in [0, 1)$$

$$\therefore |\sin x| = 1$$

$$\Rightarrow x = \frac{3\pi}{2}, \frac{5\pi}{2}$$

Case III:

$$[\cos x] = 1 \text{ or } \cos x = 1$$

$$\therefore |\sin x| = 0$$

$$\therefore x = 2\pi$$

Hence, there are three solutions.

ILLUSTRATION 4.11

If the equation $a \sin x + \cos 2x = 2a - 7$ possesses a solution, then find the values of a .

Sol. The given equation can be written as

$$a \sin x + (1 - 2 \sin^2 x) = 2a - 7$$

$$2 \sin^2 x - a \sin x + 2a - 8 = 0$$

$$\sin x = \frac{a \pm \sqrt{a^2 - 8(2a - 8)}}{4} = \frac{a \pm (a - 8)}{4}$$

$$= \frac{(a - 4)}{2} \quad (\because \sin x = 2 \text{ is not possible})$$

Equation has solution if $-1 \leq (a - 4)/2 \leq 1$. Thus,
 $-2 \leq (a - 4) \leq 2$
 $2 \leq a \leq 6$

ILLUSTRATION 4.12

Find all the possible triplets (a_1, a_2, a_3) such that $a_1 + a_2 \cos(2x) + a_3 \sin^2(x) = 0$ for all real x .

Sol. We have

$$a_1 + a_2 \cos(2x) + a_3 \sin^2 x = 0 \text{ for all real } x$$

$$a_1 + a_2(1 - 2 \sin^2 x) + a_3 \sin^2 x = 0$$

$$= (a_1 + a_2) + (-2a_2 + a_3) \sin^2 x = 0$$

Since this is true for all real x , we must have

$$a_1 + a_2 = 0 \text{ and } -2a_2 + a_3 = 0$$

$$a_2 = -a_1 \text{ and } a_3 = -2a_1$$

Thus, there exists infinite triplets $(a_1, -a_1, -2a_1)$.

IMPORTANT POINTS IN SOLVING TRIGONOMETRIC EQUATIONS

While solving trigonometric equation, we must take care of the following things:

1. Squaring the equation at any step should be avoided as far as possible. If squaring is necessary, check the solution for extraneous values.

ILLUSTRATION 4.13

Find the number of solutions of the equation $\sqrt{\cos 2x + 2} = (\sin x + \cos x)$ in $[0, \pi]$.

Sol. We have $\sqrt{\cos 2x + 2} = (\sin x + \cos x)$

Squaring both sides, we get

$$2 + \cos 2x = 1 + \sin 2x$$

$$\therefore \sin 2x - \cos 2x = 1$$

Again, squaring both sides, we get

$$1 - \sin 4x = 1$$

$$\therefore \sin 4x = 0$$

$$\Rightarrow 4x = n\pi, n \in \mathbb{Z}$$

$$\text{or } x = \frac{n\pi}{4}, \text{ where } x \in [0, \pi]$$

$$\therefore x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$$

But only $x = \frac{\pi}{4}$ and $\frac{\pi}{2}$ satisfy the original equation.

2. Never cancel the terms containing unknown terms which are in product on the two sides. It may cause the loss of a genuine solution.

ILLUSTRATION 4.14

Solve $\sin 2x = 4 \cos x$.

Sol. We have

$$\sin 2x = 4 \cos x$$

$$\therefore 2 \sin x \cos x = 4 \cos x$$

Here, we should not cancel $\cos x$.

$$\text{Now, } 2 \sin x \cos x - 4 \cos x = 0$$

$$\therefore \cos x (2 \sin x - 4) = 0$$

We have $\cos x = 0$ or $\sin x = 2$, which is not possible.

$$\therefore x = (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$$

3. The solution of the equation should not contain such values of angles which make any of the terms undefined or infinite in the original equation. Domain should not change while simplifying the equation. If it changes, necessary corrections must be made.

ILLUSTRATION 4.15

Solve $\frac{\tan 3x - \tan 2x}{1 + \tan 3x \tan 2x} = 1$.

Sol. We have

$$\frac{\tan 3x - \tan 2x}{1 + \tan 3x \tan 2x} = 1$$

$$\Rightarrow \tan(3x - 2x) = 1$$

$$\text{or } \tan x = 1$$

One of the principal solutions of the equation is $x = \frac{\pi}{4}$.

Since period of $\tan x$ is π , general solution is given by

$$x = n\pi + \frac{\pi}{4}, n \in \mathbb{Z}.$$

But for these values of x , term $(\tan 2x)$ in original equation is not defined.

So, $x = n\pi + \frac{\pi}{4}, n \in \mathbb{Z}$ is not the solution of the equation.

ILLUSTRATION 4.16

Solve $\frac{3 \sin \theta - \sin 3\theta}{\sin \theta} + \frac{\cos 3\theta}{\cos \theta} = 1$.

Sol. We have

$$\frac{3 \sin \theta - \sin 3\theta}{\sin \theta} + \frac{\cos 3\theta}{\cos \theta} = 1$$

$$\begin{aligned} \therefore \frac{4\sin^3 \theta}{\sin \theta} + \frac{(4\cos^3 \theta - 3\cos \theta)}{\cos \theta} &= 1 \\ \Rightarrow 4\sin^2 \theta + 4\cos^2 \theta - 3 &= 1 \\ \Rightarrow 1 &= 1 \end{aligned}$$

This is always true except $\theta = \frac{n\pi}{2}, n \in \mathbb{Z}$ for which either $\sin \theta$ or $\cos \theta$ is zero.

ILLUSTRATION 4.17

Solve $\tan x + \tan 2x + \tan 3x = \tan x \tan 2x \tan 3x, x \in [0, \pi]$.

Sol. We have $\tan x + \tan 2x + \tan 3x = \tan x \tan 2x \tan 3x$

We know that $\tan A + \tan B + \tan C = \tan A \tan B \tan C$,
iff $x = n\pi, n \in \mathbb{Z}$.

So from given equation, we have $x + 2x + 3x = n\pi, n \in \mathbb{Z}$

$$\therefore x = n\pi/6, n \in \mathbb{Z}$$

$$\therefore x = \pi/6, 2\pi/6, 3\pi/6, 4\pi/6, 5\pi/6, 6\pi/6$$

$$\text{or } x = \pi/6, \pi/3, \pi/2, 2\pi/3, 5\pi/6, \pi$$

But for $x = \pi/6, 5\pi/6$, $\tan 3x$ is not defined.

And for $x = \pi/2$, $\tan x$ is not defined.

$$\therefore x = \pi/3, 2\pi/3, \pi$$

CONCEPT APPLICATION EXERCISE 4.1

1. Solve $\sin^2 \theta - \cos \theta = \frac{1}{4}, 0 \leq \theta \leq 2\pi$.
2. Solve $\cos^7 x + \sin^4 x = 1$ in the interval $(-\pi, \pi)$.
3. Find the general solution of $(1 - 2\cos \theta)^2 + (\tan \theta + \sqrt{3})^2 = 0$.
4. Solve $\sin 3\theta - \sin \theta = 4\cos^2 \theta - 2$.
5. Solve $\cos 2x = |\sin x|, x \in \left(-\frac{\pi}{2}, \pi\right)$.
6. Find the number of solutions of the equation $\sin^4 x + \cos^4 x - 2\sin^2 x + \frac{3}{4}\sin^2 2x = 0$ in the interval $[0, 2\pi]$.
7. Find number of solutions of the equation $2\sin x + 5\sin^2 x + 8\sin^3 x + \dots \infty = 1$ for $x \in [0, 2\pi]$.
8. Solve the system of equations $x + y = \frac{2\pi}{3}, \cos x + \cos y = \frac{3}{2}$, where x and y are real.
9. Solve $\operatorname{cosec}^2 \theta - \cot^2 \theta = \cos \theta$.
10. Solve $\sin x \tan x - \sin x + \tan x - 1 = 0$ for $x \in [0, 2\pi]$.
11. Find number of solutions of equation $\sin^2 \theta - \frac{4}{\sin^3 \theta - 1} = 1 - \frac{4}{\sin^3 \theta - 1}, \theta \in [0, 6\pi]$.
12. Solve $\log_{|\sin x|} (1 + \cos x) = 2$.

ANSWERS

$$1. \theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$2. x = 0, \pm \frac{\pi}{2}$$

$$3. \theta = 2n\pi + \frac{5\pi}{3}, n \in \mathbb{Z}$$

$$4. \theta = (2n+1)\frac{\pi}{4}, n \in \mathbb{Z} \text{ or } \theta = (4n+1)\frac{\pi}{2}, n \in \mathbb{Z}$$

$$5. x = -\frac{\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$$

6. Four solutions

7. Two solutions

8. No solution

9. No solution

$$10. x = \frac{\pi}{4}, \frac{5\pi}{4}$$

11. Three solutions

12. No solution

TRIGONOMETRIC EQUATION
 $\sin x = \sin \alpha$

We have $\sin x = \sin \alpha$

$$\therefore \sin x - \sin \alpha = 0$$

$$\Rightarrow 2\cos \frac{x+\alpha}{2} \sin \frac{x-\alpha}{2} = 0$$

$$\Rightarrow \cos \frac{x+\alpha}{2} = 0 \text{ or } \sin \frac{x-\alpha}{2} = 0$$

$$\Rightarrow \frac{x+\alpha}{2} = (2m+1)\frac{\pi}{2} \text{ or } \frac{x-\alpha}{2} = m\pi, m \in \mathbb{Z}$$

$$\Rightarrow x = (2m+1)\pi - \alpha \text{ or } x = 2m\pi + \alpha, m \in \mathbb{Z}$$

$$\Rightarrow x = (2m+1)\pi + (-1)^{2m+1}\alpha$$

$$\text{or } x = 2m\pi + (-1)^{2m}\alpha, m \in \mathbb{Z}$$

Combining (1) and (2), we get

$$x = n\pi + (-1)^n \alpha, n \in \mathbb{Z}$$

Note:

For general solution of the equation $\sin \theta = k$, where $-1 \leq k \leq 1$ we have

$$\sin \theta = \sin (\sin^{-1} k)$$

$$\Rightarrow \theta = n\pi + (-1)^n (\sin^{-1} k), n \in \mathbb{Z}$$

ILLUSTRATION 4.18

Solve $2\cos^2 \theta + 3\sin \theta = 0$.

Sol. We have $2\cos^2 \theta + 3\sin \theta = 0$

$$\text{or } 2(1 - \sin^2 \theta) + 3\sin \theta = 0$$

$$\text{or } 2\sin^2 \theta - 3\sin \theta - 2 = 0$$

$$\text{or } (\sin \theta - 2)(2\sin \theta + 1) = 0$$

$$\text{or } 2\sin \theta + 1 = 0$$

$[\because \sin \theta \neq 2]$

$$\text{or } \sin \theta = -\frac{1}{2} = \sin \left(-\frac{\pi}{6}\right)$$

$$\Rightarrow \theta = n\pi + (-1)^n \left(-\frac{\pi}{6}\right), n \in \mathbb{Z}$$

$$= n\pi + (-1)^{n+1} \frac{\pi}{6}, n \in \mathbb{Z}$$

ILLUSTRATION 4.19

 Solve $4 \cos \theta - 3 \sec \theta = \tan \theta$.

Sol. We have $4 \cos \theta - 3 \sec \theta = \tan \theta$

$$4 \cos \theta - \frac{3}{\cos \theta} = \frac{\sin \theta}{\cos \theta}$$

$$4 \cos^2 \theta - 3 = \sin \theta$$

$$4(1 - \sin^2 \theta) - 3 = \sin \theta$$

$$4 \sin^2 \theta + \sin \theta - 1 = 0$$

$$\sin \theta = \frac{-1 \pm \sqrt{1+16}}{8} = \frac{-1 \pm \sqrt{17}}{8}$$

$$\text{Now, } \sin \theta = \frac{-1 + \sqrt{17}}{8}. \text{ Thus,}$$

$$\sin \theta = \sin \alpha, \text{ where } \sin \alpha = \frac{-1 + \sqrt{17}}{8}$$

$$\Rightarrow \theta = n\pi + (-1)^n \alpha$$

$$\text{where } \sin \alpha = \frac{-1 + \sqrt{17}}{8} \text{ and } n \in \mathbb{Z}$$

$$\text{Also, } \sin \theta = \frac{-1 - \sqrt{17}}{8}$$

$$= \sin \theta = \sin \beta, \text{ where } \sin \beta = \frac{-1 - \sqrt{17}}{8}$$

$$= \theta = n\pi + (-1)^n \beta, n \in \mathbb{Z} \text{ where } \sin \beta = \frac{-1 - \sqrt{17}}{8}$$

ILLUSTRATION 4.20

 Solve $\sin^3 \theta \cos \theta - \cos^3 \theta \sin \theta = 1/4$.

Sol. $\sin^3 \theta \cos \theta - \cos^3 \theta \sin \theta = \frac{1}{4}$

$$\text{or } 4 \sin \theta \cos \theta (\sin^2 \theta - \cos^2 \theta) = 1$$

$$\text{or } 2 \sin 2\theta (-\cos 2\theta) = 1$$

$$\text{or } -\sin 4\theta = 1$$

$$\text{or } \sin 4\theta = -1$$

$$\Rightarrow 4\theta = 2n\pi - \frac{\pi}{2}$$

$$\Rightarrow \theta = \frac{n\pi}{2} - \frac{\pi}{8}, n \in \mathbb{Z}$$

ILLUSTRATION 4.21

 Solve $\sqrt{5 - 2 \sin x} = 6 \sin x - 1$
Sol. The equation is meaningful if $\sin x \leq \frac{5}{2}$ which is always true.

 Any x for which $\sin x < \frac{1}{6}$ cannot be solution, since $\sqrt{5 - 2 \sin x} > 0$ for all x .

On squaring the equation, we get

$$5 - 2 \sin x = 36 \sin^2 x - 12 \sin x + 1$$

$$\text{or } 18 \sin^2 x - 5x - 2 = 0$$

$$\therefore (9 \sin x + 2)(2 \sin x - 1) = 0$$

$$\therefore \sin x = -\frac{2}{9} \text{ or } \sin x = \frac{1}{2}$$

$$\text{But } \sin x = -\frac{2}{9} \text{ is not possible } \left(\because \sin x < \frac{1}{6} \text{ is not possible} \right)$$

$$\therefore \sin x = \frac{1}{2}$$

$$\Rightarrow x = n\pi + (-1)^n \frac{\pi}{6}, n \in \mathbb{Z}$$

ILLUSTRATION 4.22

Solve $\frac{\sin^3 \frac{x}{2} - \cos^3 \frac{x}{2}}{2 + \sin x} = \frac{\cos x}{3}$.

Sol. $\frac{\sin^3 \frac{x}{2} - \cos^3 \frac{x}{2}}{2 + \sin x} = \frac{\cos x}{3}$

$$\text{or } \frac{\left(\sin \frac{x}{2} - \cos \frac{x}{2} \right) \left(1 + \sin \frac{x}{2} \cos \frac{x}{2} \right)}{2 \left(1 + \sin \frac{x}{2} \cos \frac{x}{2} \right)} = \frac{\cos x}{3}$$

$$\text{or } \sin \frac{x}{2} - \cos \frac{x}{2} = \frac{2}{3} \cos x$$

$$\text{or } 1 - \sin x = \frac{4}{9} \cos^2 x \quad (\text{On squaring})$$

$$\text{or } \frac{4}{9} \sin^2 x - \sin x + 1 - \frac{4}{9} = 0$$

$$\text{or } 4 \sin^2 x - 9 \sin x + 5 = 0$$

$$\text{or } (4 \sin x - 5)(\sin x - 1) = 0$$

$$\text{or } \sin x = 1$$

$$\Rightarrow x = 2n\pi + \frac{\pi}{2}, n \in \mathbb{Z}$$

ILLUSTRATION 4.23

Solve $\frac{\sqrt{5} - 1}{\sin x} + \frac{\sqrt{10 + 2\sqrt{5}}}{\cos x} = 8, x \in \left(0, \frac{\pi}{2} \right)$.

Sol. We have $\frac{\sqrt{5} - 1}{4} \times \frac{1}{\sin x} + \frac{\sqrt{10 + 2\sqrt{5}}}{4} \times \frac{1}{\cos x} = 2$

$$\Rightarrow \frac{(\sqrt{5} - 1)}{4} \cos x + \frac{\sqrt{10 + 2\sqrt{5}}}{4} \sin x = 2 \sin x \cos x$$

$$\Rightarrow \sin \left(x + \frac{\pi}{10} \right) = \sin 2x \text{ or } \sin \left(x + \frac{\pi}{10} \right) = \sin(\pi - 2x)$$

$$\Rightarrow x + \frac{\pi}{10} = 2x \quad \text{or} \quad x + \frac{\pi}{10} = \pi - 2x$$

$$\Rightarrow x = \frac{\pi}{10} \quad \text{or} \quad 3x = \pi - \frac{\pi}{10} = \frac{9\pi}{10}$$

$$\Rightarrow x = \frac{\pi}{10} \quad \text{or} \quad x = \frac{3\pi}{10}$$

ILLUSTRATION 4.24

Find the general values of x and y satisfying the equations $5 \sin x \cos y = 1$ and $4 \tan x = \tan y$.

Sol. We have

$$5 \sin x \cos y = 1 \quad \text{and} \quad 4 \tan x = \tan y$$

$$\text{or} \quad 5 \sin x \cos y = 1 \quad \text{and} \quad 4 \sin x \cos y = \sin y \cos x$$

$$\text{or} \quad \sin x \cos y = \frac{1}{5} \quad \text{and} \quad \cos x \sin y = \frac{4}{5}$$

$$\Rightarrow \sin x \cos y + \cos x \sin y = 1$$

$$\text{and} \quad \sin x \cos y - \cos x \sin y = -\frac{3}{5}$$

$$\Rightarrow \sin(x+y) = \sin \frac{\pi}{2} \quad \text{and} \quad \sin(x-y) = \sin \left\{ \sin^{-1} \left(-\frac{3}{5} \right) \right\}$$

$$\Rightarrow x+y = n\pi + (-1)^n \frac{\pi}{2} \quad n \in \mathbb{Z}$$

$$\text{and} \quad x-y = m\pi + (-1)^m \sin^{-1} \left(-\frac{3}{5} \right); m \in \mathbb{Z}$$

$$\Rightarrow x = (m+n) \frac{\pi}{2} + (-1)^n \frac{\pi}{4} + (-1)^m \frac{1}{2} \sin^{-1} \left(-\frac{3}{5} \right); m, n \in \mathbb{Z}$$

$$\text{and} \quad y = (n-m) \frac{\pi}{2} + (-1)^n \frac{\pi}{4} + (-1)^{m+1} \frac{1}{2} \sin^{-1} \left(-\frac{3}{5} \right); m, n \in \mathbb{Z}$$

CONCEPT APPLICATION EXERCISE 4.2

1. Solve $2 \sin \theta + 1 = 0$.
2. Solve $\sin^2 n\theta - \sin^2 (n-1)\theta = \sin^2 \theta$.
3. Solve $\cos \theta + \cos 7\theta + \cos 3\theta + \cos 5\theta = 0$.
4. Solve $\tan^2 \theta - 2 \sin \theta = 0$.
5. If $\sin \theta, 1, \cos 2\theta$ are in G.P., then find the general values of θ .
6. Solve $(\sin 10^\circ)^{\tan x + \tan 3x} = \tan 15^\circ + \tan 30^\circ + \tan 15^\circ \tan 30^\circ, x \in (0, \pi]$.

ANSWERS

1. $\theta = n\pi + (-1)^{n+1} \frac{\pi}{6}, n \in \mathbb{Z}$
2. $\theta = k\pi, \frac{p\pi}{n-1}, \frac{(2q+1)\pi}{2n}; k, p, q, n \in \mathbb{Z}$
3. $\theta = \frac{n\pi}{8}, n \in \mathbb{Z}$
4. $\theta = n\pi, n\pi + (-1)^n \frac{\pi}{6}, n \in \mathbb{Z}$
5. $\theta = n\pi + (-1)^{n-1} \frac{\pi}{2}$
6. $x = \frac{\pi}{4}, \frac{3\pi}{4}, \pi$

TRIGONOMETRIC EQUATION**COS $x = \cos \alpha$**

We have $\cos x = \cos \alpha$

$$\Rightarrow \cos x - \cos \alpha = 0$$

$$\Rightarrow -2 \sin \frac{x+\alpha}{2} \sin \frac{x-\alpha}{2} = 0$$

$$\Rightarrow \sin \frac{x+\alpha}{2} = 0 \quad \text{or} \quad \sin \frac{x-\alpha}{2} = 0$$

$$\Rightarrow \frac{x+\alpha}{2} = n\pi \quad \text{or} \quad \frac{x-\alpha}{2} = n\pi$$

$$\Rightarrow x = 2n\pi - \alpha \quad \text{or} \quad x = 2n\pi + \alpha$$

$$\Rightarrow x = 2n\pi \pm \alpha, n \in \mathbb{Z}$$

Note:

For general solution of the equation $\sin \theta = k$, where $-1 \leq k \leq 1$, we have

$$\cos \theta = \cos (\cos^{-1} k)$$

$$\Rightarrow \theta = 2n\pi \pm (\cos^{-1} k), n \in \mathbb{Z}$$

ILLUSTRATION 4.25

Solve $\sqrt{3} \sec 2\theta = 2$.

Sol. We have $\sqrt{3} \sec 2\theta = 2$

$$\text{or} \quad \cos 2\theta = \frac{\sqrt{3}}{2} = \cos \frac{\pi}{6}$$

$$\Rightarrow 2\theta = 2n\pi \pm \frac{\pi}{6}, n \in \mathbb{Z}$$

$$\text{or} \quad \theta = n\pi \pm \frac{\pi}{12}, n \in \mathbb{Z}$$

ILLUSTRATION 4.26

Solve $\sin 2\theta + \cos \theta = 0$.

Sol. We have $\sin 2\theta + \cos \theta = 0$

$$\text{or} \quad \cos \theta = -\sin 2\theta = \cos \left(\frac{\pi}{2} + 2\theta \right)$$

$$\Rightarrow \theta = 2n\pi \pm \left(\frac{\pi}{2} + 2\theta \right), n \in \mathbb{Z}$$

Taking positive sign, we have

$$\theta = 2n\pi + \frac{\pi}{2} + 2\theta$$

$$\theta = 2n\pi - \frac{\pi}{2}, n \in \mathbb{Z}$$

Taking negative sign, we have

$$\theta = 2n\pi - \left(\frac{\pi}{2} + 2\theta \right)$$

$$\Rightarrow \theta = \frac{2n\pi}{3} - \frac{\pi}{6}, n \in \mathbb{Z}$$

ILLUSTRATION 4.27

Solve $\cos \theta + \cos 3\theta - 2 \cos 2\theta = 0$.

Sol. We have $\cos \theta + \cos 3\theta - 2 \cos 2\theta = 0$
 $2 \cos 2\theta \cos \theta - 2 \cos 2\theta = 0$

or $2 \cos 2\theta (\cos \theta - 1) = 0$
 or $\cos 2\theta = 0$ or $\cos \theta - 1 = 0$

or $2\theta = (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$ or $\theta = 2m\pi, m \in \mathbb{Z}$

$\Rightarrow \theta = (2n+1)\frac{\pi}{4}, n \in \mathbb{Z}$ or $\theta = 2m\pi, m \in \mathbb{Z}$

ILLUSTRATION 4.28

Solve $\sec 4\theta - \sec 2\theta = 2$.

Sol. $\frac{1}{\cos 4\theta} - \frac{1}{\cos 2\theta} = 2$

or $\cos 2\theta - \cos 4\theta = 2 \cos 2\theta \cos 4\theta = \cos 2\theta + \cos 6\theta$

or $\cos 6\theta + \cos 4\theta = 0$

or $2 \cos 5\theta \cos \theta = 0$

$\Rightarrow \cos 5\theta = 0$ or $\cos \theta = 0$

$\Rightarrow 5\theta = (2n+1)\frac{\pi}{2}$ or $\theta = (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$

$\Rightarrow \theta = \left(n + \frac{1}{2}\right)\frac{\pi}{5}$ or $\theta = \left(n + \frac{1}{2}\right)\pi, n \in \mathbb{Z}$

ILLUSTRATION 4.29

Solve $5 \cos 2\theta + 2 \cos^2 \frac{\theta}{2} + 1 = 0, -\pi < \theta < \pi$.

Sol. Changing all the values in terms of $\cos \theta$, we get

$5(2 \cos^2 \theta - 1) + (1 + \cos \theta) + 1 = 0$

or $10 \cos^2 \theta + \cos \theta - 3 = 0$

or $(5 \cos \theta + 3)(2 \cos \theta - 1) = 0$

$\Rightarrow \cos \theta = \frac{1}{2}, -\frac{3}{5}$

$\Rightarrow \theta = \frac{\pi}{3}, -\frac{\pi}{3}, \cos^{-1}\left(-\frac{3}{5}\right) = \pi - \cos^{-1} \frac{3}{5}$

and $-\pi + \cos^{-1} \frac{3}{5}$ $[\because -\pi < \theta < \pi]$

ILLUSTRATION 4.30

Find the least positive value of θ for which $(\cos \theta + \cos 2\theta)^3 = \cos^3 \theta + \cos^3 2\theta$.

Sol. We have $(\cos \theta + \cos 2\theta)^3 = \cos^3 \theta + \cos^3 2\theta$

i.e., $(a+b)^3 = a^3 + b^3$

$\Rightarrow ab(a+b) = 0$

$\Rightarrow \cos \theta = 0$ or $\cos 2\theta = 0$ or $\cos \theta + \cos 2\theta = 0$

$\theta_{\min} = \frac{\pi}{2}$ or $\theta_{\min} = \frac{\pi}{4}$ or $\theta_{\min} = \frac{\pi}{3}$

$\Rightarrow \theta_{\min} = \frac{\pi}{4}$

ILLUSTRATION 4.31

Solve $\cos x \cos 2x \cos 3x = 1/4$.

Sol. $\cos x \cos 2x \cos 3x = \frac{1}{4}$

or $2(2 \cos x \cos 3x) \cos 2x = 1$

or $2(\cos 4x + \cos 2x) \cos 2x = 1$

or $2(2 \cos^2 2x - 1 + \cos 2x) \cos 2x = 1$

or $4 \cos^3 2x + 2 \cos^2 2x - 2 \cos 2x - 1 = 0$

or $(2 \cos^2 2x - 1)(2 \cos 2x + 1) = 0$

or $\cos 4x (2 \cos 2x + 1) = 0$

$\therefore \cos 4x = 0$ or $\cos 2x = -1/2$

$\therefore 4x = (2n+1)\frac{\pi}{2}, 2x = 2m\pi \pm \frac{2\pi}{3}, m, n \in \mathbb{Z}$

or $x = (2n+1)\frac{\pi}{8}, x = m\pi \pm \frac{\pi}{3}$

ILLUSTRATION 4.32

Solve the equation $\frac{\sqrt{3}}{2} \sin x - \cos x = \cos^2 x$.

Sol. We have $\sqrt{3} \sin x = 2(\cos x + \cos^2 x)$... (i)

Squaring both sides, we get

$3 \sin^2 x = 4(\cos^2 x + \cos^4 x + 2 \cos^3 x)$

or $3(1 - \cos^2 x) = 4 \cos^2 x + 4 \cos^4 x + 8 \cos^3 x$

or $4 \cos^4 x + 8 \cos^3 x + 7 \cos^2 x - 3 = 0$

or $(\cos x + 1)(2 \cos x - 1)(2 \cos^2 x + 3 \cos x + 3) = 0$

For $\cos x = -1, x = (2n+1)\pi, n \in \mathbb{Z}$

Clearly, $\cos x = -1$ satisfies eq. (i).

Now, $\cos x = \frac{1}{2}$

Clearly, for $\cos x = \frac{1}{2}$ from eq. (i), $\sin x = \frac{\sqrt{3}}{2}$

So, x lies in 1st quadrant.

$\therefore x = 2n\pi + \frac{\pi}{3}, n \in \mathbb{Z}$

ILLUSTRATION 4.33

Solve $8 \sin x = \frac{\sqrt{3}}{\cos x} + \frac{1}{\sin x}$.

Sol. $8 \sin x = \frac{\sqrt{3}}{\cos x} + \frac{1}{\sin x}$

or $8 \sin^2 x \cos x = \sqrt{3} \sin x + \cos x$

or $4(2 \sin x \cos x) \sin x = \sqrt{3} \sin x + \cos x$

or $2(2 \sin 2x \sin x) = \sqrt{3} \sin x + \cos x$

or $2 \cos x - 2 \cos 3x = \sqrt{3} \sin x + \cos x$

or $\cos x - \sqrt{3} \sin x = 2 \cos 3x$

or $\cos 3x = \cos\left(x + \frac{\pi}{3}\right)$

$$\Rightarrow 3x = 2n\pi \pm \left(x + \frac{\pi}{3}\right), n \in \mathbb{Z}$$

By taking positive sign, $x = n\pi + \frac{\pi}{6}, n \in \mathbb{Z}$

By taking negative sign, $x = \frac{n\pi}{2} - \frac{\pi}{12}, n \in \mathbb{Z}$

ILLUSTRATION 4.34

Solve the equation $2(\cos x + \cos 2x) + \sin 2x(1 + 2 \cos x) = 2 \sin x$ for $x \in [-\pi, \pi]$.

Sol. The given equation is

$$2(\cos x + 2 \cos^2 x - 1) + 2 \sin x \cos x (1 + 2 \cos x) - 2 \sin x = 0$$

$$\Rightarrow 2(2 \cos^2 x + \cos x - 1) + 2 \sin x (2 \cos^2 x + \cos x - 1) = 0$$

$$\Rightarrow (\sin x + 1)(2 \cos^2 x + \cos x - 1) = 0$$

$$\Rightarrow \sin x = -1 \quad \text{or} \quad 2 \cos^2 x + 2 \cos x - \cos x - 1 = 0$$

$$\Rightarrow x = 2n\pi - \frac{\pi}{2}, n \in \mathbb{Z} \quad \text{or} \quad (2 \cos x - 1)(\cos x + 1) = 0$$

$$\Rightarrow x = 2n\pi - \frac{\pi}{2}, n \in \mathbb{Z} \quad \text{or} \quad x = 2n\pi \pm \frac{\pi}{3}$$

$$\text{or } x = 2n\pi + \pi, n \in \mathbb{Z}$$

For $-\pi \leq x \leq \pi$, $x = -\frac{\pi}{2}, \pm \frac{\pi}{3}, \pm \pi$

CONCEPT APPLICATION EXERCISE 4.3

1. Solve $\cos \theta = 1/3$.
2. Solve $\tan \theta \tan 4\theta = 1$ for $0 < \theta < \pi$.
3. Solve $\cot(x/2) - \operatorname{cosec}(x/2) = \cot x$.
4. Solve $\cot \theta + \tan \theta = 2 \operatorname{cosec} \theta$.
5. Solve $\sin 6\theta = \sin 4\theta - \sin 2\theta$.
6. Solve $\cos \theta + \cos 2\theta + \cos 3\theta = 0$.
7. Determine the smallest positive value of x which satisfies the equation $\sqrt{1 + \sin 2x} - \sqrt{2} \cos 3x = 0$.
8. If $\cos p\theta + \cos q\theta = 0$, then prove that the different values of θ are in A.P. with common difference $2\pi/(p \pm q)$.
9. Find the number of solutions for the equation $\sin 5x + \sin 3x + \sin x = 0$ for $0 \leq x \leq \pi$.

ANSWERS

1. $\theta = 2n\pi \pm \cos^{-1} \frac{1}{3}, n \in \mathbb{Z}$
2. $\theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}$
3. $x = 4n\pi \pm \frac{2\pi}{3}, n \in \mathbb{Z}$
4. $\theta = 2n\pi \pm \frac{\pi}{3}, n \in \mathbb{Z}$
5. $\theta = \frac{n\pi}{4}$ or $\theta = m\pi \pm \frac{\pi}{6}; m, n \in \mathbb{Z}$
6. $\theta = (2n+1)\frac{\pi}{4}$ or $\theta = 2n\pi \pm \frac{2\pi}{3}, n \in \mathbb{Z}$
7. $\frac{\pi}{16}$
9. Three solutions

TRIGONOMETRIC EQUATION **$\tan x = \tan \alpha$**

We have $\tan x = \tan \alpha$

$$\Rightarrow \frac{\sin x}{\cos x} = \frac{\sin \alpha}{\cos \alpha}$$

$$\Rightarrow \sin x \cos \alpha - \cos x \sin \alpha = 0$$

$$\Rightarrow \sin(x - \alpha) = 0$$

$$\Rightarrow x - \alpha = n\pi$$

$$\Rightarrow x = n\pi + \alpha, \text{ where } n \in \mathbb{Z}$$

Note:

For general solution of the equation $\tan \theta = k$, where $k \in \mathbb{R}$ we have

$$\tan \theta = \tan(\tan^{-1} k)$$

$$\Rightarrow \theta = n\pi + (\tan^{-1} k), n \in \mathbb{Z}$$

ILLUSTRATION 4.35

Solve $\tan 3\theta = -1$.

Sol. $\tan 3\theta = -1 = \tan\left(\frac{-\pi}{4}\right)$

$$\Rightarrow 3\theta = n\pi + \left(\frac{-\pi}{4}\right), n \in \mathbb{Z}$$

$$\text{or } \theta = \frac{n\pi}{3} - \frac{\pi}{12}, n \in \mathbb{Z}$$

ILLUSTRATION 4.36

Solve $2 \tan \theta - \cot \theta = -1$.

Sol. $2 \tan \theta - \cot \theta = -1$

$$\text{or } 2 \tan \theta - \frac{1}{\tan \theta} = -1$$

$$\text{or } 2 \tan^2 \theta + \tan \theta - 1 = 0$$

$$\text{or } (\tan \theta + 1)(2 \tan \theta - 1) = 0$$

$$\text{or } \tan \theta = -1 \text{ or } \tan \theta = \frac{1}{2}$$

$$\text{or } \tan \theta = \tan\left(\frac{-\pi}{4}\right) \text{ or } \tan \theta = \tan\left(\tan^{-1} \frac{1}{2}\right)$$

$$\Rightarrow \theta = n\pi + \left(\frac{-\pi}{4}\right) \text{ or } \theta = m\pi + \alpha,$$

where $m, n \in \mathbb{Z}$ and $\tan \alpha = \frac{1}{2}$.

ILLUSTRATION 4.37

Solve $\tan 5\theta = \cot 2\theta$.

Sol. $\tan 5\theta = \cot 2\theta = \tan\left(\frac{\pi}{2} - 2\theta\right)$

$$\Rightarrow 5\theta = n\pi + \frac{\pi}{2} - 2\theta$$

$$\Rightarrow 7\theta = n\pi + \frac{\pi}{2}$$

$\theta = \frac{n\pi}{7} + \frac{\pi}{14}, \frac{(2n+1)\pi}{14}$, where $n \in \mathbb{Z}$, but $n \neq 3, 10, 17, \dots$
 where $\tan 5\theta$ is not defined.

ILLUSTRATION 4.38

Solve $(\tan^2 x + 2\sqrt{3} \tan x + 7)(\cot^2 y - 2\sqrt{3} \cot y + 8) \leq 20$ for x and y .

Sol. We have
 $((\tan x + \sqrt{3})^2 + 4)((\cot y - \sqrt{3})^2 + 5) \leq 20$

Now, L.H.S. ≥ 20 .
 Therefore, we must have

$$(\tan x + \sqrt{3})^2 = 0 \text{ and } (\cot y - \sqrt{3})^2 = 0$$

$$\tan x = -\sqrt{3} \text{ and } \cot y = \sqrt{3}$$

$$\tan x = -\sqrt{3} \text{ and } \tan y = \frac{1}{\sqrt{3}}$$

$$x = n\pi - \frac{\pi}{3}, n \in \mathbb{Z} \text{ and } y = m\pi + \frac{\pi}{6}, m \in \mathbb{Z}$$

ILLUSTRATION 4.39

Solve $\tan \theta + \tan 2\theta + \sqrt{3} \tan \theta \tan 2\theta = \sqrt{3}$.

$$\text{Sol. } \tan \theta + \tan 2\theta + \sqrt{3} \tan \theta \tan 2\theta = \sqrt{3}$$

$$\tan \theta + \tan 2\theta = \sqrt{3} (1 - \tan \theta \tan 2\theta)$$

$$\frac{\tan \theta + \tan 2\theta}{1 - \tan \theta \tan 2\theta} = \sqrt{3}$$

$$\tan 3\theta = \sqrt{3} = \tan \frac{\pi}{3}$$

$$\Rightarrow 3\theta = n\pi + \frac{\pi}{3}, n \in \mathbb{Z}$$

$$\text{or } \theta = \frac{n\pi}{3} + \frac{\pi}{9}, n \in \mathbb{Z}$$

ILLUSTRATION 4.40

Find common roots of the equations $2\sin^2 x + \sin^2 2x = 2$ and $\sin 2x + \cos 2x = \tan x$.

$$\text{Sol. We have } 2\sin^2 x + \sin^2 2x = 2 \quad \dots(i)$$

$$\text{and } \sin 2x + \cos 2x = \tan x \quad \dots(ii)$$

$$\text{Solving Eq. (i), } \sin^2 2x = 2\cos^2 x$$

$$\Rightarrow 4\sin^2 x \cos^2 x = 2\cos^2 x$$

$$\Rightarrow \cos^2 x (2\sin^2 x - 1) = 0$$

$$\Rightarrow 2\cos^2 x \cos 2x = 0$$

$$\Rightarrow \cos x = 0 \text{ or } \cos 2x = 0$$

$$\Rightarrow x = (2n+1)\frac{\pi}{2} \text{ or } x = (2n+1)\frac{\pi}{4}, n \in \mathbb{Z} \quad \dots(iii)$$

Now solving Eq. (ii),

$$\frac{2\tan x + 1 - \tan^2 x}{1 + \tan^2 x} = \tan x$$

$$\Rightarrow \tan^3 x + \tan^2 x - \tan x - 1 = 0$$

$$\Rightarrow (\tan^2 x - 1)(\tan x + 1) = 0$$

$$\Rightarrow \tan x = \pm 1$$

$$\Rightarrow x = n\pi \pm \frac{\pi}{4}, n \in \mathbb{Z} \quad \dots(iv)$$

From Eqs. (iii) and (iv), common roots are $(2n+1)\frac{\pi}{4}$.

ILLUSTRATION 4.41

Solve $2\sin^2 x - 5\sin x \cos x - 8\cos^2 x = -2$.

Sol. If $\cos x = 0$, then

$$2\sin^2 x = -2 \text{ or } \sin^2 x = -1, \text{ which is not possible.}$$

Clearly, $\cos x \neq 0$

$$\text{Now, } 2\sin^2 x - 5\sin x \cos x - 8\cos^2 x = -2$$

Dividing both sides by $\cos^2 x$, we get

$$2\tan^2 x - 5\tan x - 8 = -2\sec^2 x$$

$$\text{or } 2\tan^2 x - 5\tan x - 8 + 2(1 + \tan^2 x) = 0$$

$$\text{or } 4\tan^2 x - 5\tan x - 6 = 0$$

$$\text{or } (\tan x - 2)(4\tan x + 3) = 0$$

$$\text{Now, } \tan x - 2 = 0$$

$$\text{Let } \tan x = 2 = \tan \alpha$$

$$\Rightarrow x = n\pi + \alpha = n\pi + \tan^{-1} 2, n \in \mathbb{Z}$$

$$\text{or } 4\tan x + 3 = 0$$

$$\Rightarrow \tan x = -\frac{3}{4} = \tan \beta \text{ (let)}$$

$$\Rightarrow x = m\pi + \tan^{-1} \left(-\frac{3}{4}\right), \text{ where } m \in \mathbb{Z}$$

ILLUSTRATION 4.42

Find the number of roots of the equation $\tan \left(x + \frac{\pi}{6}\right) = 2 \tan x$, for $x \in (0, 3\pi)$.

$$\text{Sol. } \tan \left(x + \frac{\pi}{6}\right) = 2 \tan x$$

$$\Rightarrow \frac{\tan x + \tan \frac{\pi}{6}}{1 - \tan x \tan \frac{\pi}{6}} = 2 \tan x$$

Let $\tan x = y$.

$$\Rightarrow \frac{y + \frac{1}{\sqrt{3}}}{1 - \frac{y}{\sqrt{3}}} = 2y$$

$$\Rightarrow \frac{\sqrt{3}y + 1}{\sqrt{3} - y} = 2y$$

$$\Rightarrow \sqrt{3}y + 1 = 2\sqrt{3}y - 2y^2$$

$$\Rightarrow 2y^2 - \sqrt{3}y + 1 = 0, \text{ which has imaginary roots.}$$

Hence, equation has no solutions.

ILLUSTRATION 4.43Solve $\sin x - 3 \sin 2x + \sin 3x = \cos x - 3 \cos 2x + \cos 3x$.**Sol.** The given equation is

$$\begin{aligned}
 \Rightarrow \sin x - 3 \sin 2x + \sin 3x &= \cos x - 3 \cos 2x + \cos 3x \\
 \Rightarrow 2 \sin 2x \cos x - 3 \sin 2x &= 2 \cos 2x \cos x - 3 \cos 2x \\
 \Rightarrow \sin 2x (2 \cos x - 3) &= \cos 2x (2 \cos x - 3) \\
 \Rightarrow \sin 2x &= \cos 2x & (\text{As } \cos x \neq 3/2) \\
 \Rightarrow \tan 2x &= 1 \\
 \Rightarrow 2x &= n\pi + \pi/4 \\
 \Rightarrow x &= \frac{n\pi}{2} + \frac{\pi}{8}, n \in \mathbb{Z}
 \end{aligned}$$

CONCEPT APPLICATION EXERCISE 4.4

1. If $\tan a\theta - \tan b\theta = 0$, then prove that the values of θ forms an A.P.
2. Solve $\tan^2 \theta + 2\sqrt{3} \tan \theta = 1$.
3. Solve $\tan^2 x + (1 - \sqrt{3}) \tan x - \sqrt{3} = 0$.
4. Solve $3 \cos^2 \theta - 2\sqrt{3} \sin \theta \cos \theta - 3 \sin^2 \theta = 0$.
5. Solve $\tan \theta + \tan(\theta + \pi/3) + \tan(\theta + 2\pi/3) = 3$.
6. Solve $2 \sin^3 x = \cos x$.
7. Solve $7 \cos^2 x + \sin x \cos x - 3 = 0$.
8. If $\tan\left(\frac{p\pi}{4}\right) = \cot\left(\frac{q\pi}{4}\right)$, then prove that $p + q = 2(2n + 1)$, $n \in \mathbb{Z}$.
9. Solve $\sec \theta - 1 = (\sqrt{2} - 1) \tan \theta$.

ANSWERS

2. $\theta = n\pi + \frac{\pi}{12}, n \in \mathbb{Z}$ or $\theta = m\pi - \frac{5\pi}{12}, m \in \mathbb{Z}$
3. $x = n\pi + \frac{\pi}{3}, n \in \mathbb{Z}$ or $x = m\pi - \frac{\pi}{4}, m \in \mathbb{Z}$
4. $\theta = \frac{n\pi}{2} + \frac{\pi}{6}, n \in \mathbb{Z}$
5. $\theta = (4n + 1)\frac{\pi}{12}, n \in \mathbb{Z}$
6. $x = n\pi + \frac{\pi}{4}, n \in \mathbb{Z}$
7. $x = n\pi + \frac{3\pi}{4}$ and $x = k\pi + \tan^{-1} \frac{4}{3} (k, n \in \mathbb{Z})$
9. $\theta = 2n\pi$ or $\theta = 2n\pi + \frac{\pi}{4}, n \in \mathbb{Z}$

TRIGONOMETRIC EQUATION

$(f(x))^2 = (f(\alpha))^2$, WHERE $f(x)$ IS TRIGONOMETRIC FUNCTION

Consider the equations

$$\sin^2 x = \sin^2 \alpha \quad \text{or} \quad \cos^2 x = \cos^2 \alpha$$

Here, both the given equations are same.

$$\sin^2 x - \sin^2 \alpha = 0$$

$$\begin{aligned}
 \therefore \sin(x + \alpha) \sin(x - \alpha) &= 0 \\
 \Rightarrow \sin(x + \alpha) &= 0 \quad \text{or} \quad \sin(x - \alpha) = 0 \\
 \Rightarrow x + \alpha &= n\pi \quad \text{or} \quad x - \alpha = n\pi, n \in \mathbb{Z} \\
 \Rightarrow x &= n\pi \pm \alpha, n \in \mathbb{Z} \\
 \text{Also, } \tan^2 x &= \tan^2 \alpha \\
 \therefore \tan x &= \pm \tan \alpha \\
 \Rightarrow \tan x &= \tan(\pm \alpha) \\
 \Rightarrow x &= n\pi \pm \alpha, n \in \mathbb{Z}
 \end{aligned}$$

ILLUSTRATION 4.44Solve $7 \cos^2 \theta + 3 \sin^2 \theta = 4$.**Sol.** We have $7 \cos^2 \theta + 3 \sin^2 \theta = 4$

$$\Rightarrow 7(1 - \sin^2 \theta) + 3 \sin^2 \theta = 4$$

$$\text{or } 4 \sin^2 \theta = 3$$

$$\text{or } \sin^2 \theta = \frac{3}{4} = \left(\frac{\sqrt{3}}{2}\right)^2$$

$$\text{or } \sin^2 \theta = \sin^2 \frac{\pi}{3}$$

$$\text{or } \theta = n\pi \pm \frac{\pi}{3}, n \in \mathbb{Z}$$

ILLUSTRATION 4.45Solve $\frac{\sin^2 2x + 4 \sin^4 x - 4 \sin^2 x \cos^2 x}{4 - \sin^2 2x - 4 \sin^2 x} = \frac{1}{9}$.**Sol.** We have

$$\begin{aligned}
 &\frac{\sin^2 2x + 4 \sin^4 x - 4 \sin^2 x \cos^2 x}{4 - \sin^2 2x - 4 \sin^2 x} = \frac{1}{9} \\
 \Rightarrow &\frac{4 \sin^4 x}{4 \cos^2 x - 4 \sin^2 x \cos^2 x} = \frac{1}{9} \\
 \Rightarrow &\frac{\sin^4 x}{\cos^4 x} = \frac{1}{9} \\
 \Rightarrow &\tan^2 x = \left(\frac{1}{\sqrt{3}}\right)^2 = \left(\tan \frac{\pi}{6}\right)^2 \\
 \Rightarrow &x = n\pi \pm \frac{\pi}{6}
 \end{aligned}$$

ILLUSTRATION 4.46Solve $\log_{\tan x} (2 + 4 \cos^2 x) = 2$.**Sol.** $\log_{\tan x} (2 + 4 \cos^2 x) = 2$, $\tan x > 0$, $\tan x \neq 1$

$$\begin{aligned}
 \Rightarrow 2 + 4 \cos^2 x &= \tan^2 x \\
 \Rightarrow 3 + 4 \cos^2 x &= \sec^2 x \\
 \Rightarrow 4 \cos^4 x + 3 \cos^2 x - 1 &= 0 \\
 \text{Let } \cos^2 x &= t \\
 \Rightarrow 4t^2 + 3t - 1 &= 0 \\
 \Rightarrow (4t - 1)(t + 1) &= 0 \\
 \Rightarrow t = 1/4 \text{ or } t = -1 \\
 \Rightarrow \cos^2 x &= \frac{1}{4}
 \end{aligned}$$

TRIGONOMETRIC EQUATION

$$a \cos x + b \sin x = c$$

We have $a \cos x + b \sin x = c$

$$\therefore \frac{a}{\sqrt{a^2+b^2}} \cos x + \frac{b}{\sqrt{a^2+b^2}} \sin x = \frac{c}{\sqrt{a^2+b^2}} \quad \dots(1)$$

Let $\frac{a}{\sqrt{a^2+b^2}} = \cos \alpha$ then $\frac{b}{\sqrt{a^2+b^2}} = \sin \alpha$

$$\therefore \cos \alpha \cos x + \sin \alpha \sin x = \frac{c}{\sqrt{a^2+b^2}}$$

$$\Rightarrow \cos(x - \alpha) = \frac{c}{\sqrt{a^2+b^2}} = \cos \beta \text{ (say)} \quad \dots(2)$$

$$\Rightarrow x - \alpha = 2n\pi \pm \beta$$

$$\Rightarrow x = 2n\pi + \alpha \pm \beta, n \in \mathbb{Z}$$

Here, α and β are known as a, b, c are known.

Clearly, eq. (2) has solution if

$$-1 \leq \frac{c}{\sqrt{a^2+b^2}} \leq 1$$

or $\frac{|c|}{\sqrt{a^2+b^2}} \leq 1$

or $|c| \leq \sqrt{a^2+b^2}$

If $|c| > \sqrt{a^2+b^2}$, then equation has no solution.

For eq. (1),

let $\frac{a}{\sqrt{a^2+b^2}} = \sin \alpha$ then $\frac{b}{\sqrt{a^2+b^2}} = \cos \alpha$.

So, the equations reduces to

$$\sin \alpha \cos x + \cos \alpha \sin x = \frac{c}{\sqrt{a^2+b^2}}$$

or $\sin(x + \alpha) = \frac{c}{\sqrt{a^2+b^2}} = \sin \beta \text{ (say)}$

In this case, solution of the equation is

$$x + \alpha = n\pi \pm (-1)^n \beta$$

or $x = n\pi \pm (-1)^n \beta - \alpha, n \in \mathbb{Z}$

ILLUSTRATION 4.49

Solve $\sqrt{3} \cos x + \sin x = \sqrt{2}$.

Sol. We have,

$$\sqrt{3} \cos x + \sin x = \sqrt{2} \quad \dots(1)$$

Dividing both sides by $\sqrt{(\sqrt{3})^2 + 1^2} = 2$, we get

$$\frac{\sqrt{3}}{2} \cos x + \frac{1}{2} \sin x = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \cos\left(x - \frac{\pi}{6}\right) = \frac{1}{\sqrt{2}}$$

$$\cos^2 x = \cos^2 \frac{\pi}{3}$$

$$x = n\pi + \frac{\pi}{3}, n \in \mathbb{Z}$$

$$\left(\because \text{for } x = n\pi - \frac{\pi}{3}, \tan x < 0 \right)$$

ILLUSTRATION 4.47

Solve $4 \cot 2\theta = \cot^2 \theta - \tan^2 \theta$.

Sol. $\frac{4}{\tan 2\theta} = \frac{1}{\tan^2 \theta} - \tan^2 \theta$

$$\frac{4(1 - \tan^2 \theta)}{2 \tan \theta} = \frac{1 - \tan^4 \theta}{\tan^2 \theta} \quad \left[\text{put } \tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} \right]$$

$$(1 - \tan^2 \theta) [2 \tan \theta - (1 + \tan^2 \theta)] = 0$$

$$(1 - \tan^2 \theta) (\tan^2 \theta - 2 \tan \theta + 1) = 0$$

$$(1 - \tan^2 \theta) (\tan \theta - 1)^2 = 0$$

$$\tan \theta = \pm 1$$

$$\theta = n\pi \pm \frac{\pi}{4}, n \in \mathbb{Z}$$

ILLUSTRATION 4.48

Find the most general solution of

$$2^{1+|\cos x| + \cos^2 x + |\cos x|^3 + \dots \infty} = 4.$$

Sol. We have $2^{1+|\cos x| + \cos^2 x + |\cos x|^3 + \dots \infty} = 4$

$$2^{1+|\cos x| + |\cos x|^2 + |\cos x|^3 + \dots \infty} = 4$$

$$\frac{1}{2^{1-|\cos x|}} = 2^2$$

(sum of infinite G.P.)

$$\frac{1}{1 - |\cos x|} = 2$$

$$|\cos x| = \frac{1}{2} \text{ or } \cos x = \pm \frac{1}{2}$$

$$x = n\pi \pm \frac{\pi}{3}, n \in \mathbb{Z}$$

CONCEPT APPLICATION EXERCISE 4.5

1. Solve $\tan^2 \theta + \cot^2 \theta = 2$.
2. Solve $3(\sec^2 \theta + \tan^2 \theta) = 5$.
3. Solve $4 \cos^2 x + 6 \sin^2 x = 5$.
4. Solve $2^{\cos 2x} + 1 = 3 \cdot 2^{-\sin^2 x}$.
5. Find the number of solutions of the equation $\cot^2(\sin x + 3) = 1$ in $[0, 3\pi]$.

ANSWERS

1. $\theta = n\pi \pm \frac{\pi}{4}, n \in \mathbb{Z}$ 2. $\theta = n\pi \pm \frac{\pi}{6}, n \in \mathbb{Z}$

3. $x = n\pi \pm \frac{\pi}{4}, n \in \mathbb{Z}$

4. $x = n\pi, n \in \mathbb{Z}$ or $x = n\pi \pm \frac{\pi}{2}, n \in \mathbb{Z}$

5. Six solutions

$$\Rightarrow \cos\left(x - \frac{\pi}{6}\right) = \cos \frac{\pi}{4}$$

$$\Rightarrow x - \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{4}, n \in \mathbb{Z}$$

$$\Rightarrow x = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{6}$$

$$\Rightarrow x = 2n\pi + \frac{\pi}{4} + \frac{\pi}{6} \quad \text{or} \quad x = 2n\pi - \frac{\pi}{4} + \frac{\pi}{6}$$

$$\Rightarrow x = 2n\pi + \frac{5\pi}{12} \quad \text{or} \quad x = 2n\pi - \frac{\pi}{12}, \text{ where } n \in \mathbb{Z}$$

ILLUSTRATION 4.50Solve $\sqrt{3} \cos \theta - 3 \sin \theta = 4 \sin 2\theta \cos 3\theta$.**Sol.** We have $\sqrt{3} \cos \theta - 3 \sin \theta = 2(\sin 5\theta - \sin \theta)$

$$\text{or} \quad \left(\frac{\sqrt{3}}{2}\right) \cos \theta - \left(\frac{1}{2}\right) \sin \theta = \sin 5\theta$$

$$\Rightarrow \cos\left(\theta + \frac{\pi}{6}\right) = \sin 5\theta = \cos\left(\frac{\pi}{2} - 5\theta\right)$$

$$\Rightarrow \theta + \frac{\pi}{6} = 2n\pi \pm \left(\frac{\pi}{2} - 5\theta\right)$$

$$\text{or} \quad \theta = \frac{n\pi}{3} + \frac{\pi}{18} \quad \text{or} \quad \theta = -\frac{n\pi}{2} + \frac{\pi}{6}, \forall n \in \mathbb{Z}$$

ILLUSTRATION 4.51Find the number of integral values of n so that $\sin x (\sin x + \cos x) = n$ has at least one solution.**Sol.** $\sin x (\sin x + \cos x) = n$

$$\text{or} \quad \sin^2 x + \sin x \cos x = n$$

$$\text{or} \quad \frac{1 - \cos 2x}{2} + \frac{\sin 2x}{2} = n$$

$$\text{or} \quad \sin 2x - \cos 2x = 2n - 1$$

$$\Rightarrow -\sqrt{2} \leq 2n - 1 \leq \sqrt{2}$$

$$\text{or} \quad \frac{1 - \sqrt{2}}{2} \leq n \leq \frac{1 + \sqrt{2}}{2}$$

$$\text{or} \quad n = 0, 1$$

ILLUSTRATION 4.52Find the smallest positive values of x and y satisfying $x - y = \frac{\pi}{4}$ and $\cot x + \cot y = 2$.**Sol.** Given, $x - y = \frac{\pi}{4}$

$$\cot x + \cot y = 2$$

From Eq. (ii), $\sin(x + y) = 2 \sin x \cdot \sin y$

$$= \cos(x - y) - \cos(x + y)$$

$$= \cos \frac{\pi}{4} - \cos(x + y)$$

$$\text{or} \quad \sin(x + y) + \cos(x + y) = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\text{or} \quad \frac{1}{\sqrt{2}} \sin(x + y) + \frac{1}{\sqrt{2}} \cos(x + y) = \frac{1}{2}$$

$$\text{or} \quad \cos\left(x + y - \frac{\pi}{4}\right) = \cos \frac{\pi}{3}$$

$$\Rightarrow x + y - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{3}, n \in \mathbb{Z}$$

$$\text{or} \quad x + y = 2n\pi \pm \frac{\pi}{3} + \frac{\pi}{4}$$

$$\text{for } n = 0, x + y = \frac{7\pi}{12}$$

 $(\because x, y > 0)$

$$\text{From (i) and (iv), } x = \frac{5\pi}{12}, y = \frac{\pi}{6}$$

Hence, the least positive values of x and y are $\frac{5\pi}{12}$ and $\frac{\pi}{6}$ respectively.**ILLUSTRATION 4.53**For what value of $k \in (-\pi, \pi)$ the equation $\sin x + \cos(k + x) + \cos(k - x) = 2$ has real solutions?**Sol.** $\sin x + \cos(k + x) + \cos(k - x) = 2$

$$\Rightarrow 2 \cos x \cdot \cos k + \sin x = 2$$

This equation is of the form $a \cos x + b \sin x = c$ Here $a = 2 \cos k$, $b = 1$ and $c = 2$ Since for real solutions, $|c| \leq \sqrt{a^2 + b^2}$, we have

$$2 \leq \sqrt{1 + 4 \cos^2 k} \quad \text{or} \quad \cos^2 k \geq \frac{3}{4}$$

$$\Rightarrow \sin^2 k \leq \frac{1}{4} \quad \text{or} \quad \sin^2 k - \frac{1}{4} \leq 0$$

$$\text{or} \quad \left(\sin k + \frac{1}{2}\right) \left(\sin k - \frac{1}{2}\right) \leq 0$$

$$\text{or} \quad -\frac{1}{2} \leq \sin k \leq \frac{1}{2}$$

$$\Rightarrow -\frac{\pi}{6} \leq k \leq \frac{\pi}{6}$$

CONCEPT APPLICATION EXERCISE 4.6

1. Solve $\cot \theta + \operatorname{cosec} \theta = \sqrt{3}$.
2. Solve $\sin \theta + \cos \theta = \sqrt{2} \cos A$.
3. Solve $\sqrt{2} \sec \theta + \tan \theta = 1$.
4. Find the number of integral values of k for which the equation $7 \cos x + 5 \sin x = 2k + 1$ has at least one solution.

ANSWERS

1. $\theta = 2n\pi + \frac{\pi}{3}, n \in \mathbb{Z}$
2. $\theta = 2n\pi + \frac{\pi}{4} \pm A, n \in \mathbb{Z}$
3. $\theta = 2n\pi - \frac{\pi}{4}, n \in \mathbb{Z}$
4. Eight values

SOLVING TRIGONOMETRIC EQUATIONS USING MAXIMUM/MINIMUM VALUE OF FUNCTION

Many times, while solving equation $f(x) = g(x)$, $x \in A$, we come across the following situation:
 If $\forall x \in A$, $f(x) \leq a$ and $g(x) \geq a$, then $f(x) = g(x) = a$.
 Let us study few illustrations to understand this.

ILLUSTRATION 4.54

Find the number of solution(s) of the equation $\cos(\pi\sqrt{x}) \cos(\pi\sqrt{x-4}) = 1$.

Sol. We have
 $\cos(\pi\sqrt{x}) \cos(\pi\sqrt{x-4}) = 1$, where $x \geq 4$
 $\Rightarrow \cos(\pi\sqrt{x}) = 1$ and $\cos(\pi\sqrt{x-4}) = 1$
 $\Rightarrow \pi\sqrt{x} = 2m\pi$ and $(\pi\sqrt{x-4}) = 2n\pi$ ($m, n \in \mathbb{Z}$)
 $\Rightarrow x = 4m^2$ and $x - 4 = 4n^2$
 $\Rightarrow m^2 - n^2 = 1$
 Now, $m^2 - n^2 = 1$
 $\Rightarrow m = 1, n = 0$
 $\therefore x = 4$
 So, $x = 4$ is the only solution of the equation.

ILLUSTRATION 4.55

If $x, y \in [0, 2\pi]$, then find the total number of ordered pairs (x, y) satisfying the equation $\sin x \cos y = 1$.

Sol. $\sin x \cos y = 1$
 $\Rightarrow \sin x = 1, \cos y = 1$ or $\sin x = -1, \cos y = -1$
 If $\sin x = 1, \cos y = 1$; hence, $x = \pi/2, y = 0, 2\pi$
 If $\sin x = -1, \cos y = -1$; hence, $x = 3\pi/2, y = \pi$
 Thus, the possible ordered pairs are $(\frac{\pi}{2}, 0), (\frac{\pi}{2}, 2\pi),$ and $(\frac{3\pi}{2}, \pi)$.

ILLUSTRATION 4.56

Find the number of solutions of $\sin^2 x \cos^2 x = 1 + \cos^2 x \sin^4 x$ in the interval $[0, 2\pi]$.

Sol. $\sin^2 x \cos^2 x = 1 + \cos^2 x \sin^4 x$
 $\therefore \sin^2 x \cos^2 x (1 - \sin^2 x) = 1$
 $\therefore \sin^2 x \cos^4 x = 1$
 $\therefore \sin^2 x = \cos^4 x = 1$, which is not possible.

ILLUSTRATION 4.57

If $3 \sin x + 4 \cos ax = 7$ has at least one solution, then find the possible values of a .

Sol. We have $3 \sin x + 4 \cos ax = 7$ which is possible only when
 $\sin x = 1$ and $\cos ax = 1$
 or $x = (4n+1)\frac{\pi}{2}$ and $ax = 2m\pi; m, n \in \mathbb{Z}$

$$\text{or } (4n+1)\frac{\pi}{2} = \frac{2m\pi}{a}$$

$$\text{or } a = \frac{4m}{4n+1}; m, n \in \mathbb{Z}$$

ILLUSTRATION 4.58

Solve $\cos^{50} x - \sin^{50} x = 1$.

Sol. $\cos^{50} x - \sin^{50} x = 1$ or $\cos^{50} x = 1 + \sin^{50} x$
 $\text{L.H.S.} \leq 1$ and $\text{R.H.S.} \geq 1$
 Hence, we must have $\cos^{50} x = 1 + \sin^{50} x = 1$
 or $\sin x = 0$
 or $x = n\pi, n \in \mathbb{Z}$

ILLUSTRATION 4.59

Solve $\sin^2 x + \cos^2 y = 2 \sec^2 z$ for x, y , and z .

Sol. $\text{L.H.S.} = \sin^2 x + \cos^2 y \leq 2$ [$\because \sin^2 x \leq 1$ and $\cos^2 y \leq 1$]
 $\text{R.H.S.} = 2 \sec^2 z \geq 2$
 Hence, $\text{L.H.S.} = \text{R.H.S.}$ only when $\sin^2 x = 1$, $\cos^2 y = 1$, and $2 \sec^2 z = 2$.
 Thus, $\cos^2 x = 0, \sin^2 y = 0, \cos^2 z = 1$
 $\Rightarrow \cos x = 0, \sin y = 0, \sin z = 0$
 $x = (2m+1)\frac{\pi}{2}, y = n\pi$
 and $z = t\pi$, where $m, n, t \in \mathbb{Z}$.

ILLUSTRATION 4.60

Solve $1 + \sin x \sin^2 \frac{x}{2} = 0$.

Sol. $1 + \sin x \sin^2 \frac{x}{2} = 0$
 or $2 + 2 \sin x \sin^2 \frac{x}{2} = 0$
 or $2 + \sin x (1 - \cos x) = 0$
 or $4 + 2 \sin x - \sin 2x = 0$
 or $\sin 2x = 2 \sin x + 4$

The above result is not possible for any value of x as L.H.S. has maximum value 1 and R.H.S. has minimum value 2.
 Hence, there is no solution.

ILLUSTRATION 4.61

Solve $\cos 4\theta + \sin 5\theta = 2$.

Sol. The equation $\cos 4\theta + \sin 5\theta = 2$ is valid only when $\cos 4\theta = 1$ and $\sin 5\theta = 1$. Thus,
 $4\theta = 2n\pi$ and $5\theta = 2m\pi + \pi/2, n, m \in \mathbb{Z}$
 $\Rightarrow \theta = \frac{2n\pi}{4}$ and $\theta = \frac{2m\pi}{5} + \frac{\pi}{10}, n, m \in \mathbb{Z}$
 Putting $n, m = 0, \pm 1, \pm 2, \dots$, the common value in $[0, 2\pi]$ is $\theta = \pi/2$.
 Therefore, the solution is $\theta = 2k\pi + \pi/2, k \in \mathbb{Z}$.

ILLUSTRATION 4.62

Solve for x and y : $\sqrt{3} \sin x + \cos x = 8y - y^2 - 18$, where $0 \leq x \leq 4\pi$, $y \in \mathbb{R}$.

$$\begin{aligned}\text{Sol. R.H.S. } 8y - y^2 - 18 &= -(y^2 - 8y) - 18 \\ &= -[(y - 4)^2 - 16] - 18 \\ &= -2 - (y - 4)^2\end{aligned}$$

$$\therefore \text{R.H.S.} \leq -2$$

$$\text{whereas L.H.S.} \geq -2$$

$$\therefore \text{Equality is possible only when L.H.S.} = \text{R.H.S.} = -2$$

$$\text{Now, for L.H.S., } 2 \cos(x - \pi/3) = -2$$

$$\therefore \cos(x - \pi/3) = -1$$

$$\therefore x - \pi/3 = \pi \text{ or } 3\pi$$

$$\therefore x = 4\pi/3, 10\pi/3$$

$$\text{For R.H.S.} = -2, y = 4.$$

ILLUSTRATION 4.63

Solve the equation

$$\cos^2 \left[\frac{\pi}{4} (\sin x + \sqrt{2} \cos^2 x) \right] - \tan^2 \left[x + \frac{\pi}{4} \tan^2 x \right] = 1.$$

$$\begin{aligned}\text{Sol. } \cos^2 \left[\frac{\pi}{4} (\sin x + \sqrt{2} \cos^2 x) \right] - \tan^2 \left[x + \frac{\pi}{4} \tan^2 x \right] &= 1 \\ \Rightarrow \sin^2 \left[\frac{\pi}{4} (\sin x + \sqrt{2} \cos^2 x) \right] + \tan^2 \left[x + \frac{\pi}{4} \tan^2 x \right] &= 0\end{aligned}$$

It is possible only when

$$\sin^2 \left[\frac{\pi}{4} (\sin x + \sqrt{2} \cos^2 x) \right] = 0 \quad \dots(i)$$

$$\text{and } \tan^2 \left[x + \frac{\pi}{4} \tan^2 x \right] = 0 \quad \dots(ii)$$

$$\therefore \frac{\pi}{4} (\sin x + \sqrt{2} \cos^2 x) = n\pi, n \in \mathbb{I}$$

$$\text{or } \sin x + \sqrt{2} \cos^2 x = 4n$$

This equation has solution only for $n = 0$. Thus,

$$\sin x + \sqrt{2} \cos^2 x = 0$$

$$\text{i.e., } \sqrt{2} \sin^2 x - \sin x - \sqrt{2} = 0$$

$$\text{or } (\sin x - \sqrt{2}) (\sqrt{2} \sin x + 1) = 0$$

$$\therefore \sin x = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow x = 2k\pi - \pi/4, k \in \mathbb{Z}$$

Also these values of x satisfy Eq. (ii); therefore, the general solution of given equation is given by

$$x = 2k\pi - \frac{\pi}{4}, k \in \mathbb{Z}$$

CONCEPT APPLICATION EXERCISE 4.7

1. Solve $\cos x + \cos 2x + \dots + \cos(nx) = n$, $n \in \mathbb{N}$.
2. Show that $x = 0$ is the only solution satisfying the equation $1 + \sin^2 ax = \cos x$, where a is irrational.
3. Solve $\sin^4 x = 1 + \tan^8 x$.
4. Solve $\sin x \left(\cos \frac{x}{4} - 2 \sin x \right) + \left(1 + \sin \frac{x}{4} - 2 \cos x \right) \cos x = 0$.
5. Solve $12 \sin x + 5 \cos x = 2y^2 - 8y + 21$, to get the values of x and y .
6. Solve $\sin 2x + \cos 4x = 2$.
7. If the equation $\tan(P \cot x) = \cot(P \tan x)$ has a solution $x \in (0, \pi) - \left\{ \frac{\pi}{2} \right\}$, then prove that $P \leq \frac{\pi}{4}$.
8. If $\tan^2 \{ \pi(x+y) \} + \cot^2 \{ \pi(x+y) \} = 1 + \sqrt{\frac{2x}{1+x^2}}$ where $x, y \in \mathbb{R}$, then find the least possible positive value of y .
9. Find the number of real solutions of the equation $(\cos x)^3 + (\sin x)^3 = 1$ in the interval $[0, 2\pi]$.

ANSWERS

1. $x = 0$
3. No solution
4. $x = (8k - 6)\pi, k \in \mathbb{Z}$
5. $x = 2n\pi + \cos^{-1} \left(\frac{5}{13} \right), n \in \mathbb{Z}$ and $y = 2$
6. No solution
8. $1/4$
9. Three solutions

SOLVING EQUATIONS USING GRAPHS

Consider equation $f(x) = g(x)$, where $f(x)$ is trigonometric function and $g(x)$ is either trigonometric or non-trigonometric function.

In many cases, it is not possible to find the exact values of x which satisfy the equation.

However, we can find the number of values of x which satisfy the equation.

To find the number of roots of the equation, we draw graphs of $y = f(x)$ and $y = g(x)$ and find the number of points of intersection.

ILLUSTRATION 4.64

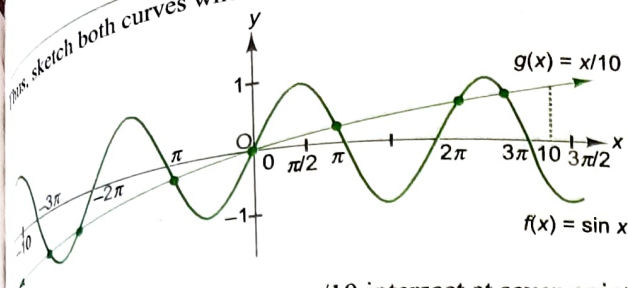
Find the number of solutions of $\sin x = \frac{x}{10}$.

Sol. Here, let $f(x) = \sin x$ and $g(x) = \frac{x}{10}$. Also, we know that $-1 \leq \sin x \leq 1$.

$$\Rightarrow -1 \leq \frac{x}{10} \leq 1$$

$$\text{or } -10 \leq x \leq 10$$

Prob. sketch both curves when $x \in [-10, 10]$.



From figure, $f(x) = \sin x$ and $g(x) = x/10$ intersect at seven points.
∴ the number of solutions is seven.

ILLUSTRATION 4.65

Find the number of roots of equation $x \sin x = 1$.

Sol. We have $x \sin x = 1$

$$\sin x = \frac{1}{x}$$

To find the number of roots of above equation, we draw the graphs of $y = \sin x$ and $y = \frac{1}{x}$ and count the number of points of intersection.

$$\text{Now for } y = \frac{1}{x},$$

$$\text{we have } \frac{1}{x} \rightarrow \infty, \text{ when } x \rightarrow 0^+$$

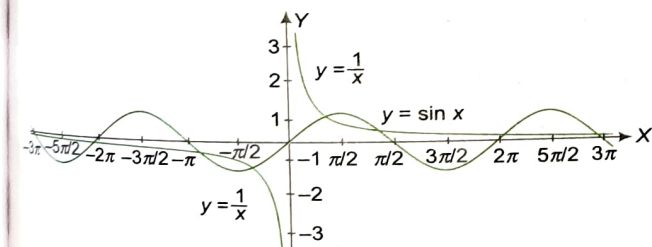
On increasing the value of x from 0^+ onwards, value of $\frac{1}{x}$ decreases.

$$\text{Also, } \frac{1}{x} \rightarrow 0^+, \text{ when } x \rightarrow \infty$$

$$\text{Similarly, } \frac{1}{x} \rightarrow -\infty, \text{ when } x \rightarrow 0^-$$

On decreasing the value of x from 0^- , value of $\frac{1}{x}$ increases.

Thus graphs of $y = \frac{1}{x}$ and $y = \sin x$ are as shown in the following figure.



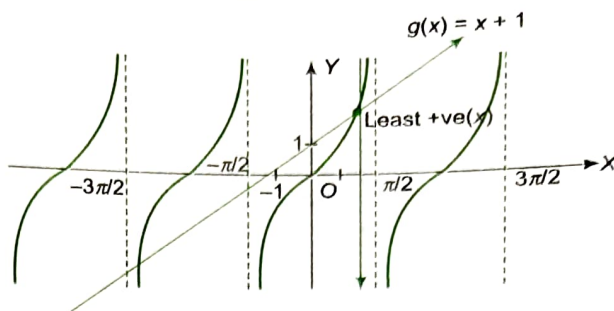
From the figure, we can observe that both graphs will intersect each other infinite number of times.

Hence, equation $\sin x = \frac{1}{x}$ or $x \sin x = 1$ has infinite roots.

ILLUSTRATION 4.66

Prove that the least positive value of x , satisfying $\tan x = x + 1$, lies in the interval $(\pi/4, \pi/2)$.

Sol. Let $f(x) = \tan x$ and $g(x) = x + 1$, which could be graphically represented as follows:



From figure, $\tan x = x + 1$ has infinitely many solutions but the least positive value of $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

ILLUSTRATION 4.67

If m and n ($n > m$) are positive integers, then find the number of solutions of the equation $n|\sin x| = m|\cos x|$ for $x \in [0, 2\pi]$. Also find the solution.

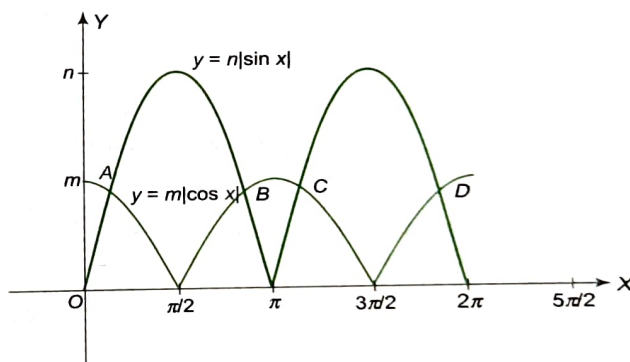
Sol. We have $n|\sin x| = m|\cos x|$

Draw the graphs of $y = n|\sin x|$ and $y = m|\cos x|$.

Range of $n|\sin x|$ and $m|\cos x|$ are $[0, n]$ and $[0, m]$, respectively.

Also, period of each of $n|\sin x|$ and $m|\cos x|$ is π .

Graphs of functions are as shown in the following figure.



From the figure graphs intersect at four points.

Hence, there are four roots of the equation.

For point A, $n \sin x = m \cos x$

$$\therefore \tan x = \frac{m}{n}$$

$$\therefore x = \tan^{-1} \frac{m}{n}$$

$$\text{For point B, } x = \pi - \tan^{-1} \frac{m}{n}$$

$$\text{For point C, } x = \pi + \tan^{-1} \frac{m}{n}$$

$$\text{For point D, } x = 2\pi - \tan^{-1} \frac{m}{n}$$

CONCEPT APPLICATION EXERCISE 4.8

- Find the number of solutions of the equation $\sin x = x^2 + x + 1$.
- Find number of solutions of equation $\sin x = \log_{10} x$.
- Find the number of solutions of the equation $2x = 3\pi(1 - \cos x)$.
- Solve $\tan x = [x]$; $x \in (0, 3\pi/2)$. Here $[.]$ represents the greatest integer function.

ANSWERS

- No solution
- Three solutions
- Five solutions
- $x = \tan^{-1} 4$

TRIGONOMETRIC INEQUALITIES

Consider inequality $f(x) > g(x)$ or $f(x) < g(x)$, where at least one of the functions $f(x)$ and $g(x)$ is trigonometric function.

To solve such inequalities, we draw the graphs of $y = f(x)$ and $y = g(x)$.

The solution set of $f(x) > g(x)$ is the set of values of x for which graph of $y = f(x)$ lies above the graph of $y = g(x)$.

The solution set of $f(x) < g(x)$ is the set of values of x for which graph of $y = f(x)$ lies below the graph of $y = g(x)$.

ILLUSTRATION 4.68

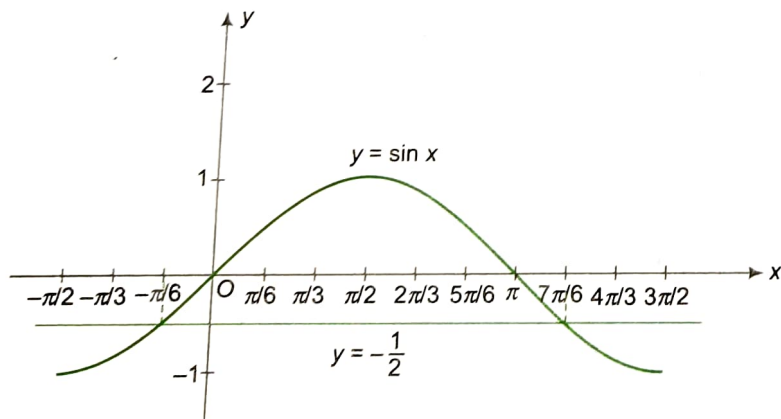
Solve $\sin x > -\frac{1}{2}$.

Sol. We have $\sin x > -\frac{1}{2}$.

Since $\sin x$ is periodic function with period 2π , we solve inequality for the length of the interval 2π and then we generalize the solution by adding $2n\pi$, $n \in \mathbb{Z}$.

Let us solve the inequality for the value of $x \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$ (we can choose any other interval of length 2π)

Draw the graph of $y = \sin x$ and $y = -\frac{1}{2}$ as shown in the following figure.



From above figure, $\sin x > -\frac{1}{2}$ when $-\frac{\pi}{6} < x < \frac{7\pi}{6}$

On generalizing above solution, we get

$$2n\pi - \frac{\pi}{6} < x < 2n\pi + \frac{7\pi}{6}, n \in \mathbb{Z}$$

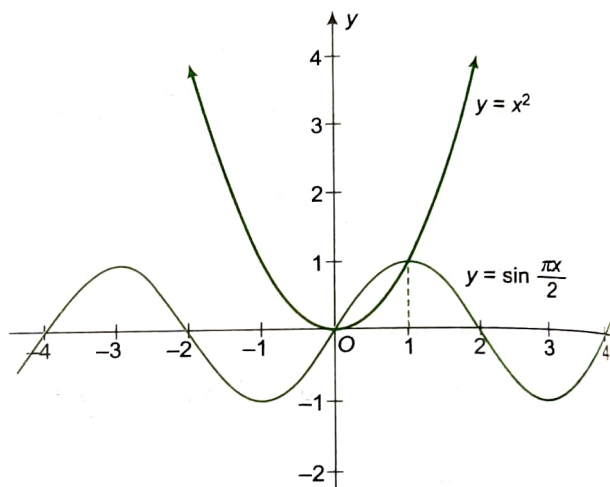
ILLUSTRATION 4.69

Solve $x^2 < \sin \frac{\pi}{2} x$.

Sol. We have $x^2 < \sin \frac{\pi}{2} x$

Function $y = \sin \frac{\pi}{2} x$ has period $\frac{2\pi}{\pi/2} = 4$.

Graphs of $y = x^2$ and $y = \sin \frac{\pi}{2} x$ are as shown in the following figure.



Graphs of the functions intersect at two points $(0, 0)$ and

From the figure, $x^2 < \sin \frac{\pi}{2} x$ for $x \in (0, 1)$.

ILLUSTRATION 4.70

Solve $\sin \theta + \sqrt{3} \cos \theta \geq 1, -\pi < \theta \leq \pi$.

Sol. We have

$$\sin \theta + \sqrt{3} \cos \theta \geq 1, -\pi < \theta \leq \pi$$

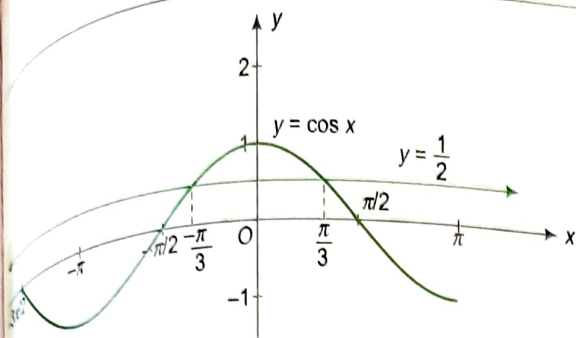
$$\therefore \frac{1}{2} \sin \theta + \frac{\sqrt{3}}{2} \cos \theta \geq \frac{1}{2}$$

$$\Rightarrow \cos(\theta - \pi/6) \geq \frac{1}{2}$$

$$\Rightarrow \cos x \geq \frac{1}{2}, \text{ where } x = \theta - \pi/6$$

$$\therefore x \in \left(-\frac{7\pi}{6}, \frac{5\pi}{6}\right)$$

Draw the graph of $y = \cos x$ and $y = \frac{1}{2}$ as shown in the following figure.



From the figure, $x \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$

$$\theta \in \left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$$

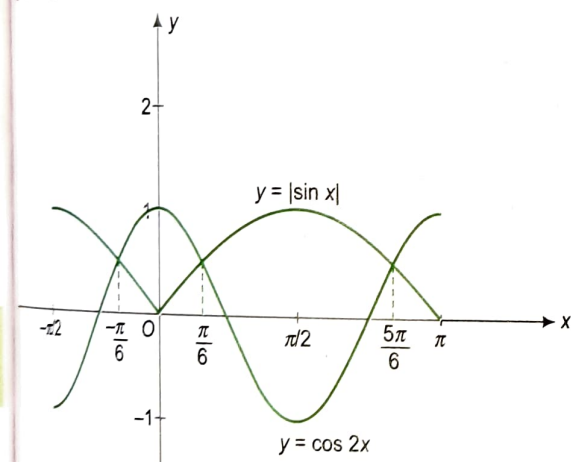
ILLUSTRATION 4.71

Solve $\cos 2x > |\sin x|$, $x \in \left(-\frac{\pi}{2}, \pi\right)$.

Sol. We have

$$\text{We have } \cos 2x > |\sin x|, x \in \left(-\frac{\pi}{2}, \pi\right)$$

Draw the graph of $y = \cos 2x$ and $y = |\sin x|$ as shown in the following figure.



Find points of intersection of graphs,

$$\cos 2x = \sin x$$

$$2 \sin^2 x + \sin x - 1 = 0$$

$$\sin x = -1, \frac{1}{2}$$

$$\sin x \neq -1.$$

$$\sin x = \frac{1}{2}.$$

Clearly from the figure, graphs of $y = |\sin x|$ and $y = \cos 2x$ intersect at $x = \pm \frac{\pi}{6}, \frac{5\pi}{6}$.

Therefore, the solutions set is $x \in \left(-\frac{\pi}{6}, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, \pi\right)$.

CONCEPT APPLICATION EXERCISE 4.9

1. Solve $\sin^2 \theta > \cos^2 \theta$.
2. Solve $\tan x < 2$.
3. Solve $\sin 2x > \sqrt{2} \sin^2 x + (2 - \sqrt{2}) \cos^2 x$.
4. Solve $\tan^3 x + 3 > 3 \tan x + \tan^2 x$.
5. Solve $2 \cos^2 x + \sin x \leq 2$, where $\pi/2 \leq x \leq 3\pi/2$.
6. Solve $\cos x > 1 - \frac{2x}{\pi}$.

ANSWERS

1. $n\pi + \frac{\pi}{4} < \theta < n\pi + \frac{3\pi}{4}, n \in \mathbb{Z}$
2. $x \in \left(n\pi - \frac{\pi}{2}, n\pi + \tan^{-1} 2\right), n \in \mathbb{Z}$
3. $x \in \left(n\pi + \frac{\pi}{8}, n\pi + \frac{\pi}{4}\right), n \in \mathbb{Z}$
4. $x \in \left(n\pi + \frac{\pi}{3}, n\pi + \frac{\pi}{2}\right) \cup \left(n\pi - \frac{\pi}{3}, n\pi + \frac{\pi}{4}\right), n \in \mathbb{Z}$
5. $x \in [\pi/2, 5\pi/6] \cup [\pi, 3\pi/2]$
6. $x \in (0, \pi/2) \cup (\pi, \infty)$

Solved Examples

EXAMPLE 4.1

Find the smallest positive root of the equation $\sqrt{\sin(1-x)} = \sqrt{\cos x}$.

Sol. The given equation is possible if $\sin(1-x) \geq 0$ and $\cos x \geq 0$.

On squaring, we get $\sin(1-x) = \cos x$

$$\text{or } \cos\left(\frac{\pi}{2} - (1-x)\right) = \cos x$$

$$\Rightarrow \frac{\pi}{2} - 1 + x = 2n\pi \pm x, n \in \mathbb{Z}$$

$$\text{or } x = \frac{2n\pi - \frac{\pi}{2} + 1}{2}, n \in \mathbb{Z}$$

For $n = 2$, $x = \frac{7\pi}{4} + \frac{1}{2}$, which is the smallest positive root of the given equation.

EXAMPLE 4.2

Solve $\sin x + \sin y = \sin(x+y)$ and $|x| + |y| = 1$.

Sol. $\sin x + \sin y = \sin(x+y)$

$$\text{or } 2 \sin \frac{x+y}{2} \left[\cos \frac{x-y}{2} - \cos \frac{x+y}{2} \right] = 0$$

$$\text{or } 4 \sin \frac{x+y}{2} \sin \frac{x}{2} \sin \frac{y}{2} = 0$$

$$(a) \sin \frac{x+y}{2} = 0$$

$$\Rightarrow x+y = 2n\pi, n \in \mathbb{Z}$$

$$\Rightarrow x+y = 0 \quad (\because |x| + |y| = 1 \Rightarrow -1 \leq x, y \leq 1)$$

$$(b) \sin \frac{x}{2} = 0$$

$$\Rightarrow x = 2m\pi, m \in \mathbb{Z}$$

$$\Rightarrow x = 0$$

$$(c) \sin \frac{y}{2} = 0$$

$$\Rightarrow y = 2p\pi, p \in \mathbb{Z}$$

$$\Rightarrow y = 0$$

$$\text{From } |x| + |y| = 1$$

$$\text{If } x = 0, \text{ then } |y| = 1$$

$$\Rightarrow y = \pm 1$$

$$\text{If } y = 0, \text{ then } |x| = 1$$

$$\Rightarrow x = \pm 1$$

$$\text{If } y = -x, \text{ then } |x| + |-x| = 2$$

$$\Rightarrow x = \pm \frac{1}{2} \text{ and } y = \mp \frac{1}{2}$$

Hence, solutions are

$$(0, 1), (0, -1), (1, 0), (-1, 0), \left(\frac{1}{2}, -\frac{1}{2}\right),$$

and $\left(-\frac{1}{2}, \frac{1}{2}\right).$

EXAMPLE 4.3

Solve the equation $\tan^4 x + \tan^4 y + 2 \cot^2 x \cot^2 y = 3 + \sin^2(x+y)$ for the values of x and y .

Sol. $\tan^4 x + \tan^4 y + 2 \cot^2 x \cot^2 y = 3 + \sin^2(x+y)$

or $\tan^4 x + \tan^4 y + 2 \cot^2 x \cot^2 y - 2 = 1 + \sin^2(x+y)$

or $(\tan^2 x - \tan^2 y)^2 + 2(\tan x \tan y - \cot x \cot y)^2$
 $= -1 + \sin^2(x+y)$

Now L.H.S. ≥ 0 and R.H.S. ≤ 0

$$\Rightarrow \text{L.H.S.} = \text{R.H.S.} = 0$$

$$\Rightarrow \tan^2 x = \tan^2 y \text{ and } \tan^2 x \tan^2 y = 1$$

and $\sin^2(x+y) = 0$

$$\Rightarrow \tan^2 y = 1 \text{ and } x+y = n\pi, n \in \mathbb{Z}$$

$$\Rightarrow x = m\pi \pm \frac{\pi}{4}, m \in \mathbb{Z} \text{ and } y = p\pi \mp \frac{\pi}{4}, p \in \mathbb{Z}.$$

EXAMPLE 4.4

Find all values of θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ satisfying the equation $(1 - \tan \theta)(1 + \tan \theta) \sec^2 \theta + 2^{\tan^2 \theta} = 0$.

Sol. Given that $(1 - \tan \theta)(1 + \tan \theta) \sec^2 \theta + 2^{\tan^2 \theta} = 0$

or $(1 - \tan^2 \theta)(1 + \tan^2 \theta) + 2^{\tan^2 \theta} = 0$

Let us put $\tan^2 \theta = t$. Then

$$(1-t)(1+t) + 2^t = 0 \quad \text{or} \quad 1-t^2 + 2^t = 0$$

It is clearly satisfied by $t = 3$. Thus, we get

$$\tan^2 \theta = 3$$

Therefore, $\theta = \pm \pi/3$ in the given interval.

EXAMPLE 4.5

For which values of a does the equation $4 \sin(x + \pi/6) \cos(x - \pi/6) = a^2 + \sqrt{3} \sin 2x - \cos 2x$ have solutions? Find solutions for $a = 0$, if any exists.

Sol. The given equation can be rewritten as

$$2[\sin(2x + \pi/6) + \sin \pi/2] = a^2 + \sqrt{3} \sin 2x - \cos 2x$$

or $\cos 2x = \frac{(a^2 - 2)}{2}$

or $2 \cos^2 x = \frac{a^2}{2} \text{ or } \cos^2 x = \left(\frac{a}{2}\right)^2$

or $a^2 \leq 4 \text{ or } -2 \leq a \leq 2$

For $a = 0$, the given equation is reduced to

$$\cos x = 0, \text{ i.e., } x = n\pi + (\pi/2), n \in \mathbb{Z}$$

EXAMPLE 4.6

Solve the following system of simultaneous equations for x and y .

$$4^{\sin x} + 3^{1/\cos y} = 11$$

$$5 \times 16^{\sin x} - 2 \times 3^{1/\cos y} = 2$$

Sol. Let $4^{\sin x} = u$ and $3^{1/\cos y} = v$.

Then, the given system of equations reduces to

$$u + v = 11$$

$$5u^2 - 2v = 2$$

Eliminating v from (i) and (ii), we get

$$5u^2 - 22 + 2u = 2$$

or $(5u + 12)(u - 2) = 0$

or $u - 2 = 0 \quad [\because u = 4^{\sin x} > 0 \text{ for all } x; \therefore 5u + 12 \neq 0]$

or $4^{\sin x} = 2$

or $2 \sin x = 1$

or $\sin x = \frac{1}{2}$

or $\sin x = \sin \frac{\pi}{6}$

or $x = n\pi + (-1)^n \frac{\pi}{6}; n \in \mathbb{Z}$

Putting the value of u in (i), we get $v = 9$

or $3^{1/\cos y} = 3^2$

or $\cos y = \frac{1}{2}$

or $\cos y = \cos \frac{\pi}{3}$

$$y = 2m\pi \pm \frac{\pi}{3}; m \in \mathbb{Z}$$

Hence, $x = n\pi + (-1)^n \frac{\pi}{6}$ and $y = 2m\pi \pm \frac{\pi}{3}$, where $m, n \in \mathbb{Z}$

EXAMPLE 4.7

Solve the equation for x , $\sin^{10} x + \cos^{10} x = \frac{29}{16} \cos^4 2x$.

$$\text{Sol. } \sin^{10} x + \cos^{10} x = \frac{29}{16} \cos^4 2x$$

$$\left(\frac{1 - \cos 2x}{2} \right)^5 + \left(\frac{1 + \cos 2x}{2} \right)^5 = \frac{29}{16} \cos^4 2x$$

Let $\cos 2x = t$. Then

$$\left(\frac{1-t}{2} \right)^5 + \left(\frac{1+t}{2} \right)^5 = \frac{29}{16} t^4$$

$$24t^4 - 10t^2 - 1 = 0$$

$$(2t^2 - 1)(12t^2 + 1) = 0$$

$$t^2 = \frac{1}{2}$$

$$\cos^2 2x = \frac{1}{2} = \left(\frac{1}{\sqrt{2}} \right)^2 = \left(\cos \frac{\pi}{4} \right)^2$$

$$2x = n\pi \pm \frac{\pi}{4}, n \in \mathbb{Z}$$

$$x = \frac{n\pi}{2} \pm \frac{\pi}{8}, n \in \mathbb{Z}$$

EXAMPLE 4.8

Find the number of solutions of the equation

$$1 + e^{\cot^2 x} = \sqrt{2} |\sin x| - 1 + \frac{1 - \cos 2x}{1 + \sin^4 x} \text{ for } x \in (0, 5\pi).$$

$$\text{Sol. L.H.S} = 1 + e^{\cot^2 x} \geq 2$$

$$\text{As } \sqrt{2} |\sin x| - 1 \leq 1$$

$$\text{mid } \frac{1 - \cos 2x}{1 + \sin^4 x} = \frac{2 \sin^2 x}{1 + \sin^4 x} = \frac{2}{\frac{1}{\sin^2 x} + \sin^2 x} \leq 1$$

$$\therefore \text{R.H.S.} = \sqrt{2} |\sin x| - 1 + \frac{1 - \cos 2x}{1 + \sin^4 x} \leq 2$$

Equation will be satisfied if L.H.S. = R.H.S. = 2.

This is possible when $\cot^2 x = 0$ and $|\sin x| = 1$.

$$\Rightarrow x = (2n+1) \frac{\pi}{2}, n \in \mathbb{Z}$$

EXAMPLE 4.9

Find the number of solutions of $\theta \in [0, 2\pi]$ satisfying the

$$\text{equation } (\log_{\sqrt{3}} \tan \theta) \left(\sqrt{(\log_{\tan \theta} 3 + \log_{\sqrt{3}} 3\sqrt{3})} \right) = -1.$$

$$\text{Sol. } (\log_{\sqrt{3}} \tan \theta) \left[\sqrt{\frac{\log_{\sqrt{3}} 3}{\log_{\sqrt{3}} \tan \theta} + \log_e (\sqrt{3})^3} \right] = -1$$

$$\Rightarrow (\log_{\sqrt{3}} \tan \theta) \left[\sqrt{\frac{2}{\log_{\sqrt{3}} \tan \theta} + 3} \right] = -1$$

$$\text{Let } \log_{\sqrt{3}} \tan \theta = y$$

$$\Rightarrow y \sqrt{\frac{2}{y} + 3} = -1$$

$$\Rightarrow \sqrt{\frac{2}{y} + 3} = \frac{-1}{y}$$

$$\Rightarrow \frac{2}{y} + 3 = \frac{1}{y^2}$$

$$\Rightarrow y [3y^2 + 2y - 1] = 0$$

$$\Rightarrow y(3y - 1)(y + 1) = 0$$

$$\Rightarrow y = -1$$

$$\Rightarrow \log_{\sqrt{3}} \tan \theta = -1$$

$$\Rightarrow \tan \theta = \frac{1}{\sqrt{3}}$$

$$\therefore \theta = \frac{\pi}{6} \text{ and } \frac{7\pi}{6}$$

Thus, there are two values of θ in $[0, 2\pi]$.

EXAMPLE 4.10

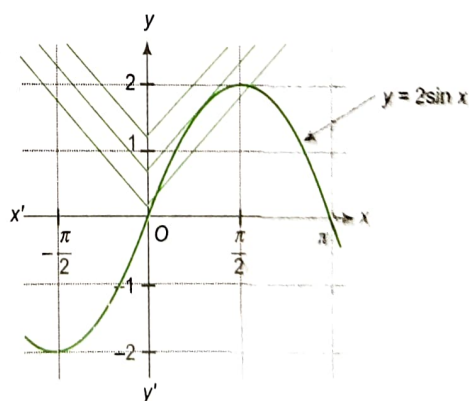
Prove that the equation $2 \sin x = |x| + a$ has no solution for

$$a \in \left(\frac{3\sqrt{3} - \pi}{3}, \infty \right).$$

Sol. We have

$$2 \sin x = |x| + a.$$

Consider graphs of $y = 2 \sin x$ and $y = |x| + a$.



Equation $2 \sin x = |x| + a$ will have a solution so long as the line $y = |x| + a$ intersects or at least touches the curve, $y = 2 \sin x$. In this case, we must have

$$dy/dx = 2 \cos x = 1 = \text{Slope of the line}$$

$$\Rightarrow x = \pi/3.$$

Hence, the solution does not exist if $\frac{\pi}{3} + a > 2 \sin \frac{\pi}{3}$

$$\Rightarrow a > \frac{3\sqrt{3} - \pi}{3}$$

EXAMPLE 4.11

Solve $\sin^2 x + \frac{1}{4} \sin^2 3x = \sin x \sin^2 3x$.

Sol. $\sin^2 x + \frac{1}{4} \sin^2 3x = \sin x \sin^2 3x$

or $\sin^2 x - \sin x \sin^2 3x + \frac{1}{4} \sin^2 3x = 0$

or $\left(\sin x - \frac{1}{2} \sin^2 3x\right)^2 + \frac{1}{4} \sin^2 3x (1 - \sin^2 3x) = 0$

or $\left(\sin x - \frac{1}{2} \sin^2 3x\right)^2 + \frac{1}{4} \sin^2 3x \cos^2 3x = 0$

or $\left(\sin x - \frac{1}{2} \sin^2 3x\right)^2 + \frac{1}{16} \sin^2 6x = 0$

or $\sin x - \frac{1}{2} \sin^2 3x = 0$ and $\sin 6x = 0$

or $2 \sin x = \sin^2 3x$ and $\sin 6x = 0$

From $\sin 6x = 0$, $x = k\pi/6$, $k \in \mathbb{Z}$

From here, we choose those values which satisfy the equation $2 \sin x = \sin^2 3x$. Now,

$$\sin^2 3 \left(\frac{k\pi}{6} \right) = \sin^2 \frac{k\pi}{2} = \begin{cases} 1, & \text{if } k \text{ is odd} \\ 0, & \text{if } k \text{ is even} \end{cases}$$

$$\Rightarrow \sin x = 0 \text{ or } \frac{1}{2}$$

$$\Rightarrow x = n\pi \text{ or } x = n\pi + \frac{\pi}{6} (-1)^n, n \in \mathbb{Z}$$

EXAMPLE 4.12

Prove that equation $2 \cos^2 \frac{x}{2} \sin^2 x = x^2 + x^{-2}$; $0 < x \leq \frac{\pi}{2}$ has no solution.

Sol. The given equation is

$$2 \cos^2 \left(\frac{x}{2} \right) \sin^2 x = x^2 + \frac{1}{x^2}$$

where $0 < x \leq \frac{\pi}{2}$

$$\text{LHS} = 2 \cos^2 \left(\frac{x}{2} \right) \sin^2 x = (1 + \cos x) \sin^2 x$$

$$\because 1 + \cos x < 2 \text{ and } \sin^2 x \leq 1 \text{ for } 0 < x < \frac{\pi}{2}$$

$$\therefore (1 + \cos x) \sin^2 x < 2$$

$$\text{Also, R.H.S.} = x^2 + \frac{1}{x^2} \geq 2$$

Therefore, for $0 < x \leq \frac{\pi}{2}$, given equation is not possible for any real value of x .

Exercises

Single Correct Answer Type

1. If $\sin \theta = \frac{1}{2}$ and $\cos \theta = -\frac{\sqrt{3}}{2}$, then the general value of θ is ($n \in \mathbb{Z}$)

(1) $2n\pi + \frac{5\pi}{6}$ (2) $2n\pi + \frac{\pi}{6}$

(3) $2n\pi + \frac{7\pi}{6}$ (4) $2n\pi + \frac{\pi}{4}$

2. The most general value for which $\tan \theta = -1$, $\cos \theta = \frac{1}{\sqrt{2}}$ is ($n \in \mathbb{Z}$)

(1) $n\pi + \frac{7\pi}{4}$ (2) $n\pi + (-1)^n \frac{7\pi}{4}$

(3) $2n\pi + \frac{7\pi}{4}$ (4) none of these

3. Sum of roots of the equation $x^4 - 2x^2 \sin^2 \frac{\pi x}{2} + 1 = 0$ is

(1) 0 (2) 2 (3) 1 (4) 3

4. The number of solutions of the pair of equations $2\sin^2 \theta - \cos 2\theta = 0$ and $2\cos^2 \theta - 3\sin \theta = 0$ in the interval $[0, 2\pi]$ is

(1) 0 (2) 1 (3) 2 (4) 4

5. Number of solutions of equation $2\sin \frac{x}{2} \cos^2 x - 2\sin \frac{x}{2} \sin^2 x = \cos^2 x - \sin^2 x$ for $x \in [0, 4\pi]$ is

(1) 6 (2) 8 (3) 10 (4) 12

6. Number of solutions of the equation $4(\cos^2 2x + \cos 2x + 1) + \tan x (\tan x - 2\sqrt{3}) = 0$ in $[0, 2\pi]$ is

(1) 0 (2) 1 (3) 2 (4) 3

7. Let $0 < \theta_1 < \theta_2 < \theta_3 < \dots$ denote the positive solution of the equation $3 + 3 \cos \theta = 2 \sin^2 \theta$. The value of $\theta_3 + \theta_7$ is

(1) 6π (2) 7π (3) 8π (4) 4π

8. Assume that θ is a rational multiple of π such that $\cos \theta$ is a distinct rational. The number of values of $\cos \theta$ is

(1) 3 (2) 4 (3) 5 (4) 6

9. If $x, y \in [0, 2\pi]$ and $\sin x + \sin y = 2$, then the value of $x + y$ is

(1) π (2) $\pi/2$
(3) 3π (4) none of these

10. Number of roots of $\cos^2 x + \frac{\sqrt{3}+1}{2} \sin x - \frac{\sqrt{3}}{4} - 1 = 0$ which lie in the interval $[-\pi, \pi]$ is

(1) 2 (2) 4 (3) 6 (4) 8

11. The sum of all the solutions of $\cot \theta = \sin 2\theta$ ($\theta \neq n\pi$, n is integer), $0 \leq \theta \leq \pi$, is

(1) $3\pi/2$ (2) π (3) $3\pi/4$ (4) 2π

12. The number of solutions of $12 \cos^3 x - 7 \cos^2 x + 4 \cos x = 9$ is

(1) 0 (2) 2
(3) infinite (4) none of these

13. Number of solutions of the equation $\sin x + \cos x - 2\sqrt{2} \sin x \cos x = 0$ for $x \in [0, \pi]$ is

(1) 3 (2) 0 (3) 1 (4) 2

14. The general solution of $\frac{\tan 5x - \tan 4x}{1 + \tan 5x \tan 4x} = 1$ is

(1) $n\pi + \frac{\pi}{4}, \forall n \in \mathbb{Z}$ (2) $n\pi \pm \frac{\pi}{4}, \forall n \in \mathbb{Z}$

(3) ϕ (4) $n\pi = \frac{\pi}{6}, \forall n \in \mathbb{Z}$

15. Consider the equations:

$$x \sin a + y \sin 2a + z \sin 3a = \sin 4a$$

$$x \sin b + y \sin 2b + z \sin 3b = \sin 4b$$

$$x \sin c + y \sin 2c + z \sin 3c = \sin 4c.$$

Then the roots of the equation

$$t^3 - \frac{z}{2}t^2 - \frac{y+2}{4}t + \frac{z-x}{8} = 0 (a, b, c \neq n\pi)$$
 are

(1) $\cos a, \cos b, \cos c$ (2) $\sin a, \sin b, \sin c$
(3) $\sin 2a, \sin 2b, \sin 2c$ (4) $\cos 2a, \cos 2b, \cos 2c$

16. The number of solutions of the equation $\sin 2\theta - 2 \cos \theta + 4 \sin \theta = 4$ in $[0, 5\pi]$ is equal to

(1) 3 (2) 4 (3) 5 (4) 6

17. The number of distinct real roots of the equation

$$\tan \frac{2\pi x}{x^2 + x + 1} = -\sqrt{3}$$
 is

(1) 4 (2) 5
(3) 6 (4) none of these

18. The smallest positive value of x (in radians) satisfying the

$$\text{equation } \log_{\cos x} \left(\frac{\sqrt{3}}{2} \sin x \right) = 2 - \log_{\sec x} (\tan x), \text{ is}$$

(1) $\frac{\pi}{12}$ (2) $\frac{\pi}{6}$ (3) $\frac{\pi}{4}$ (4) $\frac{\pi}{3}$

19. The number of solution of $\sin^4 x - \cos^2 x \sin x + 2 \sin^2 x + \sin x = 0$ in $0 \leq x \leq 3\pi$ is

(1) 3 (2) 4 (3) 5 (4) 6

20. The range of y such that the equation in x , $y + \cos x = \sin x$ has a real solution is

(1) $[-2, 2]$ (2) $[-\sqrt{2}, \sqrt{2}]$
(3) $[-1, 1]$ (4) $[-1/2, 1/2]$

21. Solution of the equation $\sin(\sqrt{1 + \sin 2\theta}) = \sin \theta + \cos \theta$ is ($n \in \mathbb{Z}$)

(1) $n\pi - \frac{\pi}{4}$ (2) $n\pi + \frac{\pi}{12}$
(3) $n\pi + \frac{\pi}{6}$ (4) none of these

22. One of the general solutions of $4 \sin \theta \sin 2\theta \sin 4\theta = \sin 3\theta$ is
 (1) $(3n \pm 1)\pi/12, \forall n \in \mathbb{Z}$
 (2) $(4n \pm 1)\pi/9, \forall n \in \mathbb{Z}$
 (3) $(3n \pm 1)\pi/9, \forall n \in \mathbb{Z}$
 (4) $(3n \pm 1)\pi/3, \forall n \in \mathbb{Z}$
23. The general solution of the equation $8 \cos x \cos 2x \cos 4x = \sin 6x / \sin x$ is
 (1) $x = (n\pi/7) + (\pi/21), \forall n \in \mathbb{Z}$
 (2) $x = (2\pi/7) + (\pi/14), \forall n \in \mathbb{Z}$
 (3) $x = (n\pi/7) + (\pi/14), \forall n \in \mathbb{Z}$
 (4) $x = (n\pi) + (\pi/14), \forall n \in \mathbb{Z}$
24. $\frac{\sin^3 \theta - \cos^3 \theta}{\sin \theta - \cos \theta} - \frac{\cos \theta}{\sqrt{1 + \cot^2 \theta}} - 2 \tan \theta \cot \theta = -1$ if
 (1) $\theta \in \left(0, \frac{\pi}{2}\right)$ (2) $\theta \in \left(\frac{\pi}{2}, \pi\right)$
 (3) $\theta \in \left(\pi, \frac{3\pi}{2}\right)$ (4) $\theta \in \left(\frac{3\pi}{2}, 2\pi\right)$
25. For $0 < x, y < \pi$, the number of ordered pairs (x, y) satisfying the system of equations $\cot^2(x - y) - (1 + \sqrt{3}) \cot(x - y) + \sqrt{3} = 0$ and $\cos y = \frac{\sqrt{3}}{2}$ is
 (1) 0 (2) 1 (3) 2 (4) 3
26. The least positive solution of $\cot\left(\frac{\pi}{3\sqrt{3}} \sin 2x\right) = \sqrt{3}$ lies in
 (1) $\left(0, \frac{\pi}{6}\right]$ (2) $\left(\frac{\pi}{9}, \frac{\pi}{6}\right)$
 (3) $\left(\frac{\pi}{12}, \frac{\pi}{9}\right]$ (4) $\left(\frac{\pi}{3}, \frac{\pi}{2}\right]$
27. The number of real roots of the equation $\operatorname{cosec} \theta + \sec \theta - \sqrt{15} = 0$ lying in $[0, \pi]$ is
 (1) 6 (2) 8 (3) 4 (4) 0
28. If $0 \leq x \leq 2\pi$, then the number of solutions of $3(\sin x + \cos x) - 2(\sin^3 x + \cos^3 x) = 8$ is
 (1) 0 (2) 1 (3) 2 (4) 4
29. If $2 \sin^2((\pi/2) \cos^2 x) = 1 - \cos(\pi \sin 2x)$, $x \neq (2n + 1)\pi/2$, $n \in \mathbb{I}$, then $\cos 2x$ is equal to
 (1) $1/5$ (2) $3/5$ (3) $4/5$ (4) 1
30. The number of solutions of the equation $\cos 6x + \tan^2 x + \cos 6x \cdot \tan^2 x = 1$ in the interval $[0, 2\pi]$ is
 (1) 4 (2) 5 (3) 6 (4) 7
31. The number of solutions of the equation $\sin^3 x \cos x + \sin^2 x \cos^2 x + \sin x \cos^3 x = 1$ in the interval $[0, 2\pi]$ is/are
 (1) 0 (2) 2
 (3) 3 (4) infinite
32. The sum of all the solutions of the equation $\cos \theta \cos\left(\frac{\pi}{3} + \theta\right) \cos\left(\frac{\pi}{3} - \theta\right) = \frac{1}{4}$, $\theta \in [0, 6\pi]$
 (1) 15π (2) 30π
 (3) $\frac{100\pi}{3}$ (4) none of these
33. General solution of $\sin^2 x - 5 \sin x \cos x - 6 \cos^2 x = 0$ is
 (1) $x = n\pi - \pi/4, n \in \mathbb{Z}$ only
 (2) $n\pi + \tan^{-1} 6, n \in \mathbb{Z}$ only
 (3) both (1) and (2)
 (4) none of these
34. The total number of solution of $\sin^4 x + \cos^4 x = \sin x \cos x$ in $[0, 2\pi]$ is equal to
 (1) 2 (2) 4
 (3) 6 (4) none of these
35. General solution of $\tan \theta + \tan 4\theta + \tan 7\theta = \tan \theta \tan 4\theta \tan 7\theta$ is
 (1) $\theta = n\pi/12$, where $n \in \mathbb{Z}$
 (2) $\theta = n\pi/9$, where $n \in \mathbb{Z}$
 (3) $\theta = n\pi + \pi/12$, where $n \in \mathbb{Z}$
 (4) none of these
36. The general solution of $\tan \theta + \tan 2\theta + \tan 3\theta = 0$ is
 (1) $\theta = n\pi/6, n \in \mathbb{Z}$ only
 (2) $\theta = n\pi \pm \alpha, n \in \mathbb{Z}$, where $\tan \alpha = 1/\sqrt{2}$ only
 (3) Both a and b
 (4) none of these
37. The number of solutions of $\sec^2 \theta + \operatorname{cosec}^2 \theta + 2 \operatorname{cosec}^2 \theta = 0 \leq \theta \leq \pi/2$ is
 (1) 4 (2) 3 (3) 0 (4) 2
38. Which of the following is true for $z = (3 + 2i \sin \theta) / (1 - 2i \sin \theta)$, where $i = \sqrt{-1}$?
 (1) z is purely real for $\theta = n\pi \pm \pi/3, n \in \mathbb{Z}$
 (2) z is purely imaginary for $\theta = n\pi \pm \pi/2, n \in \mathbb{Z}$
 (3) z is purely real for $\theta = n\pi, n \in \mathbb{Z}$
 (4) none of these
39. The number of solutions of $\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$, $0 \leq x \leq 2\pi$, is
 (1) 7 (2) 5 (3) 4 (4) 6
40. Number of solutions of the equation $\cos^4 2x + 2 \sin^2 2x = 17 (\cos x + \sin x)^8$, $0 < x < 2\pi$ is
 (1) 4 (2) 8 (3) 10 (4) 16
41. The number of values of θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ satisfying the equation $(\sqrt{3})^{\sec^2 \theta} = \tan^4 \theta + 2 \tan^2 \theta$ is
 (1) 2 (2) 4 (3) 0 (4) 1
42. The value of k if the equation $2 \cos x + \cos 2kx = 3$ has only one solution is
 (1) 0 (2) 2 (3) $\sqrt{2}$ (4) $1/2$
43. Number of solution(s) satisfying the equation $\frac{1}{\sin x} - \frac{1}{\sin 2x} = \frac{2}{\sin 4x}$ in $[0, 4\pi]$ equals
 (1) 0 (2) 2 (3) 4 (4) 6

44. The number of roots of $(1 - \tan \theta)(1 + \sin 2\theta) = 1 + \tan \theta$ for $\theta \in [0, 2\pi]$ is

- (1) 3
(2) 4
(3) 5
(4) none of these

45. If $\tan(A - B) = 1$ and $\sec(A + B) = 2\sqrt{3}$, then the smallest positive values of A and B , respectively, are

- (1) $\frac{25\pi}{24}, \frac{19\pi}{24}$
(2) $\frac{19\pi}{24}, \frac{25\pi}{24}$
(3) $\frac{31\pi}{24}, \frac{13\pi}{24}$
(4) $\frac{13\pi}{24}, \frac{31\pi}{24}$

46. If $3 \tan(\theta - 15^\circ) = \tan(\theta + 15^\circ)$, then θ is equal to ($n \in \mathbb{Z}$)

- (1) $n\pi + \frac{\pi}{4}$
(2) $n\pi + \frac{\pi}{8}$
(3) $n\pi + \frac{\pi}{3}$
(4) none of these

47. If $\tan 3\theta - \tan \theta = 2 \tan 2\theta$, then θ is equal to ($n \in \mathbb{Z}$)

- (1) $n\pi$
(2) $\frac{n\pi}{4}$
(3) $\frac{n\pi}{2}$
(4) none of these

48. The solution of $4 \sin^2 x + \tan^2 x + \operatorname{cosec}^2 x + \cot^2 x - 6 = 0$ is ($n \in \mathbb{Z}$)

- (1) $n\pi = \frac{\pi}{4}$
(2) $2n\pi \pm \frac{\pi}{4}$
(3) $n\pi = \frac{\pi}{3}$
(4) $n\pi = \frac{\pi}{6}$

49. The set of values of x satisfying the equation $\sin 3x = 4 \sin x \sin(x + \alpha) \sin(x - \alpha)$ is

- (1) $n\pi = \pi/4, \forall n \in \mathbb{Z}$
(2) $n\pi \pm \pi/3, \forall n \in \mathbb{Z}$
(3) $n\pi = \pi/9, \forall n \in \mathbb{Z}$
(4) $n\pi \pm \pi/12, \forall n \in \mathbb{Z}$

50. One of the general solutions of $4 \sin^4 x + \cos^4 x = 1$ is

- (1) $n\pi = \alpha/2, \alpha = \cos^{-1}(1/5), \forall n \in \mathbb{Z}$
(2) $n\pi = \alpha/2, \alpha = \cos^{-1}(3/5), \forall n \in \mathbb{Z}$
(3) $2n\pi \pm \alpha/2, \alpha = \cos^{-1}(1/3), \forall n \in \mathbb{Z}$
(4) none of these

51. For $n \in \mathbb{Z}$, the general solution of $(\sqrt{3} - 1) \sin \theta + (\sqrt{3} + 1) \cos \theta = 2$ is ($n \in \mathbb{Z}$)

- (1) $\theta = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{12}$
(2) $\theta = n\pi + (-1)^n \frac{\pi}{4} + \frac{\pi}{12}$
(3) $\theta = 2n\pi \pm \frac{\pi}{4}$
(4) $\theta = n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{12}$

52. The value of $\cos y \cos\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - y\right)$

- $\cos x + \sin y \cos\left(\frac{\pi}{2} - x\right) + \cos x \sin\left(\frac{\pi}{2} - y\right)$ is zero if
(1) $x = 0$
(2) $y = 0$
(3) $x = y$
(4) $n\pi + y - \frac{\pi}{4} (n \in \mathbb{Z})$

53. One of the general solutions of $\sqrt{3} \cos \theta - 3 \sin \theta = 4 \sin 2\theta$ $\cos 3\theta$ is

- (1) $m\pi + \pi/8, m \in \mathbb{Z}$
(2) $m\pi/2 + \pi/6, \forall m \in \mathbb{Z}$
(3) $m\pi/3 + \pi/8, m \in \mathbb{Z}$
(4) none of these

54. The equation $\sin^4 x + \cos^4 x + \sin 2x + \alpha = 0$ is solvable for

- (1) $-5/2 \leq \alpha \leq 1/2$
(2) $-3 \leq \alpha \leq 1$
(3) $-3/2 \leq \alpha \leq 1/2$
(4) $-1 \leq \alpha \leq 1$

55. The total number of solutions of $\cos x = \sqrt{1 - \sin 2x}$ in $[0, 2\pi]$ is equal to

- (1) 2
(2) 3
(3) 5
(4) none of these

56. The total number of solutions of $|\cot x| = \cot x + \frac{1}{\sin x}$, $x \in [0, 3\pi]$, is equal to

- (1) 1
(2) 2
(3) 3
(4) 0

57. Let α and β be any two positive values of x for which $2 \cos x, |\cos x|$, and $1 - 3 \cos^2 x$ are in G.P. The minimum value of $|\alpha - \beta|$ is

- (1) $\frac{\pi}{3}$
(2) $\frac{\pi}{4}$
(3) $\frac{\pi}{2}$
(4) none of these

58. The number of values of θ satisfying $4 \cos \theta + 3 \sin \theta = 5$ as well as $3 \cos \theta + 4 \sin \theta = 5$ is

- (1) one
(2) two
(3) zero
(4) none of these

59. The general solution of $\cos x \cos 6x = -1$ is

- (1) $x = (2n + 1)\pi, n \in \mathbb{Z}$
(2) $x = 2n\pi, n \in \mathbb{Z}$
(3) $x = n\pi, n \in \mathbb{Z}$
(4) none of these

60. The number of solutions the equation $\cos(\theta) \cdot \cos(\pi\theta) = 1$ has

- (1) 0
(2) 2
(3) 1
(4) infinite

61. Let $\theta \in [0, 4\pi]$ satisfy the equation $(\sin \theta + 2)(\sin \theta + 3)(\sin \theta + 4) = 6$. If the sum of all the values of θ is of the form $k\pi$, then the value of k is

- (1) 6
(2) 5
(3) 4
(4) 2

62. The number of solutions of $\sum_{r=1}^5 \cos r x = 5$ in the interval $[0, 2\pi]$ is

- (1) 0
(2) 2
(3) 5
(4) 10

63. If $\cos 3x + \sin\left(2x - \frac{7\pi}{6}\right) = -2$, then x is equal to ($k \in \mathbb{Z}$)

- (1) $\frac{\pi}{3}(6k + 1)$
(2) $\frac{\pi}{3}(6k - 1)$
(3) $\frac{\pi}{3}(2k + 1)$
(4) none of these

64. The general solution of the equation $\sin^{100} x - \cos^{100} x = 1$ is

- (1) $2n\pi + \frac{\pi}{3}, n \in \mathbb{I}$
(2) $n\pi + \frac{\pi}{2}, n \in \mathbb{I}$
(3) $n\pi + \frac{\pi}{4}, n \in \mathbb{I}$
(4) $2n\pi - \frac{\pi}{3}, n \in \mathbb{I}$

65. The sum of all the solutions in $[0, 4\pi]$ of the equation $\tan x + \cot x + 1 = \cos\left(x + \frac{\pi}{4}\right)$ is
 (1) 3π (2) $\pi/2$ (3) $7\pi/2$ (4) 4π
66. The total number of solutions of $\log_e |\sin x| = -x^2 + 2x$ in $[0, \pi]$ is equal to
 (1) 1 (2) 2
 (3) 4 (4) none of these
67. The total number of solutions of $\sin \{x\} = \cos \{x\}$ (where $\{ \cdot \}$ denotes the fractional part) in $[0, 2\pi]$ is equal to
 (1) 5 (2) 6
 (3) 8 (4) none of these
68. The set of all x in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ satisfying $|4 \sin x - 1| < \sqrt{5}$ is given by
 (1) $\left(-\frac{\pi}{10}, \frac{3\pi}{10}\right)$ (2) $\left(\frac{\pi}{10}, \frac{3\pi}{10}\right)$
 (3) $\left(\frac{\pi}{10}, \frac{3\pi}{10}\right)$ (4) none of these
69. If roots of the equation $2x^2 - 4x + 2 \sin \theta - 1 = 0$ are of opposite sign, then θ belongs to
 (1) $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ (2) $\left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$
 (3) $\left(\frac{13\pi}{6}, \frac{17\pi}{6}\right)$ (4) $(0, \pi)$
70. If $|2 \sin \theta - \operatorname{cosec} \theta| \geq 1$ and $\theta \neq \frac{n\pi}{2}, n \in \mathbb{Z}$, then
 (1) $\cos 2\theta \geq 1/2$ (2) $\cos 2\theta \geq 1/4$
 (3) $\cos 2\theta \leq 1/2$ (4) $\cos 2\theta \leq 1/4$
71. Which of the following is not the solution of the equation $\sin 5x = 16 \sin^5 x$ ($n \in \mathbb{Z}$)?
 (1) $n\pi$ (2) $n\pi + \frac{\pi}{6}$
 (3) $n\pi - \frac{\pi}{6}$ (4) none of these
72. The number of solutions of the equation $|2 \sin x - \sqrt{3}|^{2 \cos^2 x - 3 \cos x + 1} = 1$ in $[0, \pi]$ is
 (1) 2 (2) 3 (3) 4 (4) 5
73. One root of the equation $\cos x - x + \frac{1}{2} = 0$ lies in the interval
 (1) $\left(0, \frac{\pi}{2}\right)$ (2) $\left(-\frac{\pi}{2}, 0\right)$
 (3) $\left(\frac{\pi}{2}, \pi\right)$ (4) $\left(\pi, \frac{3\pi}{2}\right)$
74. The smallest positive x satisfying the equation $\log_{\cos x} \sin x + \log_{\sin x} \cos x = 2$ is
 (1) $\pi/2$ (2) $\pi/3$ (3) $\pi/4$ (4) $\pi/6$
75. The number of ordered pairs which satisfy the equation $x^2 + 2x \sin(xy) + 1 = 0$ are (where $y \in [0, 2\pi]$)
 (1) 1 (2) 2 (3) 3 (4) 0
76. Consider the system of linear equations in x, y , and z :
 $(\sin 3\theta)x - y + z = 0$
 $(\cos 2\theta)x + 4y + 3z = 0$
 $2x + 7y + 7z = 0$
 Which of the following can be the values of θ for which the system has a non-trivial solution?
 (1) $n\pi + (-1)^n \pi/6, \forall n \in \mathbb{Z}$
 (2) $n\pi + (-1)^n \pi/3, \forall n \in \mathbb{Z}$
 (3) $n\pi + (-1)^n \pi/9, \forall n \in \mathbb{Z}$
 (4) none of these
77. The equation $\sin^4 x - 2\cos^2 x + a^2 = 0$ can be solved if
 (1) $-\sqrt{3} \leq a \leq \sqrt{3}$ (2) $-\sqrt{2} \leq a \leq \sqrt{2}$
 (3) $-1 \leq a \leq 1$ (4) None of these
78. If the inequality $\sin^2 x + a \cos x + a^2 > 1 + \cos x$ holds for any $x \in \mathbb{R}$, then the largest negative integral value of a is
 (1) -4 (2) -3 (3) -2 (4) -1
79. $\sin x + \cos x = y^2 - y + a$ has no value of x for any value of y if a belongs to
 (1) $(0, \sqrt{3})$ (2) $(-\sqrt{3}, 0)$
 (3) $(-\infty, -\sqrt{3})$ (4) $(\sqrt{3}, \infty)$
80. The number of solutions of $[\sin x + \cos x] = 3 + [-\sin x + \cos x]$ (where $[\cdot]$ denotes the greatest integer function) $x \in [0, 2\pi]$, is
 (1) 0 (2) 4 (3) infinite (4) 1
81. The equation $\cos^8 x + b \cos^4 x + 1 = 0$ will have a solution if b belongs to
 (1) $(-\infty, 2]$ (2) $[2, \infty)$
 (3) $(-\infty, -2]$ (4) none of these
82. The number of values of y in $[-2\pi, 2\pi]$ satisfying the equation $|\sin 2x| + |\cos 2x| = |\sin y|$ is
 (1) 3 (2) 4 (3) 5 (4) 6
83. If both the distinct roots of the equation $|\sin x|^2 + |\sin x| + b = 0$ in $[0, \pi]$ are real, then the values of b are
 (1) $[-2, 0]$ (2) $(-2, 0)$
 (3) $[-2, 0)$ (4) none of these
84. $e^{|\sin x|} + e^{-|\sin x|} + 4a = 0$ will have exactly four different solutions in $[0, 2\pi]$ if
 (1) $a \in \mathbb{R}$ (2) $a \in \left[-\frac{e}{4}, -\frac{1}{4}\right]$
 (3) $a \in \left[\frac{-1-e^2}{4e}, \infty\right)$ (4) none of these
85. The equation $\tan^4 x - 2 \sec^2 x + a = 0$ will have at least one solution if
 (1) $1 < a \leq 4$ (2) $a \geq 2$
 (3) $a \leq 3$ (4) none of these
86. The total number of ordered pairs (x, y) satisfying $|x| + |y| = 2, \sin(\pi x^2/3) = 1$, is equal to
 (1) 2 (2) 3 (3) 4 (4) 6

87. If $a, b \in [0, 2\pi]$ and the equation $x^2 + 4 + 3 \sin(ax + b) - 2x = 0$ has at least one solution, then the value of $(a + b)$ can be

- (1) $\frac{7\pi}{2}$ (2) $\frac{5\pi}{2}$
(3) $\frac{9\pi}{2}$ (4) none of these

88. The sum of all roots of $\sin\left(\pi \log_3\left(\frac{1}{x}\right)\right) = 0$ in $(0, 2\pi)$ is

- (1) $3/2$ (2) 4 (3) $9/2$ (4) $13/3$

89. Find the number of pairs of integer (x, y) that satisfy the following two equations:

- $$\begin{cases} \cos(xy) = x \\ \tan(xy) = y \end{cases}$$

(1) 1 (2) 2 (3) 4 (4) 6

90. If no solution of $3 \sin y + 12 \sin^3 x = a$ lies on the line $y = 3x$, then

- (1) $a \in (-\infty, -9) \cup (9, \infty)$
(2) $a \in [-9, 9]$
(3) $a \in \{-9, 9\}$
(4) none of these

Multiple Correct Answers Type

1. If $4 \sin^4 x + \cos^4 x = 1$, then x is equal to ($n \in \mathbb{Z}$)

- (1) $n\pi$ (2) $n\pi \pm \sin^{-1} \sqrt{\frac{2}{5}}$
(3) $\frac{2n\pi}{3}$ (4) $2n\pi \pm \frac{\pi}{4}$

2. If $\sin^3 \theta + \sin \theta \cos \theta + \cos^3 \theta = 1$, then θ is equal to ($n \in \mathbb{Z}$)

- (1) $2n\pi$ (2) $2n\pi + \frac{\pi}{2}$
(3) $2n\pi - \frac{\pi}{2}$ (4) $n\pi$

3. A general solution of $\tan^2 \theta + \cos 2\theta = 1$ is ($n \in \mathbb{Z}$)

- (1) $n\pi - \frac{\pi}{4}$ (2) $2n\pi + \frac{\pi}{4}$
(3) $n\pi + \frac{\pi}{4}$ (4) $n\pi$

4. If $\sin x + \cos x = \sqrt{y + \frac{1}{y}}$ for $x \in [0, \pi]$, then

- (1) $x = \pi/4$ (2) $y = 0$
(3) $y = 1$ (4) $x = 3\pi/4$

5. The equation $2 \sin^2\left(\frac{\pi}{2} \cos^2 x\right) = 1 - \cos(\pi \sin 2x)$ is satisfied by

- (1) $x = (2n + 1) \frac{\pi}{2}, n \in \mathbb{Z}$ (2) $\tan x = \frac{1}{2}, n \in \mathbb{Z}$
(3) $\tan x = -\frac{1}{2}, n \in \mathbb{Z}$ (4) $x = \frac{n\pi}{2}, n \in \mathbb{Z}$

6. If $\sin^2 x - 2 \sin x - 1 = 0$ has exactly four different solutions in $x \in [0, n\pi]$, then value/values of n is/are ($n \in \mathbb{N}$)

- (1) 5 (2) 3 (3) 4 (4) 6

7. For the smallest positive values of x and y , the equation $2(\sin x + \sin y) - 2 \cos(x - y) = 3$ has a solution, then which of the following is/are true?

- (1) $\sin \frac{x+y}{2} = 1$

- (2) $\cos\left(\frac{x-y}{2}\right) = \frac{1}{2}$

- (3) number of ordered pairs (x, y) is 2

- (4) number of ordered pairs (x, y) is 3

8. For the equation $1 - 2x - x^2 = \tan^2(x + y) + \cot^2(x + y)$

- (1) exactly one value of x exists

- (2) exactly two values of x exists

- (3) $y = -1 + n\pi + \pi/4, n \in \mathbb{Z}$

- (4) $y = 1 + n\pi + \pi/4, n \in \mathbb{Z}$

9. If $x + y = \pi/4$ and $\tan x + \tan y = 1$, then ($n \in \mathbb{Z}$)

- (1) $\sin x = 0$ always

- (2) when $x = n\pi + \pi/4$ then $y = -n\pi$

- (3) when $x = n\pi$ then $y = n\pi + (\pi/4)$

- (4) when $x = n\pi + \pi/4$ then $y = n\pi - (\pi/4)$

10. If $x + y = 2\pi/3$ and $\sin x / \sin y = 2$, then the

- (1) number of values of $x \in [0, 4\pi]$ are 4

- (2) number of values of $x \in [0, 4\pi]$ are 2

- (3) number of values of $y \in [0, 4\pi]$ are 4

- (4) number of values of $y \in [0, 4\pi]$ are 8

11. Let $\tan x - \tan^2 x > 0$ and $|2 \sin x| < 1$. Then the intersection of which of the following two sets satisfies both the inequalities?

- (1) $x > n\pi, n \in \mathbb{Z}$ (2) $x > n\pi - \pi/6, n \in \mathbb{Z}$

- (3) $x < n\pi - \pi/4, n \in \mathbb{Z}$ (4) $x < n\pi + \pi/6, n \in \mathbb{Z}$

12. If $\cos(x + \pi/3) + \cos x = a$ has real solutions, then

- (1) number of integral values of a are 3

- (2) sum of number of integral values of a is 0

- (3) when $a = 1$, number of solutions for $x \in [0, 2\pi]$ are 3

- (4) when $a = 1$, number of solutions for $x \in [0, 2\pi]$ are 2

13. If $0 \leq x \leq 2\pi$, then $2^{\cos^2 x} \sqrt{\frac{1}{2} y^2 - y + 1} \leq \sqrt{2}$

- (1) is satisfied by exactly one value of y

- (2) is satisfied by exactly two value of x

- (3) is satisfied by x for which $\cos x = 0$

- (4) is satisfied by x for which $\sin x = 0$

14. If $\sin^2 x - a \sin x + b = 0$ has only one solution in $(0, \pi)$, then which of the following statements are correct?

- (1) $a \in (-\infty, 1] \cup [2, \infty)$

- (2) $b \in (-\infty, 0] \cup [1, \infty)$

- (3) $a = 1 + b$

- (4) none of these

15. If $(\operatorname{cosec}^2 \theta - 4)x^2 + (\cot \theta + \sqrt{3})x + \cos^2 \frac{3\pi}{2} = 0$ holds true for all real x , then the most general values of θ can be given by ($n \in \mathbb{Z}$)
- (1) $2n\pi + \frac{11\pi}{6}$ (2) $2n\pi + \frac{5\pi}{6}$
 (3) $2n\pi \pm \frac{7\pi}{6}$ (4) $n\pi \pm \frac{11\pi}{6}$
16. If $(\sin \alpha)x^2 - 2x + b \geq 2$ for all the real values of $x \leq 1$ and $\alpha \in (0, \pi/2) \cup (\pi/2, \pi)$, then the possible real values of b is/are
- (1) 2 (2) 3 (3) 4 (4) 5
17. The value of x in $(0, \pi/2)$ satisfying $\frac{\sqrt{3}-1}{\sin x} + \frac{\sqrt{3}+1}{\cos x} = 4\sqrt{2}$ is
- (1) $\frac{\pi}{12}$ (2) $\frac{5\pi}{12}$ (3) $\frac{7\pi}{24}$ (4) $\frac{11\pi}{36}$
18. If $\cos 3\theta = \cos 3\alpha$, then the value of $\sin \theta$ can be given by
- (1) $\pm \sin \alpha$ (2) $\sin \left(\frac{\pi}{3} \pm \alpha \right)$
 (3) $\sin \left(\frac{2\pi}{3} + \alpha \right)$ (4) $\sin \left(\frac{2\pi}{3} - \alpha \right)$
19. Which of the following sets can be the subset of the general solution of $1 + \cos 3x = 2 \cos 2x$ ($n \in \mathbb{Z}$)?
- (1) $n\pi + \frac{\pi}{3}$ (2) $n\pi + \frac{\pi}{6}$
 (3) $n\pi - \frac{\pi}{6}$ (4) $2n\pi$
20. The values of x , between 0 and 2π , satisfying the equation $\cos 3x + \cos 2x = \sin \frac{3x}{2} + \sin \frac{x}{2}$ are
- (1) $\frac{\pi}{7}$ (2) $\frac{5\pi}{7}$ (3) $\frac{9\pi}{7}$ (4) $\frac{13\pi}{7}$
21. Which of the following set of values of x satisfies the equation $2^{(2 \sin^2 x - 3 \sin x + 1)} + 2^{(2 - 2 \sin^2 x + 3 \sin x)} = 9$?
- (1) $x = n\pi \pm \frac{\pi}{6}, n \in \mathbb{I}$ (2) $x = n\pi \pm \frac{\pi}{3}, n \in \mathbb{I}$
 (3) $x = n\pi, n \in \mathbb{I}$ (4) $x = 2n\pi + \frac{\pi}{2}, n \in \mathbb{I}$
22. If $0 \leq x \leq 2\pi$ and $|\cos x| \leq \sin x$, then
- (1) the set of all values of x is $\left[\frac{\pi}{4}, \frac{3\pi}{4} \right]$
 (2) the number of solutions that are integral multiple of $\frac{\pi}{2}$ is four
 (3) the sum of the largest and the smallest solution is π
 (4) the set of all values of x is $x \in \left[\frac{\pi}{4}, \frac{\pi}{2} \right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4} \right]$
23. The expression $\cos 3\theta + \sin 3\theta + (2 \sin 2\theta - 3)(\sin \theta - \cos \theta)$ is positive for all θ in
- (1) $\left(2n\pi - \frac{3\pi}{4}, 2n\pi + \frac{\pi}{4} \right), n \in \mathbb{Z}$
 (2) $\left(2n\pi - \frac{\pi}{4}, 2n\pi + \frac{\pi}{6} \right), n \in \mathbb{Z}$
 (3) $\left(2n\pi - \frac{\pi}{3}, 2n\pi + \frac{\pi}{3} \right), n \in \mathbb{Z}$
 (4) $\left(2n\pi - \frac{\pi}{4}, 2n\pi + \frac{3\pi}{4} \right), n \in \mathbb{Z}$
24. The solutions of the equation $1 + (\sin x - \cos x) \sin \frac{\pi}{4} = 2 \cos^2 \frac{5x}{2}$ is/are
- (1) $x = \frac{n\pi}{3} + \frac{\pi}{8}, n \in \mathbb{Z}$ (2) $x = \frac{n\pi}{2} + \frac{5\pi}{16}, n \in \mathbb{Z}$
 (3) $x = \frac{n\pi}{3} + \frac{\pi}{4}, n \in \mathbb{Z}$ (4) $x = \frac{n\pi}{2} + \frac{7\pi}{8}, n \in \mathbb{Z}$
25. If x and y are positive acute angles such that $(x+y)$ and $(x-y)$ satisfy the equation $\tan^2 \theta - 4 \tan \theta + 1 = 0$, then
- (1) $x = \frac{\pi}{6}$ (2) $y = \frac{\pi}{4}$
 (3) $y = \frac{\pi}{6}$ (4) $x = \frac{\pi}{4}$
26. The solutions of the system of equations $\sin x \sin y = \frac{\sqrt{3}}{4}$ and $\cos x \cos y = \frac{\sqrt{3}}{4}$ are
- (1) $x = \frac{\pi}{3} + \frac{\pi}{2}(2n+k); n, k \in \mathbb{I}$
 (2) $y = \frac{\pi}{6} + \frac{\pi}{2}(k-2n); n, k \in \mathbb{I}$
 (3) $x = \frac{\pi}{6} + \frac{\pi}{2}(2n+k); n, k \in \mathbb{I}$
 (4) $y = \frac{\pi}{3} + \frac{\pi}{2}(k-2n); n, k \in \mathbb{I}$
27. Let $f(x) = \cos(a_1 + x) + \frac{1}{2} \cos(a_2 + x) + \frac{1}{2^2} \cos(a_3 + x) + \dots + \frac{1}{2^{n-1}} \cos(a_n + x)$ where $a_1, a_2, \dots, a_n \in \mathbb{R}$.
 If $f(x_1) = f(x_2) = 0$, then $|x_2 - x_1|$ may be equal to
- (1) π (2) 2π (3) 3π (4) $\pi/2$
28. The equation $2 \sin^3 \theta + (2\lambda - 3) \sin^2 \theta - (3\lambda + 2) \sin \theta - 2\lambda = 0$ has exactly three roots in $(0, 2\pi)$, then λ can be equal to
- (1) 0 (2) 2 (3) 1 (4) -1
29. The system of equations $\tan x = a \cot x, \tan 2x = b \cos x$
- (1) cannot have a solution if $a = 0$
 (2) cannot have a solution if $a = 1$
 (3) cannot have a solution if $2\sqrt{a} > |b(1-a)|$
 (4) has a solution for all a and b

9. If $\left(\cos^2 x + \frac{1}{\cos^2 x}\right) (1 + \tan^2 2y) (3 + \sin 3z) = 4$, then

- (1) x is an integral multiple of π
 - (2) x cannot be an even multiple of π
 - (3) z is an integral multiple of π
 - (4) y is an integral multiple of $\pi/2$
10. Number of real solutions of the equation $(\tan x + 1)(\tan x + 3)(\tan x + 5)(\tan x + 7) = 33$
- (1) will be two in the interval $[-\pi/2, \pi/2]$
 - (2) will be four in the interval $[-\pi/2, \pi/2]$
 - (3) will be three in the interval $[-\pi/2, \pi]$
 - (4) will be four in the interval $[-\pi/2, \pi]$

Linked Comprehension Type

For Problems 1-3

Consider the cubic equation $x^3 - (1 + \cos \theta + \sin \theta)x^2 + \cos \theta \sin \theta x - \sin \theta \cos \theta = 0$ whose roots are x_1, x_2 and x_3 .

1. The value of $x_1^2 + x_2^2 + x_3^2$ equals
 - (1) 1
 - (2) 2
 - (3) $2 \cos \theta$
 - (4) $\sin \theta (\sin \theta + \cos \theta)$
2. The number of values of θ in $[0, 2\pi]$ for which at least two roots are equal
 - (1) 3
 - (2) 4
 - (3) 5
 - (4) 6
3. The greatest possible difference between two of the roots if $\theta \in [0, 2\pi]$ is
 - (1) 2
 - (2) 1
 - (3) $\sqrt{2}$
 - (4) $2\sqrt{2}$

For Problems 4-6

Consider the equation

$$\sec \theta + \operatorname{cosec} \theta = a, \theta \in (0, 2\pi) - \{\pi/2, \pi, 3\pi/2\}$$

4. If the equation has four distinct real roots, then
 - (1) $|a| > 2\sqrt{2}$
 - (2) $|a| < 2\sqrt{2}$
 - (3) $a \geq -2\sqrt{2}$
 - (4) none of these
5. If the equation has two distinct real roots, then
 - (1) $|a| \geq 2\sqrt{2}$
 - (2) $a < 2\sqrt{2}$
 - (3) $|a| < 2\sqrt{2}$
 - (4) none of these
6. If the equation has no real roots, then
 - (1) $|a| \geq 2\sqrt{2}$
 - (2) $a < 2\sqrt{2}$
 - (3) $|a| < 2\sqrt{2}$
 - (4) none of these

For Problems 7-9

Consider the system of equations

$$\sin x \cos 2y = (a^2 - 1)^2 + 1, \cos x \sin 2y = a + 1$$

7. The number of values of a for which the system has a solution is
 - (1) 1
 - (2) 2
 - (3) 3
 - (4) infinite
8. The number of values of $x \in [0, 2\pi]$, when the system has solution for permissible values of a , is/are
 - (1) 1
 - (2) 2
 - (3) 3
 - (4) 4

9. The number of values of $y \in [0, 2\pi]$, when the system has solution for permissible values of a , are
 - (1) 2
 - (2) 3
 - (3) 4
 - (4) 5

For Problems 10-12

Consider the equation $\int_0^x (t^2 - 8t + 13)dt = x \sin(a/x)$

10. The number of real values of x for which the equation has solution is
 - (1) 1
 - (2) 2
 - (3) 3
 - (4) infinite
11. If x takes the values for which the equation has a solution, then the number of values of $a \in [0, 100]$ is
 - (1) 2
 - (2) 1
 - (3) 5
 - (4) 3
12. One of the solutions of $|y - \cos a| < x$, where x and a are values that satisfy the given equation, is
 - (1) $y \in [-5, 7]$
 - (2) $y \in [-7, 5]$
 - (3) $y \in [5, 7]$
 - (4) none of these

For Problems 13-15

Consider the system of equations

$$x \cos^3 y + 3x \cos y \sin^2 y = 14$$

$$x \sin^3 y + 3x \cos^2 y \sin y = 13$$

13. The value/values of x is/are
 - (1) $\pm 5\sqrt{5}$
 - (2) $\pm \sqrt{5}$
 - (3) $\pm 1/\sqrt{5}$
 - (4) none of these
14. The number of values of $y \in [0, 6\pi]$ is
 - (1) 5
 - (2) 3
 - (3) 4
 - (4) 6
15. The value of $\sin^2 y + 2\cos^2 y$ is
 - (1) $4/5$
 - (2) $9/5$
 - (3) 2
 - (4) none of these

For Problems 16-18

Let S_1 be the set of all those solutions of the equation $(1 + a) \cos \theta \cos(2\theta - b) = (1 + a \cos 2\theta) \cos(\theta - b)$ which are independent of a and b and S_2 be the set of all such solutions which are dependent on a and b . Then

16. The set S_1 and S_2 are

- (1) $\{n\pi, n \in \mathbb{Z}\}$ and

$$\frac{1}{2} \left\{ n\pi + (-1)^n \sin^{-1}(a \sin b) + b; n \in \mathbb{Z} \right\}$$

- (2) $\left\{ n\frac{\pi}{2}, n \in \mathbb{Z} \right\}$ and $\left\{ n\pi + (-1)^n \sin^{-1}(a \sin b); n \in \mathbb{Z} \right\}$

- (3) $\left\{ n\frac{\pi}{2}, n \in \mathbb{Z} \right\}$ and $\left\{ n\pi + (-1)^n \sin^{-1}\left(\frac{a}{2} \sin b\right); n \in \mathbb{Z} \right\}$

- (4) none of these

17. Condition that should be imposed on a and b such that S_2 is non empty is

$$(1) \left| \frac{a}{2} \sin b \right| < 1$$

$$(2) \left| \frac{a}{2} \sin b \right| \leq 1$$

$$(3) |a \sin b| \leq 1$$

$$(4) \text{ none of these}$$

18. All the permissible values of b if $a = 0$ and S_2 is a subset of $(0, \pi)$ are given by

- (1) $b \in (-n\pi, 2n\pi), n \in \mathbb{Z}$
- (2) $b \in (-n\pi, 2\pi - n\pi), n \in \mathbb{Z}$
- (3) $b \in (-n\pi, n\pi), n \in \mathbb{Z}$
- (4) none of these

For Problems 19 and 20

Let $\frac{b \cos x}{2 \cos 2x - 1} = \frac{b + \sin x}{(\cos^2 x - 3 \sin^2 x) \tan x}, b \in \mathbb{R}$.

19. Equation has solutions if

- (1) $b \in \left(-\infty, \frac{1}{2}\right) - \left\{-1, 0, \frac{1}{3}\right\}$
- (2) $b \in (-\infty, 1) - \left\{-1, 0, \frac{1}{3}\right\}$
- (3) $b \in \mathbb{R} - \left\{-1, 0, \frac{1}{3}\right\}$
- (4) none of these

20. For any value of b for which the equation has a solution, the number of solutions when $x \in (0, 2\pi)$ are always

- (1) infinite
- (2) depends upon the value of b
- (3) two
- (4) none of these

Matrix Match Type

1.

List I (Equation)	List II (Solution)
a. $\cos^2 2x + \cos^2 x = 1$	p. $x = \left\{n\pi + \frac{\pi}{4}\right\} \cup \left\{n\pi + \frac{\pi}{6}\right\}, n \in \mathbb{Z}$
b. $\cos x + \sqrt{3} \sin x = \sqrt{3}$	q. $x = \frac{n\pi}{3}, n \in \mathbb{Z}$
c. $1 + \sqrt{3} \tan^2 x = (1 + \sqrt{3}) \tan x$	r. $x = (2n - 1) \frac{\pi}{6}, n \in \mathbb{Z}$
d. $\tan 3x - \tan 2x - \tan x = 0$	s. $x = \left\{2n\pi + \frac{\pi}{2}\right\} \cup \left\{2n\pi + \frac{\pi}{6}\right\}, n \in \mathbb{Z}$

2.

List I (Equation)	List II (Number of solutions)
a. $x^3 + x^2 + 4x + 2 \sin x = 0$ in $0 \leq x \leq 2\pi$	p. 4
b. $\sin e^x \cos e^x = 2^{x-2} + 2^{-x-2}$	q. 1
c. $\sin 2x + \cos 4x = 2$	r. 2
d. $30 \sin x = x$ when $0 \leq x \leq 2\pi$	s. 0

3.

List I (Equation)	List II (Solution)
a. If $\max_{\theta \in \mathbb{R}} \{5 \sin \theta + 3 \sin(\theta - \alpha)\} = 7$, then the set of possible values of α is	p. $2n\pi + 3\pi/4, n \in \mathbb{Z}$
b. $x \neq \frac{n\pi}{2}$ and $(\cos x)^{\sin^2 x - 3 \sin x + 2} = 1$	q. $x = 2n\pi \pm \frac{\pi}{3}, n \in \mathbb{Z}$
c. $\sqrt{(\sin x)} + 2^{1/4} \cos x = 0$	r. $2n\pi + \cos^{-1}(1), n \in \mathbb{Z}$
d. $\log_5 \tan x = (\log_5 4)(\log_4 (3 \sin x))$	s. no solution

4. Match the equations in List I with the number of solutions in List II.

List I	List II
a. $\cos ax + \cos bx = 2, (a \neq 0, b \neq 0)$	p. Exactly one solution
b. $\cos x + \cos(\sqrt{2})x = 2$	q. At least one solution
c. $\cos ax + \cos bx = 1, (a \neq 0, b \neq 0)$	r. No solution
d. $\cos x + \cos \pi x = -2$	s. Infinite solutions

Codes:

- | | | | | |
|-----|---|---|---|---|
| | a | b | c | d |
| (1) | q | p | s | r |
| (2) | q | p | r | s |
| (3) | s | r | q | p |
| (4) | p | q | r | s |

5. Consider the equation $\sin^2 x + (2a - 3) \sin x + (a^2 - 3a) = 0, x \in [0, 2\pi)$ and match the following lists.

List I	List II
a. Equation has one solution if	p. $a \in (0, 1) \cup (2, 3)$
b. Equation has two solutions if	q. $a = 0, 3$
c. Equation has three solutions if	r. $a = 1, 2$
d. Equation has four solutions if	s. $a \in (1, 2)$

Codes:

- | | | | | |
|-----|---|---|---|---|
| | a | b | c | d |
| (1) | q | p | s | r |
| (2) | q | p | r | s |
| (3) | s | r | p | q |
| (4) | p | q | r | s |

6.

List I	List II
a. The least positive root of the equation $\tan x = -x$ lies in	p. $(0, \pi/2)$
b. The least positive root of the equation $\tan x = 0.5x$ lies in	q. $(\pi/2, \pi)$
c. The least positive root of the equation $\tan x = x$ lies in	r. $(\pi, 3\pi/2)$
d. The least positive root of the equation $\tan x = 2x$ lies in	s. $(3\pi/2, 2\pi)$

Codes:

- | | | | | |
|-----|---|---|---|---|
| | a | b | c | d |
| (1) | r | p | s | q |
| (2) | r | q | s | p |
| (3) | s | r | p | q |
| (4) | q | r | r | p |

Numerical Value Type

- Number of values of p for which equation $\sin^3 x + 1 + p^3 - 3p \sin x = 0$ ($p > 0$) has a root is _____.
- If $\log_{0.5} \sin x = 1 - \log_{0.5} \cos x$, then number of values of $x \in [-2\pi, 2\pi]$ is _____.
- Number of roots of the equation $(3 + \cos x)^2 = 4 - 2 \sin^8 x$, $x \in [0, 5\pi]$, are _____.
- Number of solution(s) of the equation $\frac{\sin x}{\cos 3x} + \frac{\sin 3x}{\cos 9x} + \frac{\sin 9x}{\cos 27x} = 0$ in the interval $\left(0, \frac{\pi}{4}\right)$ is _____.
- Number of solutions of the equation $(\sqrt{3} + 1)^{2x} + (\sqrt{3} - 1)^{2x} = 2^{3x}$ is _____.
- Number of integral value(s) of m for which the equation $\sin x - \sqrt{3} \cos x = \frac{4m-6}{4-m}$ has solutions, $x \in [0, 2\pi]$, is _____.
- The number of solutions of the equation $\cos^2\left(x + \frac{\pi}{6}\right) + \cos^2 x - 2 \cos\left(x + \frac{\pi}{6}\right) \cdot \cos \frac{\pi}{6} = \sin^2 \frac{\pi}{6}$ in interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is _____.
- If $\cos 4x = a_0 + a_1 \cos^2 x + a_2 \cos^4 x$ is true for all values of $x \in R$, then the value of $5a_0 + a_1 + a_2$ is _____.
- Number of integral values of a for which the equation $\cos^2 x - \sin x + a = 0$ has roots when $x \in (0, \pi/2)$ is _____.
- Number of roots of the equation $2^{\tan\left(x - \frac{\pi}{4}\right)} - 2(0.25)^{\frac{\sin^2\left(x - \frac{\pi}{4}\right)}{\cos 2x}} + 1 = 0$ is _____.

- Number of solutions of the equation $\sin^4 x - \cos^2 x \sin x + 2 \sin^2 x + \sin x = 0$ in $0 \leq x \leq 3\pi$ is _____.
- Let k be sum of all x in the interval $[0, 2\pi]$ such that $3 \cot^2 x + 8 \cot x + 3 = 0$, then the value of k/π is _____.
- If $\theta \in [0, 5\pi]$ and $r \in R$ such that $2 \sin \theta = r^4 - 2r^2 + 3$ then the maximum number of values of the pair (r, θ) is _____.
- If $2 \tan^2 x - 5 \sec x = 1$ is satisfied by exactly seven distinct values of $x \in \left[0, (2n+1)\frac{\pi}{2}\right]$, $n \in N$, then the greatest value of n is _____.
- If $\sin x + \sin y \geq \cos \alpha \cos x \forall x \in R$, then $\sin y + \cos \alpha$ is equal to _____.
- If $\sin(\sin x + \cos x) = \cos(\cos x - \sin x)$, and largest possible value of $\sin x$ is $\frac{\pi}{k}$, then the value of k is _____.
- The number of solutions of the equation $1 + \cos x + \cos 2x + \sin x + \sin 2x + \sin 3x = 0$, which satisfy the condition $\frac{\pi}{2} < \left|3x - \frac{\pi}{2}\right| \leq \pi$ is _____.
- The least value of 'a' for which the equation $2\sqrt{a} \sin^2 x + \sqrt{a-3} \sin 2x = 5 + \sqrt{a}$ has at least one solution is _____.
- The number of ordered pairs (x, y) satisfying the equation $\sin^2(x+y) + \cos^2(x-y) = 1$ which lie on the circle $x^2 + y^2 = \pi^2$ is _____.
- The number of ordered pairs (x, y) satisfying $x\left(\sin^2 x + \frac{1}{x^2}\right) = 2 \sin x \sin^2 y$, where $x \in (-\pi, 0) \cup (0, \pi)$ and $y \in [0, 2\pi]$ is _____.
- Number of solutions of the equation $\cos 5x \times \tan(6|x|) + \sin 5x = 0$ lying in $[-2\pi, \pi]$ is _____.

Archives

JEE MAIN

Single Correct Answer Type

- If $0 \leq x < 2\pi$, then the number of real values of x , which satisfy the equation $\cos x + \cos 2x + \cos 3x + \cos 4x = 0$, is
(1) 5 (2) 7 (3) 9 (4) 3
(JEE Main 2016)
- If sum of all the solutions of the equation $8 \cos x \left(\cos\left(\frac{\pi}{6} + x\right) \cos\left(\frac{\pi}{6} - x\right) - \frac{1}{2} \right) = 1$ in $[0, \pi]$ is $k\pi$, then k is equal to
(1) 20/9 (2) 2/3 (3) 13/9 (4) 8/9
(JEE Main 2018)

JEE ADVANCED

Single Correct Answer Type

- Let $f(x) = x^2$ and $g(x) = \sin x$ for all $x \in R$. Then the set of all x satisfying $(f \circ g \circ g \circ f)(x) = (g \circ g \circ f)(x)$, where $(f \circ g)(x) = f(g(x))$, is
(1) $\pm \sqrt{n\pi}$, $n \in \{0, 1, 2, \dots\}$
(2) $\pm \sqrt{n\pi}$, $n \in \{1, 2, \dots\}$
(3) $\frac{\pi}{2} + 2n\pi$, $n \in \{\dots, -2, -1, 0, 1, 2, \dots\}$
(4) $2n\pi$, $n \in \{\dots, -2, -1, 0, 1, 2, \dots\}$

(IIT-JEE 2011)

2. For $x \in (0, \pi)$, the equation $\sin x + 2 \sin 2x - \sin 3x = 3$ has
 (1) infinitely many solutions
 (2) three solutions
 (3) one solution
 (4) no solution

(JEE Advanced 2014)

3. Let $S = \left\{ x \in (-\pi, \pi) : x \neq 0, \pm \frac{\pi}{2} \right\}$. The sum of all distinct solutions of the equation $\sqrt{3} \sec x + \operatorname{cosec} x + 2(\tan x - \cot x) = 0$ in the set S is equal to

- (1) $-\frac{7\pi}{9}$ (2) $-\frac{2\pi}{9}$
 (3) 0 (4) $\frac{5\pi}{9}$

(JEE Advanced 2016)

Multiple Correct Answers Type

1. For $0 < \theta < \frac{\pi}{2}$, the solution(s) of

$$\sum_{m=1}^6 \operatorname{cosec} \left(\theta + \frac{(m-1)}{4} \pi \right) \operatorname{cosec} \left(\theta + \frac{m\pi}{4} \right) = 4\sqrt{2} \text{ is (are)}$$

- (1) $\pi/4$ (2) $\pi/6$ (3) $\pi/12$ (4) $5\pi/12$

(IIT-JEE 2009)

2. Let $\theta, \varphi \in [0, 2\pi]$ be such that $2 \cos \theta (1 - \sin \varphi) = \sin^2 \theta \left(\tan \frac{\theta}{2} + \cot \frac{\theta}{2} \right) \cos \varphi - 1$, $\tan(2\pi - \theta) > 0$ and $-1 < \sin \theta < -\frac{\sqrt{3}}{2}$. Then φ cannot satisfy

- (1) $0 < \varphi < \frac{\pi}{2}$ (2) $\frac{\pi}{2} < \varphi < \frac{4\pi}{3}$
 (3) $\frac{4\pi}{3} < \varphi < \frac{3\pi}{2}$ (4) $\frac{3\pi}{2} < \varphi < 2\pi$

(IIT-JEE 2012)

Linked Comprehension Type

1. Let $f(x) = (1-x)^2 \sin^2 x + x^2$ for all $x \in R$.

Consider the statements:

 P : There exists some $x \in R$ such that $f(x) + 2x = 2(1+x^2)$. Q : There exists some $x \in R$ such that $2f(x) + 1 = 2x(1+x)$. Then

- (1) both P and Q are true (2) P is true and Q is false
 (3) P is false and Q is true (4) both P and Q are false

(IIT-JEE 2012)

Matrix Match Type

1. Match the statements/expressions in List I with the statements/expressions in List II.

List I	List II
a. Root(s) of the equation $2 \sin^2 \theta + \sin^2 2\theta = 2$	p. $\frac{\pi}{6}$
b. Points of discontinuity of the function $f(x) = \left[\frac{6x}{\pi} \right] \cos \left[\frac{3x}{\pi} \right]$, where $[y]$ denotes the largest integer less than or equal to y	q. $\frac{\pi}{4}$
c. Volume of the parallelepiped with its edges represented by the vectors $\hat{i} + \hat{j}$, $\hat{i} + 2\hat{j}$ and $\hat{i} + \hat{j} + \pi\hat{k}$	r. $\frac{\pi}{3}$
d. Angle between vectors $\vec{a}$ and $\vec{b}$ where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors satisfying $\vec{a} + \vec{b} + \sqrt{3}\vec{c} = \vec{0}$	s. $\frac{\pi}{2}$
	t. π

(IIT-JEE 2009)

Numerical Value Type

1. The number of all possible values of θ , where $0 < \theta < \pi$ for which the system of equations
 $(y+z) \cos 3\theta = (xyz) \sin 3\theta$
 $x \sin 3\theta = \frac{2 \cos 3\theta}{y} + \frac{2 \sin 3\theta}{z}$
 $(xyz) \sin 3\theta = (y+2z) \cos 3\theta + y \sin 3\theta$
 has a solution (x_0, y_0, z_0) with $y_0 z_0 \neq 0$ is _____.

(IIT-JEE 2010)

2. The number of values of θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\theta \neq \frac{n\pi}{5}$ for $n = 0, \pm 1, \pm 2$ and $\tan \theta = \cot 5\theta$ as well as $\sin 2\theta = \cos 4\theta$ is _____.

(IIT-JEE 2010)

3. The number of distinct solutions of the equation $\frac{5}{4} \cos^2 2x + \cos^4 x + \sin^4 x + \cos^6 x + \sin^6 x = 2$ in the interval $[0, 2\pi]$ is _____.

(JEE Advanced 2012)

4. Let a, b, c be three non-zero real numbers such that the equation $\sqrt{3}a \cos x + 2b \sin x = c$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, has two distinct real roots α and β with $\alpha + \beta = \frac{\pi}{3}$. Then, the value of $\frac{b}{a}$ is _____.

(JEE Advanced 2010)

Answers Key

EXERCISES

Single Correct Answer Type

- | | | | | |
|---------|---------|---------|---------|---------|
| 1. (1) | 2. (3) | 3. (1) | 4. (3) | 5. (3) |
| 6. (3) | 7. (1) | 8. (3) | 9. (1) | 10. (2) |
| 11. (1) | 12. (3) | 13. (4) | 14. (1) | 15. (1) |
| 16. (1) | 17. (2) | 18. (2) | 19. (2) | 20. (2) |
| 21. (1) | 22. (3) | 23. (3) | 24. (2) | 25. (3) |
| 26. (1) | 27. (3) | 28. (1) | 29. (2) | 30. (4) |
| 31. (1) | 32. (2) | 33. (3) | 34. (1) | 35. (4) |
| 36. (2) | 37. (4) | 38. (3) | 39. (4) | 40. (1) |
| 41. (1) | 42. (3) | 43. (3) | 44. (3) | 45. (1) |
| 46. (1) | 47. (1) | 48. (1) | 49. (2) | 50. (1) |
| 51. (1) | 52. (4) | 53. (3) | 54. (3) | 55. (1) |
| 56. (2) | 57. (4) | 58. (3) | 59. (1) | 60. (3) |
| 61. (2) | 62. (2) | 63. (4) | 64. (2) | 65. (3) |
| 66. (2) | 67. (2) | 68. (1) | 69. (2) | 70. (1) |
| 71. (4) | 72. (2) | 73. (1) | 74. (3) | 75. (2) |
| 76. (1) | 77. (2) | 78. (2) | 79. (4) | 80. (3) |
| 81. (3) | 82. (2) | 83. (2) | 84. (4) | 85. (3) |
| 86. (3) | 87. (1) | 88. (3) | 89. (1) | 90. (1) |

Multiple Correct Answers Type

- | | |
|-------------------|------------------------|
| 1. (1), (2) | 2. (1), (2) |
| 3. (1), (3), (4) | 4. (1), (3) |
| 5. (1), (2), (3) | 6. (1), (3) |
| 7. (1), (2), (3) | 8. (1), (4) |
| 9. (2), (3) | 10. (1), (3) |
| 11. (1), (4) | 12. (1), (2), (4) |
| 13. (1), (2), (3) | 14. (1), (2), (3) |
| 15. (1), (2) | 16. (3), (4) |
| 17. (1), (4) | 18. (1), (3), (4) |
| 19. (2), (3), (4) | 20. (1), (2), (3), (4) |
| 21. (1), (4) | 22. (1), (3) |
| 23. (1), (2) | 24. (1), (2) |
| 25. (3), (4) | 26. (1), (2), (3), (4) |
| 27. (1), (2), (3) | 28. (1), (3), (4) |
| 29. (2), (3) | 30. (1), (4) |
| 31. (1), (4) | |

Linked Comprehension Type

- | | | | | |
|---------|---------|---------|---------|---------|
| 1. (2) | 2. (3) | 3. (1) | 4. (1) | 5. (3) |
| 6. (4) | 7. (1) | 8. (2) | 9. (4) | 10. (1) |
| 11. (4) | 12. (2) | 13. (1) | 14. (4) | 15. (2) |
| 16. (1) | 17. (3) | 18. (2) | 19. (1) | 20. (3) |

Matrix Match Type

- $a \rightarrow r, b \rightarrow s, c \rightarrow p, d \rightarrow q.$
- $a \rightarrow q; b \rightarrow s; c \rightarrow s; d \rightarrow p.$
- $a \rightarrow q; b \rightarrow s; c \rightarrow p; d \rightarrow r.$
- (1)
- (3)
- (4)

Numerical Value Type

- | | | | | |
|----------|---------|----------|----------|---------|
| 1. (1) | 2. (2) | 3. (3) | 4. (6) | 5. (1) |
| 6. (4) | 7. (2) | 8. (5) | 9. (1) | 10. (0) |
| 11. (4) | 12. (5) | 13. (6) | 14. (7) | 15. (1) |
| 16. (4) | 17. (2) | 18. (14) | 19. (12) | 20. (8) |
| 21. (13) | | | | |

ARCHIVES

JEE MAIN

Single Correct Answer Type

- (2)
- (3)

JEE ADVANCED

Single Correct Answer Type

- (1)
- (4)
- (3)

Multiple Correct Answers Type

- (3), (4)
- (1), (3), (4)

Linked Comprehension Type

- (3)

Matrix Match Type

- $a \rightarrow q, s$

Numerical Value Type

- | | | | |
|--------|--------|--------|----------|
| 1. (3) | 2. (3) | 3. (8) | 4. (0.5) |
|--------|--------|--------|----------|

5

Properties and Solutions of Triangle

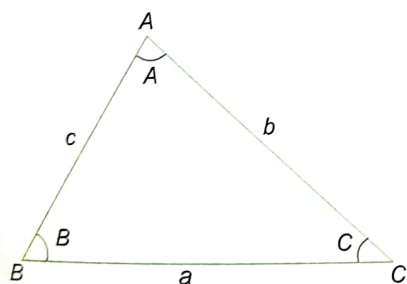
INTRODUCTION

Whether it is navigation, architecture, medical imaging or computer graphics, principles of trigonometry are applicable in various fields of science and technology. The presence of triangles is very evident around us in the shape of a field, a mountain side, a flag, a pizza slice etc. Triangle is more special as compared to other polygons as it is the polygon having the least number of sides. A triangle has six main elements, three sides and three angles. There are different rules and theorems for triangles which relate their sides and angles. We will apply the previously studied rules and will study few new ones. Some of these rules are Sine rule, Cosine rule, Tangent rule, Projection rule etc. These rules can be applied to analyse triangles involved in different cases.

Solution of triangles involves the application of properties, rules and theorems to analyse them more deeply. Now, we will be facing more advanced trigonometric problems of finding the elements of a triangle, when some of these are known. Depending upon which elements are given, appropriate rules are used to solve the triangle.

Few standard symbols to represent elements of triangle:

Consider $\triangle ABC$ as shown below.



Following symbols in relation to $\triangle ABC$ are universally adopted.

- $m\angle BAC = A$; $m\angle ABC = B$; $m\angle BCA = C$
- $AB = c$, $BC = a$, $CA = b$
- Semi perimeter of the triangle is denoted by s .

$$s = \frac{a + b + c}{2}$$

So, $a + b + c = 2s$

- Area of triangle represented by S or Δ .

We shall be using these symbols in subsequent discussions.

SINE RULE

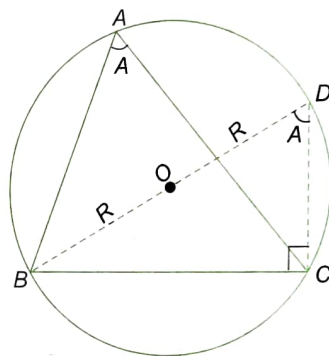
According to the sine rule, in triangle ABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

Here, R is circumradius of triangle.

We shall prove here that $\frac{a}{\sin A} = 2R$.

Case I: When triangle is acute angled



In the given figure, triangle ABC is acute angled. O is the circumcentre of $\triangle ABC$.

$\overline{BO}$ intersects the circumcircle at D .

Here, $BD = 2OB = 2R$

and $\angle BDC = \angle BAC = A$

... (1)

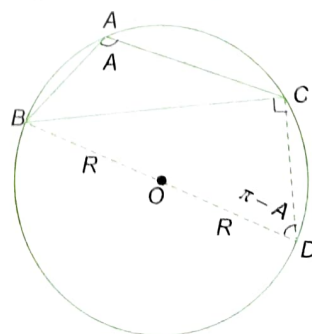
In right angled $\triangle BCD$,

$$\sin(\angle BDC) = \frac{BC}{BD} = \frac{a}{2R}$$

$$\Rightarrow \sin A = \frac{a}{2R}$$

$$\text{or } \frac{a}{\sin A} = 2R$$

Case II: When triangle is obtuse angled



5.2 Trigonometry

In the given figure, $\angle A$ is obtuse.

$\overline{BO}$ intersects the circumcircle at D .

$\therefore \angle BDC = \pi - A$, which is acute angle.

In right angled $\triangle BCD$,

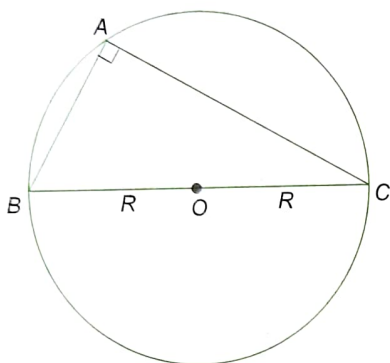
$$\sin(\angle BDC) = \frac{BC}{BD}$$

$$\therefore \sin(\pi - A) = \frac{a}{2R}$$

$$\Rightarrow \sin A = \frac{a}{2R}$$

$$\text{or } \frac{a}{\sin A} = 2R$$

Case III: When triangle is right angled



In the given figure, triangle ABC is right angled at vertex A .

$\overline{BC}$ is a diameter of the circumcircle.

$\therefore BC = 2R$ or $a = 2R$

$$\therefore a = 2R \sin \frac{\pi}{2} = 2R \sin A$$

$$\therefore \frac{a}{\sin A} = 2R$$

So, in each case, $\frac{a}{\sin A} = 2R$.

Similarly, it can be proved that $\frac{b}{\sin B} = 2R$ and $\frac{c}{\sin C} = 2R$.

$$\text{Thus, } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

ILLUSTRATION 5.1

If in a triangle ABC , $b = 3c$ and $C - B = 90^\circ$, then find the value of $\tan B$.

Sol. Since $\frac{b}{\sin B} = \frac{c}{\sin C}$, we have

$$\frac{\sin B}{\sin C} = \frac{b}{c}$$

$$\text{or } \frac{\sin B}{\sin(90^\circ + B)} = \frac{b}{c}$$

$$\text{or } \tan B = \frac{b}{c} = 3$$

ILLUSTRATION 5.2

In a triangle ABC if $BC = 1$ and $AC = 2$, then what is the maximum possible value of angle A ?

Sol. Using Sine Rule, $\frac{1}{\sin A} = \frac{2}{\sin B}$

$$\text{or } \sin A = \frac{\sin B}{2}$$

$$\Rightarrow \text{Maximum value of } \sin A \text{ is } \frac{1}{2}$$

$$\text{Hence, } A = \frac{\pi}{6}$$

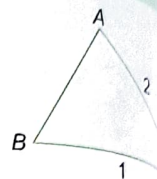


ILLUSTRATION 5.3

The perimeter of a triangle ABC is six times the arithmetic mean of the sines of its angles. If the side a is 1, then find angle A .

Sol. Given that $a + b + c = 6 \times \frac{\sin A + \sin B + \sin C}{3}$

$$\text{or } 2R(\sin A + \sin B + \sin C) = 2(\sin A + \sin B + \sin C)$$

$$\Rightarrow R = 1$$

$$\text{Now, } \frac{a}{\sin A} = 2R$$

$$\text{or } \sin A = \frac{1}{2} \quad (\because a = 1) \Rightarrow A = 30^\circ$$

ILLUSTRATION 5.4

If $A = 75^\circ$, $B = 45^\circ$, then prove that $b + c\sqrt{2} = 2a$.

Sol. $A = 75^\circ$, $B = 45^\circ \Rightarrow C = 60^\circ$

$$\text{Now, } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

$$\text{or } \frac{a}{\sin 75^\circ} = \frac{b}{\sin 45^\circ} = \frac{c}{\sin 60^\circ} = 2R$$

$$\Rightarrow b + c\sqrt{2} = \frac{\sin 45^\circ}{\sin 75^\circ} a + \sqrt{2} \frac{\sin 60^\circ}{\sin 75^\circ} a$$

$$= \frac{1}{\frac{\sqrt{2}}{\sqrt{3}+1}} a + \sqrt{2} \frac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{2}}{\sqrt{3}+1}} a$$

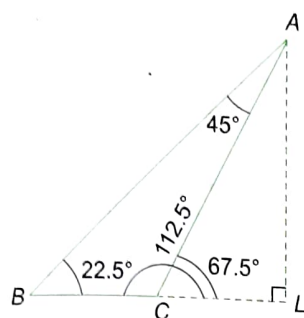
$$= \frac{2}{\sqrt{3}+1} a + \frac{2\sqrt{3}a}{\sqrt{3}+1} = 2a$$

ILLUSTRATION 5.5

If the base angles of a triangle are $22\frac{1}{2}^\circ$ and $112\frac{1}{2}^\circ$, then prove that the altitude of the triangle is equal to $\frac{1}{2}$ of its base.

Sol. In $\triangle ABC$, $\frac{BC}{\sin 45^\circ} = \frac{AC}{\sin 22\frac{1}{2}^\circ}$

In $\triangle ALC$, $\frac{AL}{AC} = \sin 67\frac{1}{2}^\circ$



$$AL = AC \cos 22\frac{1}{2}^\circ$$

$$= \frac{BC}{\sin 45^\circ} \sin 22\frac{1}{2}^\circ \cos 22\frac{1}{2}^\circ$$

[using Eq. (i)]

$$= \frac{1}{2} BC \frac{\sin 45^\circ}{\sin 45^\circ}$$

$$= \frac{1}{2} BC = \frac{1}{2} \times \text{Base}$$

ILLUSTRATION 5.6

If a^2, b^2, c^2 are in A.P., then prove that $\tan A, \tan B, \tan C$ are in H.P.

Sol. a^2, b^2, c^2 are in A.P.

$$\Rightarrow b^2 - a^2 = c^2 - b^2$$

$$\Rightarrow \sin^2 B - \sin^2 A = \sin^2 C - \sin^2 B \quad (\text{Using Sine Rule})$$

$$\Rightarrow \sin(B+A) \sin(B-A) = \sin(C+B) \sin(C-B)$$

$$\Rightarrow \sin C (\sin B \cos A - \cos B \sin A)$$

$$= \sin A (\sin C \cos B - \cos C \sin B)$$

Dividing both sides by $\sin A \sin B \sin C$, we get

$$\cot A - \cot B = \cot B - \cot C$$

$$\Rightarrow \cot A, \cot B, \cot C \text{ are in A.P.}$$

$$\Rightarrow \tan A, \tan B, \tan C \text{ are in H.P.}$$

ILLUSTRATION 5.7

Prove that $\frac{a^2 \sin(B-C)}{\sin B + \sin C} + \frac{b^2 \sin(C-A)}{\sin C + \sin A} + \frac{c^2 \sin(A-B)}{\sin A + \sin B} = 0$.

Sol. $\frac{a^2 \sin(B-C)}{\sin B + \sin C} = \frac{4R^2 \sin^2 A \sin(B-C)}{\sin B + \sin C}$ ($\because a = 2R \sin A$)

$$= \frac{4R^2 \sin A \sin(B+C) \sin(B-C)}{\sin B + \sin C}$$

$$= \frac{4R^2 \sin A (\sin^2 B - \sin^2 C)}{\sin B + \sin C}$$

$$= 4R^2 \sin A (\sin B - \sin C)$$

(i) Similarly, $\frac{b^2 \sin(C-A)}{\sin C + \sin A} = 4R^2 \sin B (\sin C - \sin A)$

and $\frac{c^2 \sin(A-B)}{\sin A + \sin B} = 4R^2 \sin C (\sin A - \sin B)$

Adding, we get

$$\frac{a^2 \sin(B-C)}{\sin B + \sin C} + \frac{b^2 \sin(C-A)}{\sin C + \sin A} + \frac{c^2 \sin(A-B)}{\sin A + \sin B} = 0$$

ILLUSTRATION 5.8

In any triangle, if $\frac{a^2 - b^2}{a^2 + b^2} = \frac{\sin(A-B)}{\sin(A+B)}$, then prove that the triangle is either right angled or isosceles.

Sol. $\frac{a^2 - b^2}{a^2 + b^2} = \frac{\sin(A-B)}{\sin(A+B)}$

or $\frac{4R^2 \sin^2 A - 4R^2 \sin^2 B}{4R^2 \sin^2 A + 4R^2 \sin^2 B} = \frac{\sin(A-B)}{\sin(A+B)} \quad (\text{Using Sine Rule})$

or $\frac{\sin(A+B) \sin(A-B)}{\sin^2 A + \sin^2 B} = \frac{\sin(A-B)}{\sin(A+B)}$

$$\Rightarrow \sin(A-B) = 0 \text{ or } \frac{\sin(\pi - C)}{\sin^2 A + \sin^2 B} = \frac{1}{\sin(\pi - C)}$$

or $A = B \text{ or } \sin^2 C = \sin^2 A + \sin^2 B$

or $A = B \text{ or } c^2 = a^2 + b^2 \quad [\text{from the sine rule}]$

Therefore, the triangle is isosceles or right angled.

ILLUSTRATION 5.9

$ABCD$ is a trapezium such that $AB \parallel CD$ and CB is perpendicular to them. If $\angle ADB = \theta$, $BC = p$, and $CD = q$, show that

$$AB = \frac{(p^2 + q^2) \sin \theta}{p \cos \theta + q \sin \theta}$$

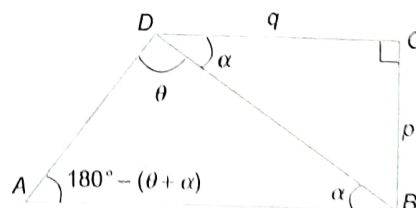
Sol. Let $\angle ABD = \angle BDC = \alpha$

$$\therefore \angle BAD = 180^\circ - (\theta + \alpha)$$

By the sine formula, in $\triangle ABD$, we have

$$\frac{AB}{\sin \theta} = \frac{BD}{\sin(180^\circ - (\theta + \alpha))}$$

or $AB = \frac{BD \sin \theta}{\sin(\theta + \alpha)} = \frac{BD \sin \theta}{\sin \theta \cos \alpha + \cos \theta \sin \alpha} \quad (i)$



In $\triangle BCD$, $\sin \alpha = p/BD$

and $\cos \alpha = q/BD$

Also $BD^2 = p^2 + q^2$

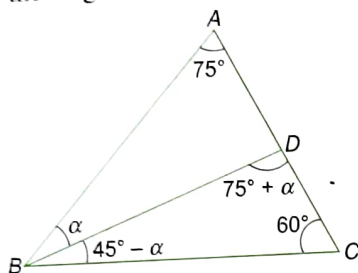
Therefore, from Eq. (i), we have

$$\begin{aligned} AB &= \frac{BD \sin \theta}{\sin \theta (q/BD) + \cos \theta (p/BD)} \\ &= \frac{BD^2 \sin \theta}{q \sin \theta + p \cos \theta} = \frac{(p^2 + q^2) \sin \theta}{p \cos \theta + q \sin \theta} \end{aligned}$$

ILLUSTRATION 5.10

In a triangle ABC , $\angle C = 60^\circ$ and $\angle A = 75^\circ$. If D is a point on AC such that the area of the $\triangle ABD$ is $\sqrt{3}$ times the area of the $\triangle BCD$, find the $\angle ABD$.

Sol. Let h be the length of perpendicular from B on AC



Given that $\frac{\Delta BAD}{\Delta BCD} = \sqrt{3}$

$$\Rightarrow \frac{\frac{1}{2}h \cdot AD}{\frac{1}{2}h \cdot DC} = \frac{AD}{CD} = \sqrt{3} \quad \dots(1)$$

In $\triangle BAD$, taking $\angle ABD = \alpha$, we have

$$\frac{AD}{\sin \alpha} = \frac{BD}{\sin 75^\circ} \quad \dots(2)$$

And in $\triangle BCD$, we have $\frac{CD}{\sin (45^\circ - \alpha)} = \frac{BD}{\sin 60^\circ} \quad \dots(3)$

$\therefore$ From (2) and (3), we get

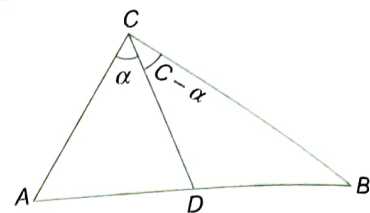
$$\begin{aligned} \frac{AD \sin (45^\circ - \alpha)}{CD \sin \alpha} &= \frac{\sin 60^\circ}{\sin 75^\circ} \\ \Rightarrow \sqrt{3} \sin 75^\circ \sin (45^\circ - \alpha) &= \sin 60^\circ \cdot \sin \alpha \\ \Rightarrow \sqrt{3} \left(\frac{\sqrt{3} + 1}{2\sqrt{2}} \right) \left(\frac{\cos \alpha - \sin \alpha}{\sqrt{2}} \right) &= \frac{\sqrt{3}}{2} \sin \alpha \\ \Rightarrow (\sqrt{3} + 1) \cos \alpha &= (3 + \sqrt{3}) \sin \alpha \\ \Rightarrow \tan \alpha &= 1/\sqrt{3} \\ \Rightarrow \alpha &= \pi/6 = 30^\circ \\ \text{Hence } \angle ABD &= 30^\circ \end{aligned}$$

ILLUSTRATION 5.11

In a scalene triangle ABC , D is a point on the side AB such that $CD^2 = AD \cdot DB$, if $\sin A \cdot \sin B = \sin^2 \frac{C}{2}$ then prove that CD is internal bisector of $\angle C$.

Sol. Let $\angle ACD = \alpha$

$$\Rightarrow \angle DCB = (C - \alpha)$$



Applying the sine rule in $\triangle ACD$ and in $\triangle DCB$ respectively, we

$$\begin{aligned} \frac{AD}{\sin \alpha} &= \frac{CD}{\sin A} \quad \text{and} \quad \frac{BD}{\sin (C - \alpha)} = \frac{CD}{\sin B} \\ \Rightarrow \frac{AD \cdot BD}{\sin \alpha \cdot \sin (C - \alpha)} &= \frac{CD^2}{\sin A \cdot \sin B} \\ \Rightarrow \frac{1}{2} [\cos (2\alpha - C) - \cos C] &= \sin^2 \frac{C}{2} \\ \Rightarrow \frac{1}{2} [\cos (2\alpha - C) - 1 + 2 \sin^2 \frac{C}{2}] &= \sin^2 \frac{C}{2} \\ \Rightarrow \cos (2\alpha - C) &= 1 \\ \Rightarrow \alpha &= \frac{C}{2} \end{aligned}$$

Thus, CD is the internal angle bisector of angle C .

NAPIER FORMULA (tan RULE)

In triangle ABC , we have

$$\begin{aligned} \text{(i)} \quad \tan \left(\frac{B - C}{2} \right) &= \frac{b - c}{b + c} \cot \frac{A}{2} \\ \text{(ii)} \quad \tan \left(\frac{C - A}{2} \right) &= \frac{c - a}{c + a} \cot \frac{B}{2} \\ \text{(iii)} \quad \tan \left(\frac{A - B}{2} \right) &= \frac{a - b}{a + b} \cot \frac{C}{2} \end{aligned}$$

Proof:

$$\begin{aligned} \text{(i)} \quad \text{From the sine rule, we have } \frac{b}{\sin B} &= \frac{c}{\sin C} \\ \Rightarrow \frac{\sin B}{\sin C} &= \frac{b}{c} \\ \Rightarrow \frac{\sin B - \sin C}{\sin B + \sin C} &= \frac{b - c}{b + c} \\ \Rightarrow \frac{2 \cos \left(\frac{B + C}{2} \right) \sin \left(\frac{B - C}{2} \right)}{2 \sin \left(\frac{B + C}{2} \right) \cos \left(\frac{B - C}{2} \right)} &= \frac{b - c}{b + c} \\ \Rightarrow \cot \left(\frac{B + C}{2} \right) \tan \left(\frac{B - C}{2} \right) &= \frac{b - c}{b + c} \\ \Rightarrow \tan \frac{A}{2} \tan \left(\frac{B - C}{2} \right) &= \frac{b - c}{b + c} \\ \Rightarrow \frac{\tan \left(\frac{B - C}{2} \right)}{\cot \frac{A}{2}} &= \frac{b - c}{b + c} \\ \Rightarrow \tan \left(\frac{B - C}{2} \right) &= \frac{b - c}{b + c} \cot \frac{A}{2} \end{aligned}$$

Similarly, other formulas can also be proved.

This formula is also known as tangent rule. This is useful in calculating the remaining parts of a triangle when two sides and included angle are given.

ILLUSTRATION 5.12

In a triangle ABC , $\angle A = 60^\circ$ and $b : c = (\sqrt{3} + 1) : 2$, then find the value of $(\angle B - \angle C)$.

$$\frac{b}{c} = \frac{\sqrt{3} + 1}{2} \Rightarrow \frac{b - c}{b + c} = \frac{\sqrt{3} + 1 - 2}{\sqrt{3} + 1 + 2} = \frac{\sqrt{3} - 1}{(\sqrt{3} + 1)\sqrt{3}}$$

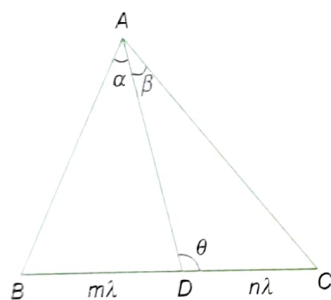
Now using $\tan \frac{B - C}{2} = \frac{b - c}{b + c} \cot \frac{A}{2}$, we get

$$\tan \frac{B - C}{2} = \frac{\sqrt{3} - 1}{(\sqrt{3} + 1)\sqrt{3}} \cdot \frac{\sqrt{3}}{2} = 2 - \sqrt{3} \Rightarrow \frac{B - C}{2} = 15^\circ$$

$$B - C = 30^\circ$$

m-n THEOREM

Let D be a point on the side BC of $\triangle ABC$ such that $BD : DC = m : n$ and $\angle ADC = \theta$, $\angle BAD = \alpha$ and $\angle DAC = \beta$.



Then

$$(i) (m - n) \cot \theta = m \cot \alpha - n \cot \beta$$

$$(ii) (m - n) \cot \theta = n \cot B - m \cot C$$

Proof:

$$(i) \text{ Given that } \frac{BD}{DC} = \frac{m}{n}, \angle ADC = \theta,$$

$$\angle BAD = \alpha \text{ and } \angle DAC = \beta.$$

$$\angle ADB = (180^\circ - \theta),$$

$$\angle ABD = 180^\circ - (\alpha + 180^\circ - \theta) = \theta - \alpha$$

$$\text{Also, } \angle ACD = 180^\circ - (\theta + \beta)$$

In $\triangle ABD$ using sine rule, we get

$$\frac{BD}{\sin \alpha} = \frac{AD}{\sin (\theta - \alpha)} \quad \dots (1)$$

In $\triangle ADC$ using sine rule, we get

$$\frac{DC}{\sin \beta} = \frac{AD}{\sin (\theta + \beta)} \quad \dots (2)$$

Dividing (1) by (2), we get

$$\frac{BD \sin \beta}{DC \sin \alpha} = \frac{\sin (\theta + \beta)}{\sin (\theta - \alpha)} \quad \dots (3)$$

$$\Rightarrow \frac{m}{n} \cdot \frac{\sin \beta}{\sin \alpha} = \frac{\sin \theta \cos \beta + \cos \theta \sin \beta}{\sin \theta \cos \alpha - \cos \theta \sin \alpha}$$

$$\Rightarrow m \sin \beta (\sin \theta \cos \alpha - \cos \theta \sin \alpha) = n \sin \alpha (\sin \theta \cos \beta + \cos \theta \sin \beta)$$

Dividing both sides by $\sin \alpha \sin \beta \sin \theta$, we get

$$m \cot \alpha - m \cot \theta = n \cot \beta + n \cot \theta$$

$$\therefore (m + n) \cot \theta = m \cot \alpha - n \cot \beta$$

(ii) We have,

$$\angle CAD = \beta = 180^\circ - (\theta + C)$$

$$\angle BAD = \alpha = (\theta - B)$$

Putting these values in (3), we get

$$m \sin (\theta + C) \sin B = n \sin C \sin (\theta - B)$$

$$\Rightarrow m (\sin \theta \cos C + \cos \theta \sin C) \sin B = n \sin C (\sin \theta \cos B - \cos \theta \sin B)$$

Dividing both sides by $\sin \theta \sin B \sin C$, we get

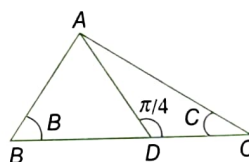
$$m (\cot C + \cot \theta) = n (\cot B - \cot \theta)$$

$$\therefore (m + n) \cot \theta = n \cot B - m \cot C$$

ILLUSTRATION 5.13

If the median AD of triangle ABC makes an angle $\pi/4$ with the side BC , then find the value of $|\cot B - \cot C|$.

Sol.



By m-n theorem,

$$(BD + DC) \cot \frac{\pi}{4} = DC \cot B - BD \cot C$$

$$\Rightarrow |\cot B - \cot C| = 2$$

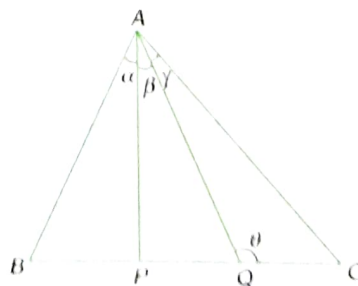
ILLUSTRATION 5.14

The base of a triangle is divided into three equal parts. If t_1, t_2, t_3 are the tangents of the angles subtended by these parts at the

opposite vertex, prove that $\left(\frac{1}{t_1} + \frac{1}{t_2}\right)\left(\frac{1}{t_2} + \frac{1}{t_3}\right) = 4\left(1 + \frac{1}{t_2^2}\right)$.

Sol. Let the points P and Q divide the side BC in three equal parts such that $BP = PQ = QC = x$

Also let $\angle BAP = \alpha$, $\angle PAQ = \beta$, $\angle QAC = \gamma$ and $\angle AQC = \theta$



From question,

$$\tan \alpha = t_1, \tan \beta = t_2, \tan \gamma = t_3$$

Applying, m : n rule in triangle ABC , we get

$$(2x + x) \cot \theta = 2x \cot (\alpha + \beta) - x \cot \gamma$$

(i)

From $\triangle APC$, we get

$$(x + x) \cot \theta = x \cot \beta - x \cot \gamma \quad (\text{ii})$$

Dividing (i) by (ii), we get

$$\frac{3}{2} = \frac{2 \cot(\alpha + \beta) - \cot \gamma}{\cot \beta - \cot \gamma}$$

$$\text{or } 3 \cot \beta - \cot \gamma = \frac{4(\cot \alpha \cdot \cot \beta - 1)}{\cot \beta + \cot \alpha}$$

$$\text{or } 3 \cot^2 \beta - \cot \beta \cot \gamma + 3 \cot \alpha \cdot \cot \beta - \cot \alpha \cdot \cot \gamma = 4 \cot \alpha \cdot \cot \beta - 4$$

$$\text{or } 4 + 4 \cot^2 \beta = \cot^2 \beta + \cot \alpha \cdot \cot \beta + \cot \beta \cdot \cot \gamma + \cot \gamma \cdot \cot \alpha$$

$$\text{or } 4(1 + \cot^2 \beta) = (\cot \beta + \cot \alpha)(\cot \beta + \cot \gamma)$$

$$\text{or } 4 \left(1 + \frac{1}{t_2^2} \right) = \left(\frac{1}{t_1} + \frac{1}{t_2} \right) \left(\frac{1}{t_2} + \frac{1}{t_3} \right)$$

CONCEPT APPLICATION EXERCISE 5.1

- Find the value of $\frac{a^2 + b^2 + c^2}{R^2}$ in any right-angled triangle.
- If angles A , B , and C of a triangle ABC are in A.P. and if $\frac{b}{c} = \frac{\sqrt{3}}{\sqrt{2}}$, then find angle A .
- In $\triangle ABC$ if $(\sqrt{3} - 1)a = 2b$, $A = 3B$ then find angle C .
- If $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$ and the side $a = 2$, then find the area of the triangle.
- In triangle ABC , if $\cos^2 A + \cos^2 B - \cos^2 C = 1$, then identify the type of the triangle.
- Prove that $b^2 \cos 2A - a^2 \cos 2B = b^2 - a^2$.
- Prove that $\frac{c}{a+b} = \frac{1 - \tan \frac{1}{2} A \tan \frac{1}{2} B}{1 + \tan \frac{1}{2} A \tan \frac{1}{2} B}$.
- In any $\triangle ABC$, prove that $(b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0$.
- In a triangle ABC , prove that $\frac{b+c}{a} \leq \operatorname{cosec} \frac{A}{2}$.
- Prove that $\frac{1 + \cos(A-B)\cos C}{1 + \cos(A-C)\cos B} = \frac{a^2 + b^2}{a^2 + c^2}$.
- In a triangle ABC , if a , b , c are in A.P. and $\frac{b}{c} \sin 2C + \frac{c}{b} \sin 2B + \frac{b}{a} \sin 2A + \frac{a}{b} \sin 2B = 2$, then find the value of $\sin B$.
- Prove that $a \cos A + b \cos B + c \cos C = 4R \sin A \sin B \sin C$.

ANSWERS

- 8
- $\frac{5\pi}{12}$
- 120°
- $\sqrt{3}$ sq. unit
- Right angled triangle
- $\frac{1}{2}$

COSINE RULE

In $\triangle ABC$, we have

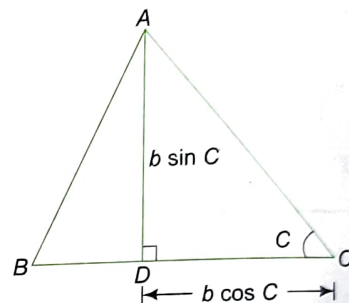
$$(i) \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$(ii) \cos B = \frac{c^2 + a^2 - b^2}{2ca}$$

$$(iii) \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Proof:

$$(iii) \text{ We shall prove that } \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$



In the figure, $BD = a - b \cos C$

In triangle ABD ,

$$\begin{aligned} AB^2 &= AD^2 + BD^2 \\ \Rightarrow c^2 &= (b \sin C)^2 + (a - b \cos C)^2 \\ \Rightarrow c^2 &= b^2 \sin^2 C + a^2 + b^2 \cos^2 C - 2ab \cos C \\ \Rightarrow c^2 &= (b^2 \sin^2 C + b^2 \cos^2 C) + a^2 - 2ab \cos C \\ \Rightarrow c^2 &= b^2 + a^2 - 2ab \cos C \\ \Rightarrow \cos C &= \frac{a^2 + b^2 - c^2}{2ab} \end{aligned}$$

The above proof will not change even if $\angle A$ is a right angle or an obtuse angle.

Similarly, other formulas can be proved.

Note:

- If the lengths of the three sides of a triangle are known, we can find all the angles by using cosine rule.
- If two sides (say b and c) and the included angle A are given,

the cosine rule $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ will give us a and then knowing a , b , c we can find B and C by the cosine rule.

ILLUSTRATION 5.15

In $\triangle ABC$, prove that $(a-b)^2 \cos^2 \frac{C}{2} + (a+b)^2 \sin^2 \frac{C}{2} = c^2$.

$$\text{Sol. L.H.S.} = (a^2 + b^2 - 2ab) \cos^2 \frac{C}{2} + (a^2 + b^2 + 2ab) \sin^2 \frac{C}{2}$$

$$= a^2 + b^2 + 2ab \left(\sin^2 \frac{C}{2} - \cos^2 \frac{C}{2} \right)$$

$$= a^2 + b^2 - 2ab \cos C$$

$$= c^2$$

(Using Cosine Rule)

ILLUSTRATION 5.16

In $\triangle ABC$, if $(a + b + c)(a - b + c) = 3ac$, then find $\angle B$.

Sol. We have $(a + c)^2 - b^2 = 3ac$

$$\text{or } a^2 + c^2 - b^2 = ac$$

$$\text{But } \cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{1}{2}$$

$$\text{So, } B = \frac{\pi}{3}$$

ILLUSTRATION 5.17

If $a = \sqrt{3}$, $b = \frac{1}{2}(\sqrt{6} + \sqrt{2})$, and $c = \sqrt{2}$, then find $\angle A$.

Sol. Using Cosine Rule,

$$\begin{aligned} \cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{(1/4)(8 + 4\sqrt{3}) + 2 - 3}{\sqrt{12} + \sqrt{4}} = \frac{1 + \sqrt{3}}{2(1 + \sqrt{3})} = \frac{1}{2} \end{aligned}$$

$$\text{So, } A = \frac{\pi}{3}$$

ILLUSTRATION 5.18

The sides of a triangle are $x^2 + x + 1$, $2x + 1$, and $x^2 - 1$. Prove that the greatest angle is 120° .

Sol. Let $a = x^2 + x + 1$, $b = 2x + 1$, and $c = x^2 - 1$.

First, we have to decide which side is the greatest. We know that in a triangle, the length of each side is greater than zero. Therefore, we have $b = 2x + 1 > 0$ and $c = x^2 - 1 > 0$. Thus,

$$x > -\frac{1}{2} \text{ and } x^2 > 1$$

$$\Rightarrow x > -\frac{1}{2} \text{ and } x < -1 \text{ or } x > 1$$

$$\Rightarrow x > 1$$

$$a = x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \left(\frac{3}{4}\right) \text{ is always positive.}$$

Thus, all sides a , b , and c are positive when $x > 1$. Now,

$$x > 1 \text{ or } x^2 > x$$

$$\text{or } x^2 + x + 1 > x + x + 1$$

$$\text{or } x^2 + x + 1 > 2x + 1 \Rightarrow a > b$$

Also, when $x > 1$,

$$x^2 + x + 1 > x^2 - 1 \Rightarrow a > c$$

Thus, $a = x^2 + x + 1$ is the greatest side and the angle A opposite to this side is the greatest angle.

$$\begin{aligned} \therefore \cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{(2x + 1)^2 + (x^2 - 1)^2 - (x^2 + x + 1)^2}{2(2x + 1)(x^2 - 1)} \end{aligned}$$

$$= \frac{-2x^3 - x^2 + 2x + 1}{2(2x^3 + x^2 - 2x - 1)} = -\frac{1}{2} = \cos 120^\circ$$

$$\Rightarrow A = 120^\circ$$

ILLUSTRATION 5.19

If the angles A, B, C of a triangle are in A.P. and sides a, b, c are in G.P., then prove that a^2, b^2, c^2 are in A.P.

Sol. Given, $2B = A + C$

$$\text{or } 3B = \pi \text{ or } B = \pi/3$$

$$\text{Also } a, b, c \text{ are in G.P. } \Rightarrow b^2 = ac$$

$$\text{Now, } \cos B = \cos 60^\circ = \frac{1}{2} = \frac{c^2 + a^2 - b^2}{2ca}$$

$$\text{or } ca = c^2 + a^2 - b^2$$

$$\text{or } 2b^2 = c^2 + a^2$$

[by using Eq. (ii)]

Hence, a^2, b^2, c^2 are in A.P.

ILLUSTRATION 5.20

Let a, b , and c be the three sides of a triangle, then prove that the equation $b^2x^2 + (b^2 + c^2 - a^2)x + c^2 = 0$ has imaginary roots.

$$\text{Sol. } b^2x^2 + (b^2 + c^2 - a^2)x + c^2 = 0$$

$$\text{Let } f(x) = b^2x^2 + (2bc \cos A)x + c^2 = 0$$

Also in $\triangle ABC$, where $A \in (0, \pi)$ in a triangle, we find $\cos A \in (-1, 1)$.

$$\text{Now, } D = (2bc \cos A)^2 - 4b^2c^2 = 4b^2c^2(\cos^2 A - 1) < 0$$

Hence, the roots are imaginary.

ILLUSTRATION 5.21

Let $a \leq b \leq c$ be the lengths of the sides of a triangle. If $a^2 + b^2 < c^2$, then prove that Δ is obtuse angled.

$$\text{Sol. } a^2 + b^2 < c^2$$

$$\Rightarrow a^2 + b^2 < a^2 + b^2 - 2ab \cos C$$

$$\Rightarrow \cos C < 0$$

Hence, C is an obtuse angle.

ILLUSTRATION 5.22

In a triangle ABC , if the sides a, b, c are roots of $x^3 - 11x^2 + 38x - 40 = 0$, then find the value of $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$.

Sol. Here a, b, c are roots of equation $x^3 - 11x^2 + 38x - 40 = 0$. Therefore,

$$a + b + c = 11, ab + bc + ac = 38, \text{ and } abc = 40$$

$$\begin{aligned} \Rightarrow \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} &= \frac{a^2 + b^2 + c^2}{2abc} \\ &= \frac{(a + b + c)^2 - 2(ab + bc + ac)}{2abc} \\ &= \frac{11^2 - 76}{80} = \frac{45}{80} = \frac{9}{16} \end{aligned}$$

ILLUSTRATION 5.23

If in a triangle ABC , $\angle C = 60^\circ$, then prove that

$$\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}.$$

Sol. By the cosine formula, we have

$$c^2 = a^2 + b^2 - 2ab \cos C$$

or $c^2 = a^2 + b^2 - 2ab \cos 60^\circ = a^2 + b^2 - ab$ (i)

Now,
$$\frac{1}{a+c} + \frac{1}{b+c} - \frac{3}{a+b+c}$$

$$= \left[\frac{(b+c)(a+b+c) + (a+c)(a+b+c) - 3(a+c)(b+c)}{(a+b)(b+c)(a+b+c)} \right]$$

$$= \frac{(a^2 + b^2 - ab) - c^2}{(a+b)(b+c)(a+b+c)} = 0 \quad [\text{from Eq. (i)}]$$

or
$$\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$$

ILLUSTRATION 5.24

In a triangle, if the angles A , B , and C are in A.P., show that

$$2 \cos \frac{1}{2}(A-C) = \frac{a+c}{\sqrt{a^2 - ac + c^2}}.$$

Sol. Since angles A , B , and C are in A.P., we have

$$A + C = 2B$$

But, $A + B + C = 180^\circ$ or $3B = 180^\circ$ or $B = 60^\circ$

Now,
$$\cos B = \frac{1}{2} = \frac{a^2 + c^2 - b^2}{2ac}$$

or $a^2 + c^2 - b^2 = ac$

or $a^2 - ac + c^2 = b^2$

$$\therefore \frac{a+c}{\sqrt{a^2 - ac + c^2}} = \frac{a+c}{b}$$

$$= \frac{2R(\sin A + \sin C)}{2R \sin B}$$

$$= \frac{2 \sin \left(\frac{A+C}{2} \right) \cos \left(\frac{A-C}{2} \right)}{\sin B}$$

$$= \frac{2 \sin 60^\circ}{\sin 60^\circ} \cos \left(\frac{A-C}{2} \right)$$

$$= 2 \cos \left(\frac{A-C}{2} \right)$$

ILLUSTRATION 5.25

In $\triangle ABC$, if $a = 9$, $b = 4$ and $c = 8$ then find the distance between the middle point of BC and the foot of the perpendicular from A .

Sol. Let D be the middle point of BC and AE be the perpendicular on BC from A

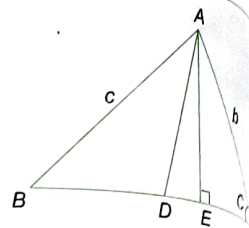
$$DE = CD - CE$$

$$= \frac{a}{2} - b \cos C$$

$$= \frac{a}{2} - b \frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{c^2 - b^2}{2a}$$

$$= \frac{64 - 16}{18} = \frac{8}{3}$$

**ILLUSTRATION 5.26**

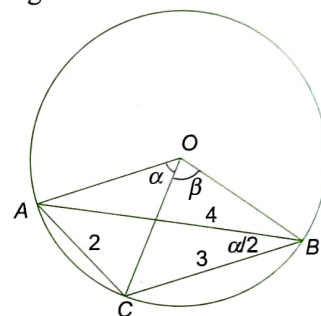
Three parallel chords of a circle have lengths 2, 3, 4 units and subtend angles $\alpha, \beta, \alpha + \beta$ at the center, respectively ($\alpha + \beta < \pi$), then find the value of $\cos \alpha$

Sol. From the given information we have the following diagram.

$$AB = 4, AC = 2, BC = 3$$

Using the geometry of the circle, we have $\angle ABC = \alpha/2$

Now in $\triangle ABC$, using cosine rule



$$\cos(\alpha/2) = \frac{(3)^2 + (4)^2 - (2)^2}{2 \times (3) \times (4)} = \frac{7}{8}$$

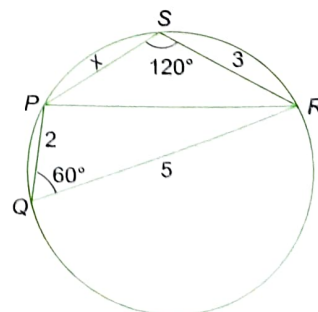
$$\Rightarrow \cos \alpha = 2 \cos^2(\alpha/2) - 1$$

$$= 2 \times \frac{49}{64} - 1 = \frac{17}{32}.$$

ILLUSTRATION 5.27

In a cyclic quadrilateral $PQRS$, $PQ = 2$ units, $QR = 5$ units, $RS = 3$ units and $\angle PQR = 60^\circ$, then what is the measure of SP ?

Sol. From $\triangle PQR$, $PR^2 = 2^2 + 5^2 - 2 \times 2 \times 5 \times \cos 60^\circ = 12$



In $\triangle PRS$, $PR^2 = 3^2 + x^2 - 2 \times 3x \times \cos 120^\circ$

$$\Rightarrow x^2 + 3x + 9 = 19$$

$$\Rightarrow (x+5)(x-2) = 0$$

Now, $x+5 = 0$ not possible

$$\therefore x = 2$$

(where $SP = x$)

CONCEPT APPLICATION EXERCISE 5.2

- If the sides of a triangle are a , b and $\sqrt{a^2 + ab + b^2}$, then find the greatest angle.
- If the line segment joining the points $A(a, b)$ and $B(c, d)$ subtends an angle θ at the origin, then prove that $\cos \theta = \frac{ac + bd}{\sqrt{(a^2 + b^2)(c^2 + d^2)}}$.
- If $x, y > 0$, then prove that the triangle whose sides are given by $3x + 4y$, $4x + 3y$, and $5x + 5y$ units is obtuse angled.
- In $\triangle ABC$, angle A is 120° , $BC + CA = 20$, and $AB + BC = 21$. Find the length of the side BC .
- In $\triangle ABC$, $AB = 1$, $BC = 1$, and $AC = 1/\sqrt{2}$. In $\triangle MNP$, $MN = 1$, $NP = 1$, and $\angle MNP = 2\angle ABC$. Find the side MP .
- If in a triangle ABC , $\frac{bc}{2 \cos A} = b^2 + c^2 - 2bc \cos A$ then prove that the triangle must be isosceles.
- With usual notation, if in triangle ABC , $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$, then prove that $\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$.
- The sides of a triangle are three consecutive natural numbers and its largest angle is twice the smallest one. Determine the sides of the triangle.

ANSWERS

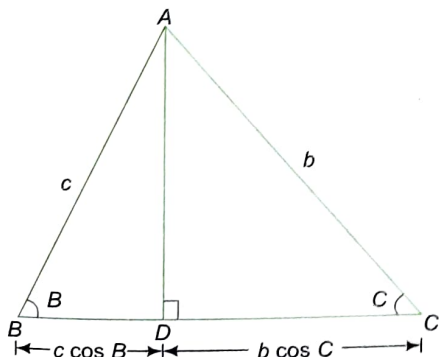
1. 120° 4. 13 units 5. $\frac{\sqrt{7}}{2}$ 8. 4, 5, 6

PROJECTION RULE

In triangle ABC , we have

- $b \cos C + c \cos B = a$
- $a \cos C + c \cos A = b$
- $a \cos B + b \cos A = c$

Proof:



In the figure,

$BD = \text{Projection of } AB \text{ on } BC = c \cos B$

$CD = \text{Projection of } AC \text{ on } BC = b \cos C$

$\therefore BC = a = BD + CD = c \cos B + b \cos C$
Similarly, other formulas can be proved.

ILLUSTRATION 5.28

Prove that $a(b \cos C - c \cos B) = b^2 - c^2$.

Sol. $a(b \cos C - c \cos B)$
 $= (b \cos C + c \cos B)(b \cos C - c \cos B)$
 $= b^2 \cos^2 C - c^2 \cos^2 B$
 $= b^2 (1 - \sin^2 C) - c^2 (1 - \sin^2 B)$
 $= b^2 - c^2 - (b^2 \sin^2 C - c^2 \sin^2 B)$
 $= b^2 - c^2 \quad [\text{as by the sine rule, } b \sin C = c \sin B]$

Alternatively, using $\cos B$ and $\cos C$ formulae, we can prove the result.

ILLUSTRATION 5.29

If in a triangle $a \cos^2 \frac{C}{2} + c \cos^2 \frac{A}{2} = \frac{3b}{2}$, then find the relation between the sides of the triangle.

Sol. $a \cos^2 \frac{C}{2} + c \cos^2 \frac{A}{2} = \frac{3b}{2}$
 or $a(1 + \cos C) + c(1 + \cos A) = 3b$
 or $a + c + (a \cos C + c \cos A) = 3b$
 or $a + c + b = 3b \quad [\text{by the projection formula}]$
 or $a + c = 2b$
 Hence, a, b, c are in A.P.

ILLUSTRATION 5.30

Prove that $(b + c) \cos A + (c + a) \cos B + (a + b) \cos C = 2s$.

Sol. $(b + c) \cos A + (c + a) \cos B + (a + b) \cos C$
 $= (b \cos A + a \cos B) + (c \cos A + a \cos C)$
 $+ (b \cos C + c \cos B) = c + b + a = 2s$

CONCEPT APPLICATION EXERCISE 5.3

- In $\triangle ABC$, prove that $c \cos(A - \alpha) + a \cos(C + \alpha) = b \cos \alpha$.
- Prove that $\frac{\cos C + \cos A}{c + a} + \frac{\cos B}{b} = \frac{1}{b}$.
- Prove that $a(b^2 + c^2) \cos A + b(c^2 + a^2) \cos B + c(a^2 + b^2) \cos C = 3abc$.

HALF ANGLE FORMULAS

HALF ANGLE FORMULAS FOR SINE

We have following half angle formulas for sine in triangle ABC .

(i) $\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$

(ii) $\sin \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$

$$(iii) \sin \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

Proof:

$$(i) \sin^2 \frac{A}{2} = \frac{1 - \cos A}{2}$$

$$\Rightarrow \sin^2 \frac{A}{2} = \frac{1}{2} \left[1 - \frac{b^2 + c^2 - a^2}{2bc} \right]$$

$$\Rightarrow \sin^2 \frac{A}{2} = \frac{1}{2} \left[\frac{2bc - b^2 - c^2 + a^2}{2bc} \right]$$

$$\Rightarrow \sin^2 \frac{A}{2} = \frac{1}{2} \left[\frac{a^2 - (b-c)^2}{2bc} \right]$$

$$\Rightarrow \sin^2 \frac{A}{2} = \frac{(a-b+c)(a+b-c)}{4bc}$$

$$\Rightarrow \sin^2 \frac{A}{2} = \frac{(2s-2b)(2s-2c)}{4bc} \quad (\text{As } a+b+c=2s)$$

$$\Rightarrow \sin^2 \frac{A}{2} = \frac{(s-b)(s-c)}{bc}$$

In a triangle, $0 < \frac{A}{2} < \frac{\pi}{2}$, so $\sin \frac{A}{2} > 0$.

$$\therefore \sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

Similarly, we can prove other formulas also.

HALF ANGLE FORMULAS FOR COSINE

We have following half angle formulas for cosine in triangle ABC.

$$(i) \cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$(ii) \cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$$

$$(iii) \cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

Proof:

$$(i) \cos^2 \frac{A}{2} = \frac{1 + \cos A}{2}$$

$$\Rightarrow \cos^2 \frac{A}{2} = \frac{1}{2} \left[1 + \frac{b^2 + c^2 - a^2}{2bc} \right]$$

$$\Rightarrow \cos^2 \frac{A}{2} = \frac{1}{2} \left[\frac{2bc + b^2 + c^2 - a^2}{2bc} \right]$$

$$\Rightarrow \cos^2 \frac{A}{2} = \frac{1}{2} \left[\frac{(b+c)^2 - a^2}{2bc} \right]$$

$$\Rightarrow \cos^2 \frac{A}{2} = \frac{1}{2} \left[\frac{(b+c+a)(b+c-a)}{2bc} \right]$$

$$\Rightarrow \cos^2 \frac{A}{2} = \frac{2s(2s-2a)}{4bc}$$

(As $a+b+c=2s$)

$$\Rightarrow \cos^2 \frac{A}{2} = \frac{s(s-a)}{bc}$$

In a triangle, $0 < \frac{A}{2} < \frac{\pi}{2}$, so $\cos \frac{A}{2} > 0$.

$$\therefore \cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

Similarly, we can prove other formulas also.

From the above formulas, we have

$$(i) \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$(ii) \tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$$

$$(iii) \tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

ILLUSTRATION 5.31

If $\cos \frac{A}{2} = \sqrt{\frac{b+c}{2c}}$, then prove that $a^2 + b^2 = c^2$.

Sol. $\cos \frac{A}{2} = \sqrt{\frac{b+c}{2c}}$

$$\Rightarrow \frac{s(s-a)}{bc} = \frac{b+c}{2c}$$

[squaring]

$$\text{or } 2s(2s-2a) = 2b(b+c)$$

$$\text{or } (b+c+a)(b+c-a) = 2b^2 + 2bc$$

$$\text{or } (b+c)^2 - a^2 = 2b^2 + 2bc$$

$$\text{or } c^2 = a^2 + b^2$$

ILLUSTRATION 5.32

If the cotangents of half the angles of a triangle are in A.P. then prove that the sides are in A.P.

Sol. $\cot \frac{A}{2}, \cot \frac{B}{2}, \cot \frac{C}{2}$ are in A.P.

$$\Rightarrow 2 \cot \frac{B}{2} = \cot \frac{A}{2} + \cot \frac{C}{2}$$

$$\Rightarrow 2 \sqrt{\frac{s(s-b)}{(s-a)(s-c)}} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} + \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$\Rightarrow 2(s-b) = s-a + s-c$$

$$\Rightarrow 2b = a + c$$

$$\Rightarrow a, b, c \text{ are in A.P.}$$

ILLUSTRATION 5.33

If the sides a , b and c of $\triangle ABC$ are in A.P., prove that

$$(i) \quad 2 \sin \frac{A}{2} \sin \frac{C}{2} = \sin \frac{B}{2}$$

$$(ii) \quad a \cos^2 \frac{C}{2} + c \cos^2 \frac{A}{2} = \frac{3b}{2}$$

Sol.

$$\begin{aligned} (i) \quad 2 \sin \frac{A}{2} \sin \frac{C}{2} &= 2 \sqrt{\frac{(s-b)(s-c)}{bc}} \cdot \sqrt{\frac{(s-a)(s-b)}{ab}} \\ &= \frac{2(s-b)}{b} \sqrt{\frac{(s-a)(s-c)}{ac}} \\ &= \frac{2s-2b}{b} \cdot \sin \frac{B}{2} \\ &= \frac{a+c-b}{b} \cdot \sin \frac{B}{2} \\ &= \frac{2b-b}{b} \cdot \sin \frac{B}{2} \quad (\text{as } a, b, c \text{ are in A.P.}) \\ &= \sin \frac{B}{2} \end{aligned}$$

$$\begin{aligned} (ii) \quad a \cos^2 \frac{C}{2} + c \cos^2 \frac{A}{2} &= \frac{a s(s-c)}{ab} + \frac{c s(s-a)}{bc} \\ &= \frac{s}{b} (s-c+s-a) \\ &= \frac{a+b+c}{2b} b \\ &= \frac{2b+b}{2} \\ &= \frac{3b}{2} \end{aligned}$$

ILLUSTRATION 5.34

Prove that $\left(\cot \frac{A}{2} + \cot \frac{B}{2} \right) \left(a \sin^2 \frac{B}{2} + b \sin^2 \frac{A}{2} \right) = c \cot \frac{C}{2}$.

Sol.

$$\begin{aligned} &\left(\cot \frac{A}{2} + \cot \frac{B}{2} \right) \left(a \sin^2 \frac{B}{2} + b \sin^2 \frac{A}{2} \right) \\ &= \left[\sqrt{\frac{s(s-a)}{(s-b)(s-c)}} + \sqrt{\frac{s(s-b)}{(s-a)(s-c)}} \right] \\ &\quad \times \left[\frac{a(s-c)(s-a)}{ca} + \frac{b(s-b)(s-c)}{bc} \right] \\ &= \sqrt{s} \cdot \left[\frac{(s-a)+(s-b)}{\sqrt{(s-a)(s-b)(s-c)}} \right] \frac{(s-c)}{c} [s-a+s-b] \\ &= \sqrt{s} \cdot \left[\frac{c(s-c)}{\sqrt{(s-a)(s-b)(s-c)}} \right] \\ &= c \sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = c \cot \frac{C}{2} \end{aligned}$$

CONCEPT APPLICATION EXERCISE 5.4

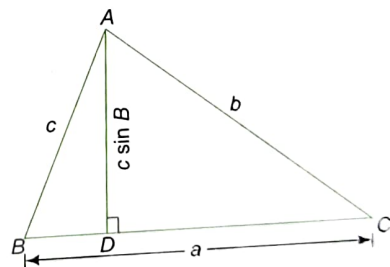
1. If $b+c=3a$, then find the value of $\cot \frac{B}{2} \cot \frac{C}{2}$.
2. Prove that $bc \cos^2 \frac{A}{2} + ca \cos^2 \frac{B}{2} + ab \cos^2 \frac{C}{2} = s^2$.
3. If in $\triangle ABC$, $\tan \frac{A}{2} = \frac{5}{6}$, $\tan \frac{C}{2} = \frac{2}{5}$, then prove that a , b , and c are in A.P.
4. Prove that $(b+c-a) \left(\cot \frac{B}{2} + \cot \frac{C}{2} \right) = 2a \cot \frac{A}{2}$.
5. If $\sin^2 \frac{A}{2}$, $\sin^2 \frac{B}{2}$, and $\sin^2 \frac{C}{2}$ are in H.P., then prove that the sides of triangle are in H.P.

ANSWERS

1. 2

AREA OF TRIANGLE

Area of triangle is usually denoted by Δ or S . There are many different formulas to find the area of triangle. Consider a triangle ABC . One of the most common formulas we have been using till now is Herons formula, that is $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, where s is semi-perimeter of triangle given by $2s = a+b+c$.



$$\text{Also, } \Delta = \frac{1}{2} (\text{base} \times \text{height})$$

$$= \frac{1}{2} (BC \times AD)$$

$$\therefore \Delta = \frac{1}{2} ac \sin B$$

... (1)

Similarly, we get $\Delta = \frac{1}{2} ab \sin C$ and $\Delta = \frac{1}{2} bc \sin A$

In (1), using sine rule, we get

$$\Delta = \frac{1}{2} ac \frac{b}{2R}$$

$$\therefore \Delta = \frac{abc}{4R}$$

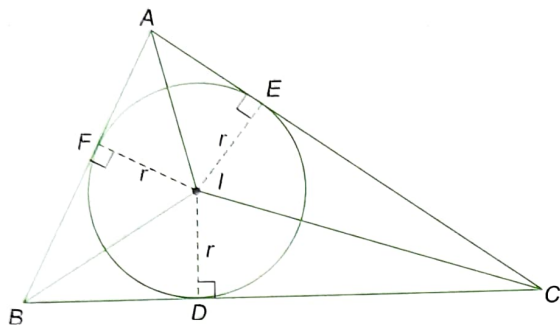
$$\text{Also, } \Delta = \frac{(2R \sin A)(2R \sin B)(2R \sin C)}{4R}$$

$$\therefore \Delta = 2R^2 \sin A \sin B \sin C$$

Further from (1), $\Delta = \frac{1}{2}ac \sin B$

$$\begin{aligned}
 &= ac \sin \frac{B}{2} \cos \frac{B}{2} \\
 &= ac \sqrt{\frac{(s-c)(s-a)}{ca}} \sqrt{\frac{s(s-b)}{ca}} \\
 &= \sqrt{s(s-a)(s-b)(s-c)}
 \end{aligned}$$

Let us get the area in terms of in radius r of the triangle.



If I is the centre of the circle inscribed in triangle, then

Area of ΔABC

$$\begin{aligned}
 &= \text{Area of } \Delta IBC + \text{area of } \Delta IAC + \text{area of } \Delta IAB \\
 &= \frac{1}{2} r \times a + \frac{1}{2} r \times b + \frac{1}{2} r \times c \\
 &= \frac{1}{2} r \times (a + b + c) \\
 &= rs
 \end{aligned}$$

ILLUSTRATION 5.35

If in triangle ABC , $\Delta = a^2 - (b - c)^2$, then find the value of $\tan^2 A$.

Sol. $\Delta = (a + b - c)(a - b + c)$

$$\Rightarrow \Delta^2 = [2(s - b) 2(s - c)]^2$$

or $s(s - a)(s - b)(s - c) = 16(s - b)^2(s - c)^2$

or $\frac{(s - b)(s - c)}{s(s - a)} = \frac{1}{16}$

or $\tan \frac{A}{2} = \frac{1}{4}$

$$\Rightarrow \tan^2 A = \frac{2 \tan(A/2)}{1 - \tan^2(A/2)} = \frac{2(1/4)}{1 - (1/16)} = \frac{8}{15}$$

ILLUSTRATION 5.36

Prove that $a^2 \sin 2B + b^2 \sin 2A = 4\Delta$.

Sol. $a^2 \sin 2B + b^2 \sin 2A$

$$= 4R^2 [\sin^2 A (2 \sin B \cos B) + \sin^2 B (2 \sin A \cos A)]$$

[using sine rule]

$$= 8R^2 \sin A \sin B (\sin A \cos B + \sin B \cos A)$$

$$= 8R^2 \sin A \sin B \sin(A + B)$$

$$= 8R^2 \sin A \sin B \sin C = 4\Delta$$

ILLUSTRATION 5.37

Prove that $\frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4b^2c^2} = \sin^2 A$

Sol.

$$\begin{aligned}
 &\frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4b^2c^2} \\
 &= \frac{2s 2(s-a) 2(s-b) 2(s-c)}{4b^2c^2} \\
 &= \frac{4\Delta^2}{b^2c^2} \\
 &= \frac{4}{b^2c^2} \left(\frac{1}{2} bc \sin A \right)^2 = \sin^2 A
 \end{aligned}$$

ILLUSTRATION 5.38

If the sides of a triangle are 17, 25, and 28, then find the greatest length of the altitude.

Sol. We know from geometry that the greatest altitude is perpendicular to the shortest side.

Let $a = 17$, $b = 25$,
and $c = 28$.

Now, $\Delta = \frac{1}{2} AD \times BC$

or $AD = \frac{2\Delta}{17}$

where $\Delta^2 = s(s-a)(s-b)(s-c)$

$$= 210^2$$

$$\Rightarrow AD = \frac{420}{17}$$

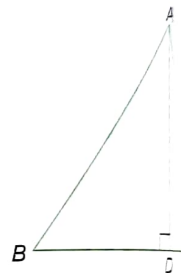


ILLUSTRATION 5.39

In equilateral triangle ABC with interior point D , if the perpendicular distances from D to the sides of 4, 5, and 6 respectively then find the area of ΔABC .

Sol. Let the side of equilateral triangle ABC be a .

Area of triangle, $\Delta = \frac{a \times 4 + a \times 5 + a \times 6}{2}$

or $\frac{a(4+5+6)}{2} = \frac{\sqrt{3}}{4} a^2$

or $\frac{15}{2} = \frac{\sqrt{3}}{4} a$

or $a = \frac{30}{\sqrt{3}} = 10\sqrt{3}$

or $\Delta = \frac{\sqrt{3}}{4} \times 100 \times 3$

$$= 75\sqrt{3}$$

ILLUSTRATION 5.40

If area of a triangle is 2 sq. units, then find the value of the product of the arithmetic mean of the lengths of the sides of a triangle and harmonic mean of the lengths of the altitudes of the triangle.

Sol. Let h_1, h_2 and h_3 be the lengths of altitudes through vertices A, B and C , respectively.

$$a h_1 = b h_2 = c h_3 = 2 \Delta$$

$$\text{or } \frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} = \frac{a+b+c}{2\Delta}$$

$$\text{or } \frac{a+b+c}{3} \times \frac{3}{\frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3}} = 2\Delta = 4$$

ILLUSTRATION 5.41

A triangle has sides 6, 7, and 8. The line through its incentre parallel to the shortest side is drawn to meet the other two sides at P and Q . Then find the length of the segment PQ .

Sol. $\Delta = r \times s$

$$\therefore \frac{21 \times r}{2} = \frac{6 \times h}{2} = 3h$$

$$\text{or } \frac{r}{h} = \frac{2}{7}$$

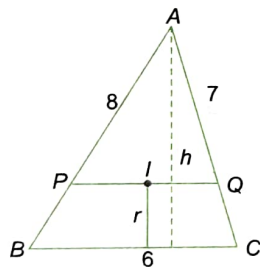
Now APQ and ABC are similar. Thus,

$$\frac{h-r}{h} = \frac{PQ}{6}$$

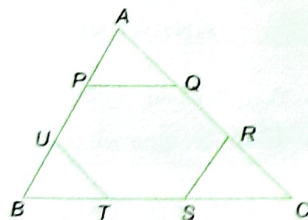
$$\text{or } 1 - \frac{r}{h} = \frac{PQ}{6}$$

$$\text{or } 1 - \frac{2}{7} = \frac{PQ}{6}$$

$$\text{or } \frac{5}{7} = \frac{PQ}{6} \text{ or } PQ = \frac{30}{7}$$


ILLUSTRATION 5.42

Each side of triangle ABC is divided into three equal parts as shown in the figure. Find the ratio of the area of hexagon $PQRSTU$ to the area of the triangle ABC .



Sol. Area of hexagon $PQRSTU$,

$$\Delta_1 = \text{area of } \Delta ABC - \text{area of } \Delta APQ -$$

$$\text{area of } \Delta BTU - \text{area of } \Delta CRS$$

$$= \Delta - \frac{1}{2} \left(\frac{b}{3} \cdot \frac{c}{3} \sin A \right) - \frac{1}{2} \left(\frac{a}{3} \cdot \frac{c}{3} \sin B \right) - \frac{1}{2} \left(\frac{a}{3} \cdot \frac{b}{3} \sin C \right)$$

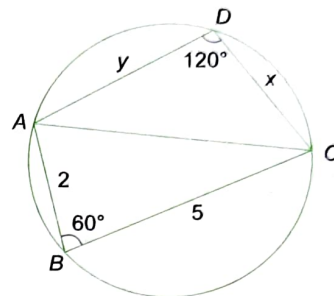
$$= \Delta - \frac{\Delta}{3} = \frac{2\Delta}{3}$$

$$\therefore \frac{\Delta_1}{\Delta} = \frac{2}{3}$$

ILLUSTRATION 5.43

The two adjacent sides of a cyclic quadrilateral are 2 and 5 and the angle between them is 60° . If the area of the quadrilateral is $4\sqrt{3}$, find the remaining two sides.

Sol.



Let $ABCD$ be the cyclic quadrilateral in which

$$AB = 2 \text{ and } BC = 5, \angle ABC = 60^\circ$$

$$\angle ADC = 180^\circ - 60^\circ = 120^\circ$$

Area of cyclic quadrilateral $ABCD$

$$= \text{Area of } \Delta ABC + \text{Area of } \Delta ACD$$

$$\begin{aligned} \text{or } 4\sqrt{3} &= \frac{1}{2} AB \times BC \sin 60^\circ + \frac{1}{2} CD \times DA \sin 120^\circ \\ &= \frac{1}{2} 2 \times 5 \times \left(\frac{\sqrt{3}}{2} \right) + \frac{1}{2} xy \left(\frac{\sqrt{3}}{2} \right), \end{aligned}$$

where $CD = x, AD = y$

$$\therefore xy = 6 \quad \dots(i)$$

From ΔABC , we get

$$AC^2 = AB^2 + BC^2 - 2AB \times BC \cos 60^\circ$$

$$= 4 + 25 - 20 \left(\frac{1}{2} \right) = 19 \quad \dots(ii)$$

Also from ΔACD , we get

$$\begin{aligned} AC^2 &= CD^2 + DA^2 - 2CD \cdot DA \cos 120^\circ \\ &= x^2 + y^2 + xy = x^2 + y^2 + 6 \quad [\text{using Eq. (i)}] \quad \dots(iii) \end{aligned}$$

Now from Eqs. (ii) and (iii), we have

$$x^2 + y^2 + 6 = 19 \text{ or } x^2 + y^2 = 13 \quad \dots(iv)$$

Solving Eqs. (i) and (iv), we get

$$x^4 - 13x^2 + 36 = 0$$

$$\text{or } x^2 = 4, 9$$

$$\text{or } x = 2, 3 \Rightarrow y = 3, 2$$

Hence, the other two sides of the cyclic quadrilateral are 3 and 2.

ILLUSTRATION 5.44

In triangle ABC , $a : b : c = 4 : 5 : 6$. Then find the ratio of the radius of the circumcircle to that of the incircle.

Sol. $a = 4k, b = 5k, c = 6k$

$$s = \frac{15}{2} k,$$

$$s - a = \frac{7}{2} k,$$

$$s - b = \frac{5}{2} k,$$

$$s - c = \frac{3}{2} k$$

$$\Rightarrow \Delta^2 = 15 \times 7 \times 5 \times 3 \left(\frac{k}{2}\right)^4$$

$$\text{or } \Delta = 15\sqrt{7} \left(\frac{k}{2}\right)^2$$

$$\text{Now, } r = \frac{\Delta}{s} = \frac{15\sqrt{7}(k/2)^2}{(15/2)k} = \sqrt{7} \frac{k}{2}$$

$$\text{and } R = \frac{abc}{4\Delta} = \frac{4 \times 5 \times 6 \times k^3}{4 \times 15\sqrt{7} \times (k^2/4)} = \frac{8}{\sqrt{7}} k$$

$$\therefore \frac{R}{r} = \frac{16}{7}$$

ILLUSTRATION 5.45

Given a triangle ABC with sides $a = 7$, $b = 8$ and $c = 5$. Find the value of the expression $(\sin A + \sin B + \sin C)$

$$\times \left(\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} \right).$$

Sol. $(\sin A + \sin B + \sin C) \left(\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} \right)$

$$= \left(\frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R} \right) \left(\frac{s(s-a)}{\Delta} + \frac{s(s-b)}{\Delta} + \frac{s(s-c)}{\Delta} \right)$$

$$= \left(\frac{a+b+c}{2R} \right) \left(\frac{s(3s-a-b-c)}{\Delta} \right)$$

$$= \left(\frac{2s}{2R} \right) \left(\frac{s^2}{\Delta} \right) = \frac{s^3}{R\Delta}$$

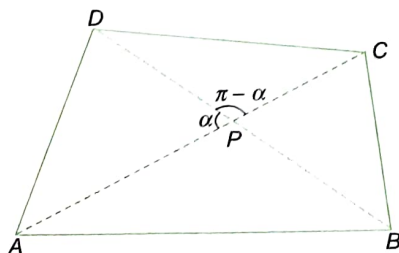
$$= \frac{s^3}{R \frac{abc}{4R}} = \frac{4s^3}{abc}$$

$$= \frac{4 \left(\frac{a+b+c}{2} \right)^3}{abc} = \frac{4 \left(\frac{7+8+5}{2} \right)^3}{7 \times 8 \times 5} = \frac{100}{7}$$

AREA OF QUADRILATERAL

Consider the quadrilateral $ABCD$.

Let S denote the area of the quadrilateral.



Now, area of triangle DAC ,

$$\Delta_1 = \text{area of } \triangle APD + \text{area of } \triangle DPC$$

$$= \frac{1}{2} \times DP \times AP \times \sin \alpha + \frac{1}{2} \times DP \times PC \times \sin(\pi - \alpha)$$

$$= \frac{1}{2} \times DP (AP + PC) \sin \alpha$$

$$\therefore \Delta_1 = \frac{1}{2} \times DP \times AC \times \sin \alpha$$

Similarly, area of $\triangle ABC$,

$$\Delta_2 = \frac{1}{2} \times BP \times AC \times \sin \alpha$$

Now, area of quadrilateral

$$S = \Delta_1 + \Delta_2$$

$$= \frac{1}{2} \times DP \times AC \times \sin \alpha + \frac{1}{2} \times BP \times AC \times \sin \alpha$$

$$= \frac{1}{2} \times (DP + BP) \times AC \times \sin \alpha$$

$$= \frac{1}{2} \times BD \times AC \times \sin \alpha$$

$$\text{Thus, area of quadrilateral} = \frac{1}{2} (\text{Product of the diagonals}) \times (\text{Sine of included angle})$$

CONCEPT APPLICATION EXERCISE 5.5

1. If $c^2 = a^2 + b^2$, then prove that $4s(s-a)(s-b)(s-c) = a^2 b^2$.
2. If the sides of a triangle are in the ratio 3:7:8, then find $R:r$.
3. In triangle ABC , if $a = 2$ and $bc = 9$, then prove that $R = 9/2\Delta$.
4. In $\triangle ABC$, if lengths of medians BE and CF are 12 and 5 respectively, find the maximum value of Δ .
5. Let the lengths of the altitudes drawn from the vertices of $\triangle ABC$ to the opposite sides are 2, 2, and 3. If the area of $\triangle ABC$ is Δ , then find the area of triangle.
6. A triangle with integral sides has perimeter 8 cm. Then find the area of the triangle.
7. The sides of a triangle are in A.P. and its area is $\frac{3}{5}$ th of an equilateral triangle of the same perimeter. Find the greatest angle of the triangle.

ANSWERS

2. 7 : 2 4. $\Delta_{\max} = 72$ sq. units 5. $\frac{9}{2\sqrt{2}}$ sq. unit
6. $2\sqrt{2} \text{ cm}^2$ 7. 120°

SOLUTIONS OF TRIANGLE

We have learned about different rules and properties of triangle. Let us see how to use these rules to solve the triangle in different situations. Let us consider the different cases when the components of triangle are given.

THREE SIDES (SSS)

If the three side lengths a , b and c are given. To find the two angles, the cosine rule can be used.

Using $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$, we can find angle A .

Similarly, we can find angle B .

Angle C can be obtained using $C = 180^\circ - A - B$.

Alternatively, after finding angle A using cosine rule, one can use sine rule to find the angle B , i.e., $\sin B = \frac{b}{a} \sin A$, but there is a risk of confusing an acute angle value with an obtuse one.

Angles can also be obtained using half angle formulas.

$$\text{i.e., } \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \text{ etc.}$$

TWO SIDES AND THE INCLUDED ANGLE (SAS)

Let the two sides b and c and included angle A are given. The side opposite to angle A can be determined using cosine rule.

$$\text{i.e., } a^2 = b^2 + c^2 - 2bc \cos A.$$

After that, using cosine rule again, the second angle can be obtained.

$$\text{i.e., } \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Finally, $B = 180^\circ - A - C$.

A SIDE AND THE TWO ANGLES ADJACENT TO IT (ASA)

Let side c and adjacent angles A and B are given.

The third angle is $C = 180^\circ - A - B$.

Sides a and b can be determined using sine rule

$$\text{i.e., } a = c \frac{\sin A}{\sin C} \text{ and } b = c \frac{\sin B}{\sin C}$$

A SIDE, THE ANGLE OPPOSITE TO IT AND AN ANGLE ADJACENT TO IT (AAS)

The procedure for solving an AAS triangle is same as that for an ASA triangle. First, find the third angle by using the angle sum property of triangle, then find the other two sides using the sine rule.

TWO SIDES AND AN ANGLE NOT INCLUDED BETWEEN THEM (SSA)

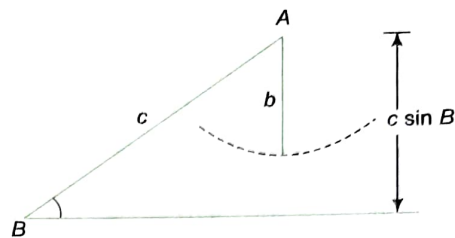
Let the two sides b , c and the angle B are given. Using sine rule, we get

$$\sin C = \frac{c}{b} \sin B.$$

Now, following cases occur:

Case I: When $\frac{c}{b} \sin B > 1$ or $b < c \sin B$.

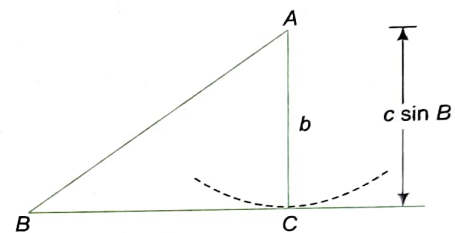
No such triangle exists because the side b does not reach line BC .



For the same reason, a solution does not exist if angle $B \geq 90^\circ$ and $b \leq c$ as triangle cannot have two right angles or two obtuse angles.

Case II: When $\frac{c}{b} \sin B = 1$, then $\sin C = 1$, i.e., $C = 90^\circ$.

So, a unique triangle exists, with $C = 90^\circ$.

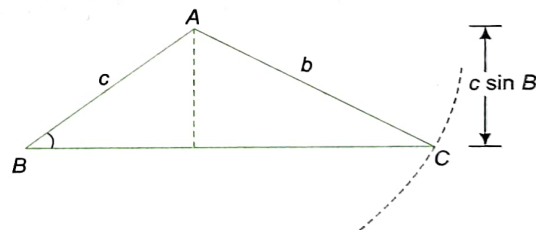


Case III: When $\frac{c}{b} \sin B < 1$.

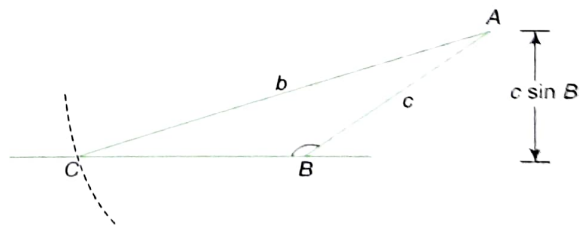
Now, two alternatives are possible.

- (i) If $b \geq c$ then $B \geq C$ (as larger side corresponds to larger angle). Since no triangle can have two obtuse angles, C must be an acute angle and there exists a unique triangle. For $b > c$, see the following figure.

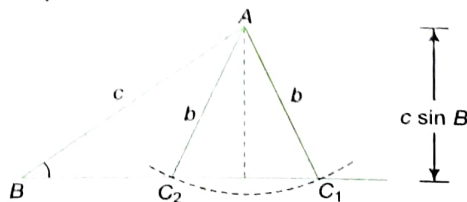
When B is acute



When B is obtuse



- (ii) If $b < c$, the angle C may be acute at C_1 or obtuse at C_2 ($= 180^\circ - C_1$).



Alternative Method:

Using cosine rule, we have $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$

$$\begin{aligned}\therefore a^2 - (2c \cos B) a + (c^2 - b^2) &= 0 \\ \Rightarrow a &= c \cos B \pm \sqrt{(c \cos B)^2 - (c^2 - b^2)} \\ \Rightarrow a &= c \cos B \pm \sqrt{b^2 - (c \sin B)^2}\end{aligned}$$

Now there are following possibilities:

- (i) If $b < c \sin B$, then no triangle is possible.
- (ii) If $b = c \sin B$, then we get $a = c \cos B$. For B being obtuse, $\cos B$ will be negative, so B must be acute. In that case, there exists only one triangle.
- (iii) If $b > c \sin B$, then there are further following cases:
 - (a) B is an acute angle. So, $\cos B$ is positive.
In this case two values of a will exist if and only if $c \cos B > \sqrt{b^2 - (c \sin B)^2}$ or $c > b$. Two such triangles are possible.
If $c < b$, only one triangle is possible.
 - (b) B is an obtuse angle. So, $\cos B$ is negative.
In this case, triangle will exist if and only if $\sqrt{b^2 - (c \sin B)^2} > |c \cos B|$.
This gives $b > c$. In this case, only one triangle is possible.
If $b < c$, then there exists no triangle.

THREE ANGLES (AAA)

If the three angles A , B , and C are given, we can only find the ratios of the sides a , b and c by using sine rule (since there are infinite similar triangles possible).

ILLUSTRATION 5.46

If $b = 3$, $c = 4$, and $B = \pi/3$, then find the number of triangles that can be constructed.

Sol. We have,

$$\frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{or} \quad \frac{\sin(\pi/3)}{3} = \frac{\sin C}{4}$$

$$\text{or} \quad \sin C = \frac{2}{\sqrt{3}} > 1, \text{ which is not possible.}$$

Hence, no triangle is possible.

ILLUSTRATION 5.47

If $A = 30^\circ$, $a = 7$, and $b = 8$ in $\triangle ABC$, then find the number of triangles that can be constructed.

Sol. We have $\frac{a}{\sin A} = \frac{b}{\sin B}$

$$\text{or} \quad \sin B = \frac{b \sin A}{a} = \frac{8 \sin 30^\circ}{7} = \frac{4}{7}$$

Thus, we have, $b > a > b \sin A$.

Hence, angle B has two values given by $\sin B = 4/7$.

ILLUSTRATION 5.48

If in triangle ABC , $a = (1 + \sqrt{3})$ cm, $b = 2$ cm, and $\angle C = 60^\circ$, then find the other two angles and the third side.

Sol. From $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$, we have

$$\frac{1}{2} = \frac{(1 + \sqrt{3})^2 + 4 - c^2}{2(1 + \sqrt{3})2}$$

$$\text{or} \quad 2 + 2\sqrt{3} = 1 + 3 + 2\sqrt{3} + 4 - c^2$$

$$\text{or} \quad c^2 = 6 \text{ or } c = \sqrt{6} \text{ cm}$$

$$\text{Also, } \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\text{or} \quad \frac{\sin A}{1 + \sqrt{3}} = \frac{\sin B}{2} = \frac{\sqrt{3}/2}{\sqrt{6}}$$

$$\Rightarrow \sin B = \frac{1}{\sqrt{2}} \quad \text{or } B = 45^\circ$$

$$\text{and } A = 180^\circ - (60^\circ + 45^\circ) = 75^\circ$$

ILLUSTRATION 5.49

In $\triangle ABC$, sides b , c , and angle B are given such that a has two values a_1 and a_2 . Then prove that $|a_1 - a_2| = 2\sqrt{b^2 - c^2 \sin^2 B}$.

Sol. $\cos B = \frac{c^2 + a^2 - b^2}{2ca}$

$$\text{or} \quad a^2 - (2c \cos B) a + c^2 - b^2 = 0$$

This equation has roots a_1 and a_2 .

$$\Rightarrow a_1 + a_2 = 2c \cos B, \quad a_1 a_2 = c^2 - b^2$$

$$\begin{aligned}\Rightarrow (a_1 - a_2)^2 &= (a_1 + a_2)^2 - 4a_1 a_2 = 4c^2 \cos^2 B - 4(c^2 - b^2) \\ &= 4b^2 - 4c^2 \sin^2 B = 4(b^2 - c^2 \sin^2 B)\end{aligned}$$

$$\text{or} \quad |a_1 - a_2| = 2\sqrt{b^2 - c^2 \sin^2 B}$$

ILLUSTRATION 5.50

In $\triangle ABC$, a , c , and A are given and b_1, b_2 are two values of the third side b such that $b_2 = 2b_1$. Then prove that

$$\sin A = \sqrt{\frac{9a^2 - c^2}{8c^2}}$$

Sol. We have $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

$$\text{or} \quad b^2 - 2bc \cos A + (c^2 - a^2) = 0$$

It is given that b_1 and b_2 are the roots of this equation. Therefore

$$\begin{aligned}b_1 + b_2 &= 2c \cos A \text{ and } b_1 b_2 = c^2 - a^2 \\ \Rightarrow 3b_1 &= 2c \cos A, \quad 2b_1^2 = c^2 - a^2 \quad (\because b_2 = 2b_1 \text{ given})\end{aligned}$$

$$\text{or} \quad 2\left(\frac{2c}{3} \cos A\right)^2 = c^2 - a^2$$

$$\text{or} \quad 8c^2(1 - \sin^2 A) = 9c^2 - 9a^2$$

$$\text{or} \quad \sin A = \sqrt{\frac{9a^2 - c^2}{8c^2}}$$

CONCEPT APPLICATION EXERCISE 5.6

- In which of the following cases, there exists a triangle ABC ?
 (a) $b \sin A = a, A < \pi/2$
 (b) $b \sin A > a, A > \pi/2$
 (c) $b \sin A > a, A < \pi/2$
 (d) $b \sin A < a, A < \pi/2, b > a$
 (e) $b \sin A < a, A > \pi/2, b = a$
- If in $\triangle ABC$, $b = 3$ cm, $c = 4$ cm and the length of the perpendicular from A to the side BC is 2 cm, then how many such triangles are possible?
- In a triangle ABC , $\frac{a}{b} = \frac{2}{3}$ and $\sec^2 A = \frac{8}{5}$. Find the number of triangles satisfying these conditions.
- In a triangle, the lengths of the two longer sides are 10 and 9, respectively. If the angles are in A.P., then find the possible length of the third side.
- If a, b and A are given in a triangle and c_1, c_2 are possible values of the third side, then prove that $c_1^2 + c_2^2 - 2c_1c_2 \cos 2A = 4a^2 \cos^2 A$.
- In $\triangle ABC$, a, b and A are given and c_1, c_2 are two values of the third side c . Prove that the sum of the areas of two triangles with sides a, b, c_1 and a, b, c_2 is $\frac{1}{2} b^2 \sin 2A$.

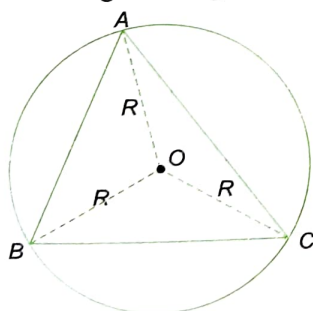
ANSWERS

1. (a), (d) 2. Two 3. Two 4. $5 + \sqrt{6}$ or $5 - \sqrt{6}$

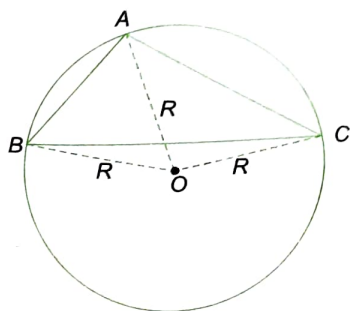
CIRCUMCIRCLE AND CIRCUMCENTRE

The circle which passes through the vertices of a triangle is called its circumcircle. The centre 'O' of the circle is called the circumcentre. Radius of the circumcircle of triangle is called its circumradius and usually denoted by R .

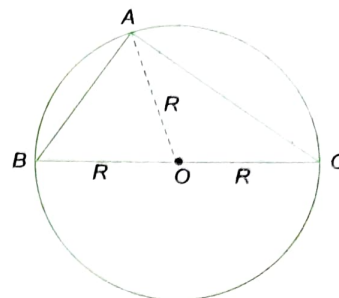
Circumcentre of acute angled triangle lies inside the triangle.



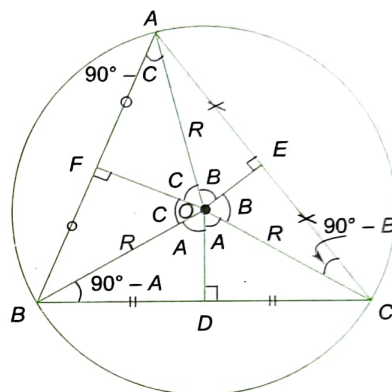
Circumcentre of obtuse angled triangle lies outside the triangle.



Circumcentre of right angled triangle is mid-point of its hypotenuse.



PROPERTIES OF CIRCUMCIRCLE AND CIRCUMCENTRE



- We know that the perpendicular from the centre of the circle to any chord bisects the chord. Thus, circumcentre of triangle is the point of concurrency of perpendicular bisectors of sides of triangle.
- Distance of circumcentre from all the vertices of triangle is same which is equal to circumradius of triangle.
- We know that the angle subtended by a chord of circle at the centre is twice the angle subtended by the same chord at any point on the circumference.
 In the figure, $\angle BOC = 2A$. Also, $OB = OC = R$ and $BD = DC = a/2$. So, triangles ODB and ODC are congruent. Therefore $\angle BOD = \angle COD = A$.
- Distance of circumcentre from sides of triangle.**
 In the figure in $\triangle ODB$, $OD = R \cos A$. Similarly, we have $OE = R \cos B$ and $OF = R \cos C$.

ILLUSTRATION 5.51

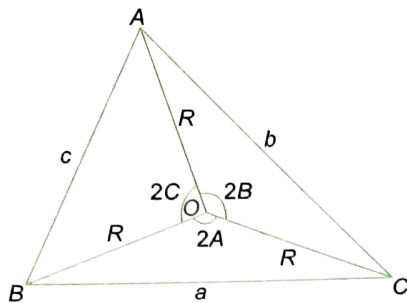
O is the circumcenter of $\triangle ABC$ and R_1, R_2, R_3 are, respectively, the radii of the circumcircles of the triangles OBC, OCA , and OAB . Prove that $\frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3} = \frac{abc}{R^3}$.

Sol. If O is the circumcenter of $\triangle ABC$, then

$$OA = OB = OC = R$$

Given that R_1, R_2 , and R_3 be the circumradii of $\triangle OBC, \triangle OCA$, and $\triangle OAB$, respectively.

In $\triangle OBC$, using sine rule, $2R_1 = \frac{a}{\sin 2A}$ or $\frac{a}{R_1} = 2 \sin 2A$



Similarly, $\frac{b}{R_2} = 2 \sin 2B$ and $\frac{c}{R_3} = 2 \sin 2C$

$$\begin{aligned} \Rightarrow \frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3} &= 2(\sin 2A + \sin 2B + \sin 2C) \\ &= 8 \sin A \sin B \sin C \\ &= 8 \frac{a}{2R} \frac{b}{2R} \frac{c}{2R} \\ &= \frac{abc}{R^3} \end{aligned}$$

ILLUSTRATION 5.52

In $\triangle ABC$, $C = 60^\circ$ and $B = 45^\circ$. Line joining vertex A of triangle and its circumcenter (O) meets the side BC in D .

- Find the ratio $BD : DC$
- Find the ratio $AO : OD$

Sol. In the figure, $\angle AOC = 2B$

$$\Rightarrow \angle DOC = \pi - 2B$$

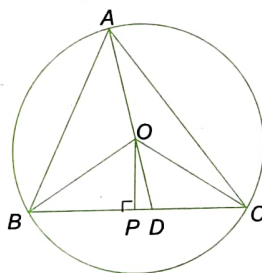
$$\text{Similarly, } \angle BOD = \pi - 2C$$

$$\begin{aligned} \text{Now, } \frac{BD}{CD} &= \frac{\frac{1}{2} BD \cdot OP}{\frac{1}{2} CD \cdot OP} \\ &= \frac{\text{Area of } \triangle BOD}{\text{Area of } \triangle COD} \\ &= \frac{\frac{1}{2} OB \cdot OD \sin \angle BOD}{\frac{1}{2} OC \cdot OD \sin \angle COD} \\ &= \frac{\sin(\pi - 2C)}{\sin(\pi - 2B)} = \frac{\sin 2C}{\sin 2B} \\ &= \frac{\sin 120^\circ}{\sin 90^\circ} \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

$$\text{Now in } \triangle OPB, OP = OB \cos A = R \cos A$$

$$\begin{aligned} \text{Also } \angle POD &= \angle POC - \angle DOC \\ &= A - (\pi - 2B) \\ &= \pi - B - C - \pi + 2B \\ &= B - C \end{aligned}$$

$$\text{Now in } \triangle OPD, OD = \frac{OP}{\cos(B-C)} = \frac{R \cos A}{\cos(B-C)}$$



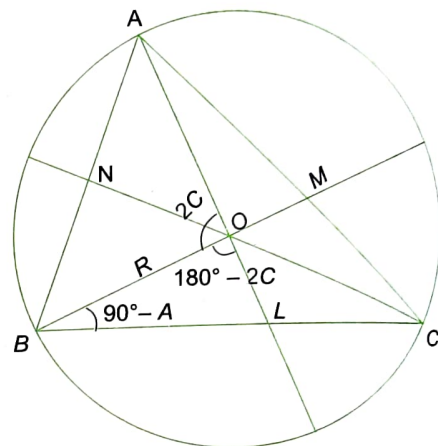
$$\begin{aligned} \Rightarrow \frac{AO}{OD} &= \frac{R}{\frac{R \cos A}{\cos(B-C)}} \\ &= \frac{\cos(B-C)}{\cos A} = \frac{\cos 15^\circ}{\cos 75^\circ} = \frac{\cos 15^\circ}{\sin 15^\circ} = 2 + \sqrt{3} \end{aligned}$$

ILLUSTRATION 5.53

The diameters of the circumcircle of triangle ABC drawn from A , B and C meet BC , CA and AB , respectively, in L , M and N .

Prove that $\frac{1}{AL} + \frac{1}{BM} + \frac{1}{CN} = \frac{2}{R}$.

Sol.



Using sine rule in $\triangle OBL$, we get

$$\begin{aligned} \frac{R}{\sin \left[\frac{\pi}{2} + C - B \right]} &= \frac{OL}{\sin \left(\frac{\pi}{2} - A \right)} \\ \therefore OL &= \frac{R \cos A}{\cos(C-B)} \\ \Rightarrow AL &= R + \frac{R \cos A}{\cos(C-B)} \\ &= R \frac{\cos(C-B) + \cos A}{\cos(C-B)} \\ &= R \frac{\cos(C-B) - \cos(B+C)}{\cos(C-B)} \\ &= \frac{2R \sin B \sin C}{\cos(C-B)} \\ \Rightarrow \frac{1}{AL} &= \frac{\cos(C-B)}{2R \sin B \sin C} = \frac{2 \cos(C-B) \sin A}{4R \sin B \sin C \sin A} \\ &= \frac{\sin 2B + \sin 2C}{4R \sin B \sin C \sin A} \\ \Rightarrow \frac{1}{AL} + \frac{1}{BM} + \frac{1}{CN} &= \frac{2(\sin 2A + \sin 2B + \sin 2C)}{4R \sin A \sin B \sin C} \\ &= \frac{2}{R} \end{aligned}$$

ILLUSTRATION 5.54

Find the lengths of chords of the circumcircle of triangle ABC , made by its altitudes

Sol. From the figure, in $\triangle ABL$,
 $\angle ABL = \angle ABD + \angle DBL$
 $= B + (90^\circ - C)$
 $= 90^\circ + (B - C)$

Let $AL = x$

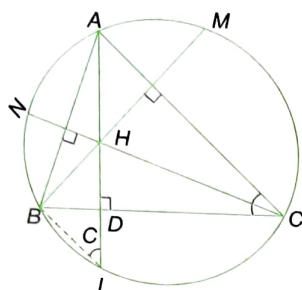
So, using sine rule in $\triangle ABL$, we get

$$\frac{x}{\sin(90^\circ + B - C)} = 2R$$

$$\therefore x = 2R \cos(B - C)$$

$$\text{Similarly, } y = 2R \cos(C - A)$$

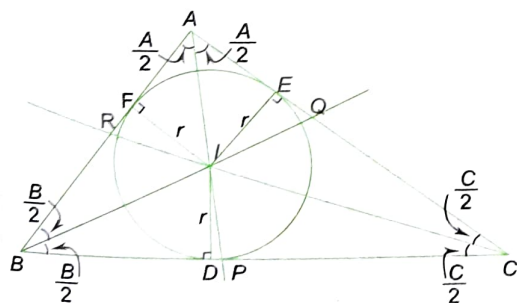
$$\text{and } z = 2R \cos(A - B)$$


CONCEPT APPLICATION EXERCISE 5.7

- Let f , g , and h be the lengths of the perpendiculars from the circumcenter of $\triangle ABC$ on the sides a , b , and c , respectively. Prove that $\frac{a}{f} + \frac{b}{g} + \frac{c}{h} = \frac{1}{4} \frac{abc}{fgh}$.
- If AD , BE , CF are the diameters of circumcircle of $\triangle ABC$, then prove that the area of hexagon $AFBDCE$ is 2Δ .
- If the sides of triangle are in the ratio $3 : 5 : 7$, then prove that the minimum distance of the circumcentre from the side of triangle is half the circumradius.
- If circumradius of triangle ABC is 4 cm, then prove that sum of perpendicular distances from circumcentre to the sides of triangle cannot exceed 6 cm.

INCIRCLE AND INCENTRE

The incircle or circle inscribed in a triangle is the largest circle contained in the triangle; it touches (or it is tangent to) the three sides of triangle. The centre ' I ' of the incircle is called the **incentre** of the triangle. The radius of incircle is called **inradius** and usually denoted by r .



In the figure, incircle of triangle touches the sides BC , CA and AB at D , E and F , respectively.

Clearly, $ID = IE = IF = r$ (inradius)

We know that the lengths of the two tangents drawn from an external point to a circle are equal. So, in the figure $AF = AE$.

Hence, AI is internal angle bisector of angle A . Similarly, BI and CI are internal angle bisectors of angles B and C , respectively.

Therefore, incentre of triangle is the point of concurrency of internal angle bisectors.

LENGTH OF TANGENT FROM VERTICES TO THE INCIRCLE

In the figure, $AF = AE = x$ (say), $BF = BD = y$ (say)

and $CE = CD = z$ (say).

Now, $AB + BC + CA = AF + FB + BD + DC + CE + EA$

$$\Rightarrow a + b + c = 2x + 2y + 2z$$

$$\Rightarrow 2s = 2x + 2y + 2z$$

$$\text{or } x + y + z = s$$

Now, $y + z = BC = a$. So, $x = s - a$.

Similarly, $y = s - b$ and $z = s - c$.

Thus, lengths of tangents to incircle from the vertices A , B and C are $(s - a)$, $(s - b)$ and $(s - c)$, respectively.

DISTANCE OF INCENTRE FROM THE VERTICES OF TRIANGLE

In the figure, in $\triangle AFI$,

$$\sin \frac{A}{2} = \frac{FI}{AI} = \frac{r}{AI}$$

$$\therefore AI = \frac{r}{\sin \frac{A}{2}}$$

$$\text{Similarly, } BI = \frac{r}{\sin \frac{B}{2}} \text{ and } CI = \frac{r}{\sin \frac{C}{2}}$$

DIFFERENT FORMULAS FOR INRADIUS

We have following different formulas for inradius:

$$\bullet r = (s - a) \tan \frac{A}{2} = (s - b) \tan \frac{B}{2} = (s - c) \tan \frac{C}{2}$$

$$\text{In the figure, in } \triangle AFI, \tan \frac{A}{2} = \frac{FI}{AF} = \frac{r}{s - a}$$

$$\therefore r = (s - a) \tan \frac{A}{2}$$

$$\text{Similarly, } r = (s - b) \tan \frac{B}{2} = (s - c) \tan \frac{C}{2}$$

$$\bullet r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

$$4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

$$= 4R \sqrt{\frac{(s - b)(s - c)}{bc}} \sqrt{\frac{(s - c)(s - a)}{ca}} \sqrt{\frac{(s - a)(s - b)}{ab}}$$

$$= 4R \frac{(s - a)(s - b)(s - c)}{abc}$$

$$= \frac{4R}{abc} \cdot \frac{s(s - a)(s - b)(s - c)}{s}$$

$$\begin{aligned}
 &= \frac{1}{\Delta} \cdot \frac{\Delta^2}{s} \\
 &= \frac{\Delta}{s} \\
 &= r
 \end{aligned}
 \quad \left(\because \Delta = \frac{abc}{4R} \right)$$

$$(\because \Delta = rs)$$

Note:

We know that in triangle ABC ,

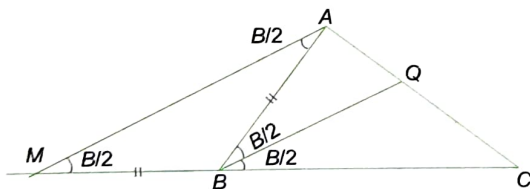
$$\begin{aligned}
 \cos A + \cos B + \cos C &\leq \frac{3}{2} \\
 \therefore 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} &\leq \frac{3}{2} \\
 \Rightarrow \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} &\leq \frac{1}{8} \\
 \therefore \text{Using } r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}, &\text{ we get } 2r \leq R \\
 R = 2r &\text{ holds when triangle is equilateral.}
 \end{aligned}$$

LENGTH OF ANGLE BISECTOR

Let us find the length of the angle bisector AP .

$$\begin{aligned}
 &\text{Area of } \triangle ABP + \text{Area of } \triangle ACP = \text{Area of } \triangle ABC \\
 \text{or } (1/2) AB \times AP \sin (A/2) + (1/2) AC \times AP \sin (A/2) &= (1/2) AB \times AC \sin A \\
 \text{or } (1/2) (c + b) AP \sin (A/2) &= (1/2) [cb (2 \sin (A/2) \cos (A/2))] \\
 \text{or } AP = \left(\frac{2bc}{b+c} \right) \cos (A/2)
 \end{aligned}$$

$$\text{Similarly, } BQ = \left(\frac{2ac}{a+c} \right) \cos \frac{B}{2} \text{ and } CR = \left(\frac{2ab}{a+b} \right) \cos \frac{C}{2}$$

DIVISION OF OPPOSITE SIDE BY THE INTERNAL ANGLE BISECTOR

In $\triangle ABC$, internal angle bisector of angle B meets opposite side AC at Q .

$$\text{Let us prove that } \frac{AQ}{CQ} = \frac{AB}{BC}$$

Extend BC to the left of B .

Draw a line parallel to BQ from vertex A , which meets BC produced at M .

Now, $\triangle BCQ$ and $\triangle MCA$ are similar.

$$\begin{aligned}
 \therefore \frac{AC}{QC} &= \frac{MC}{BC} \\
 \Rightarrow \frac{AQ + QC}{QC} &= \frac{MB + BC}{BC}
 \end{aligned}$$

$$\Rightarrow \frac{AQ}{QC} = \frac{MB}{BC}$$

Now, we have to prove that $MB = AB$.

Since BQ and AM are parallel.

Therefore, $\angle MAB = \angle ABQ = \frac{B}{2}$ and $\angle AMB = \angle QBC = \frac{B}{2}$.
So, in $\triangle ABM$, $MB = AB$.

$$\text{So, } \frac{AQ}{QC} = \frac{AB}{BC} = \frac{c}{a}$$

Therefore, angle bisector of a triangle will divide the opposite side into two segments that are proportional to the other two sides of the triangle.

Similarly, if angle bisector through vertex A meets opposite side at P then $\frac{BP}{CP} = \frac{AB}{AC} = \frac{c}{b}$ and if angle bisector through vertex

C meets opposite side at R then $\frac{AR}{BR} = \frac{AC}{BC} = \frac{b}{a}$

Now, from $\frac{BP}{CP} = \frac{AB}{AC} = \frac{c}{b}$, we have $BP = ck$, $CP = bk$, where

k is constant.

$$\text{But } BP + CP = a$$

$$\Rightarrow ck + bk = a$$

$$\Rightarrow k = \frac{a}{b+c}$$

$$\therefore BP = \frac{ac}{b+c} \text{ and } CP = \frac{ab}{b+c}$$

$$\text{Similarly, } AQ = \frac{cb}{a+c}, CQ = \frac{ab}{a+c}$$

$$\text{and } AR = \frac{bc}{a+b}, BR = \frac{ac}{a+b}$$

ILLUSTRATION 5.55

Let ABC be a triangle with $\angle B = 90^\circ$. Let AD be the bisector of $\angle A$ with D on BC . Suppose $AC = 6$ cm and the area of the triangle ADC is 10 cm^2 . Find the length of BD .

Sol. From angle bisector theorem,

$$\frac{r}{6} = \frac{p}{q}$$

$$\Rightarrow qr = 6p \quad \dots (1)$$

Now, area of $\triangle ADC = 10 \text{ cm}^2$

$$\frac{1}{2} (DC) (AB) = 10$$

$$\frac{1}{2} (q) (r) = 10$$

$$qr = 20$$

$$\text{From (1), } 20 = 6p$$

$$\Rightarrow p = \frac{20}{6} = \frac{10}{3}$$

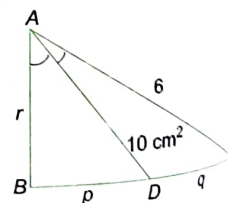


ILLUSTRATION 5.56

If the distances of the vertices of a triangle ABC from the points of contact of the incircle with sides are α , β , and γ , then prove that $r^2 = \frac{\alpha\beta\gamma}{\alpha + \beta + \gamma}$.

Sol. We know that distances of the vertices of a triangle from the points of contact of the incircle are $s - a$, $s - b$, $s - c$. Thus,

$$\alpha = s - a, \beta = s - b, \text{ and } \gamma = s - c$$

$$\alpha\beta\gamma = (s - a)(s - b)(s - c)$$

$$\alpha + \beta + \gamma = 3s - a - b - c = s$$

$$\Rightarrow \frac{\alpha\beta\gamma}{\alpha + \beta + \gamma} = \frac{(s - a)(s - b)(s - c)}{s}$$

$$= \frac{s(s - a)(s - b)(s - c)}{s^2}$$

$$= \frac{\Delta^2}{s^2}$$

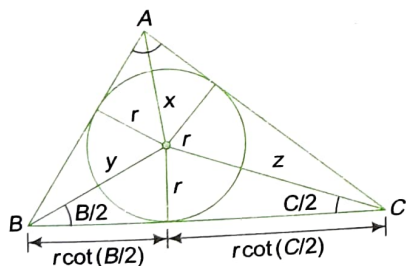
$$= r^2$$

ILLUSTRATION 5.57

If x , y , and z are the distances of incenter from the vertices of the triangle ABC , respectively, then prove that

$$\frac{abc}{xyz} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}.$$

Sol.



$$x = r \operatorname{cosec} \frac{A}{2} \text{ and } a = r \left(\cot \frac{B}{2} + \cot \frac{C}{2} \right)$$

$$\Rightarrow \frac{a}{x} = \left(\cot \frac{B}{2} + \cot \frac{C}{2} \right) \sin \frac{A}{2} = \frac{\sin \frac{A}{2} \cos \frac{A}{2}}{\sin \frac{B}{2} \sin \frac{C}{2}}$$

$$\text{Similarly, } \frac{b}{y} = \frac{\sin \frac{B}{2} \cos \frac{B}{2}}{\sin \frac{A}{2} \sin \frac{C}{2}} \text{ and } \frac{c}{z} = \frac{\sin \frac{C}{2} \cos \frac{C}{2}}{\sin \frac{A}{2} \sin \frac{B}{2}}$$

$$\therefore \frac{abc}{xyz} = \frac{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}{\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}$$

$$= \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$$

ILLUSTRATION 5.58

Prove that $\cos A + \cos B + \cos C = 1 + r/R$.

$$\text{Sol. } \cos A + \cos B + \cos C = 1 + 4 \sin \left(\frac{A}{2} \right) \sin \left(\frac{B}{2} \right) \sin \left(\frac{C}{2} \right)$$

$$= 1 + \frac{[4R \sin(A/2) \sin(B/2) \sin(C/2)]}{R} = 1 + \frac{r}{R}$$

ILLUSTRATION 5.59

Prove that $\frac{a \cos A + b \cos B + c \cos C}{a + b + c} = \frac{r}{R}$.

Sol. We have,

$$a \cos A + b \cos B + c \cos C$$

$$= R(2 \sin A \cos A + 2 \sin B \cos B + 2 \sin C \cos C)$$

$$= R(\sin 2A + \sin 2B + \sin 2C)$$

$$= 4R \sin A \sin B \sin C$$

$$\text{and } a + b + c = 2R(\sin A + \sin B + \sin C)$$

$$= 8R \cos(A/2) \cos(B/2) \cos(C/2)$$

$$\Rightarrow \frac{a \cos A + b \cos B + c \cos C}{a + b + c} = \frac{4R \sin A \sin B \sin C}{8R \cos A/2 \cos B/2 \cos C/2}$$

$$= \frac{\left(2 \sin \frac{A}{2} \cos \frac{A}{2} \right) \left(2 \sin \frac{B}{2} \cos \frac{B}{2} \right) \left(2 \sin \frac{C}{2} \cos \frac{C}{2} \right)}{2 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}$$

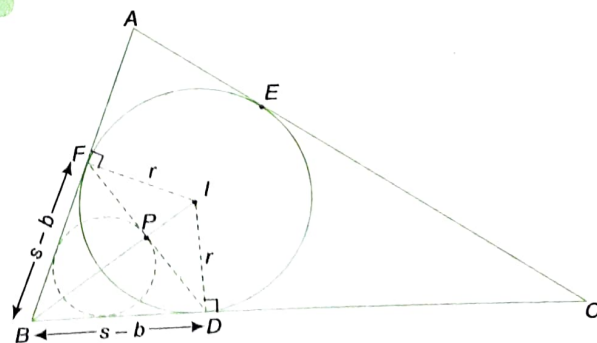
$$= 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = \frac{r}{R}$$

ILLUSTRATION 5.60

Incircle of ΔABC touches the sides BC , CA and AB at D , E and F , respectively. Let r_1 be the radius of incircle of ΔBDF . Then

$$\text{prove that } r_1 = \frac{1}{2} \frac{(s - b) \sin B}{1 + \sin \frac{B}{2}}.$$

Sol.



$$\angle DIF = \pi - B$$

$$\text{Now, } FD = 2FP = 2r \sin \left(\frac{\pi - B}{2} \right) = 2r \cos \frac{B}{2}$$

$$\text{Also, } BD = BF = s - b$$

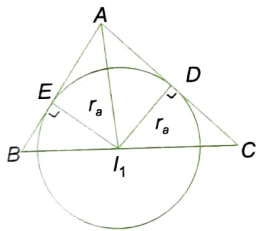
Now, in-radius of $\triangle BDF$ is r_1 .

$$\begin{aligned}\therefore r_1 &= \frac{\text{Area of } \triangle BDF}{\text{Semiperimeter of } \triangle BDF} \\ &= \frac{\frac{1}{2}(s-b)^2 \sin B}{\frac{1}{2}\left(2(s-b) + 2r \cos \frac{B}{2}\right)} \\ &= \frac{(s-b) \sin B}{2\left(1 + \tan \frac{B}{2} \cos \frac{B}{2}\right)} \quad \left(\because r = (s-b) \tan \frac{B}{2}\right) \\ &= \frac{(s-b) \sin B}{2\left(1 + \sin \frac{B}{2}\right)} \quad \left(\because r = (s-b) \tan \frac{B}{2}\right)\end{aligned}$$

ILLUSTRATION 5.61

In an acute angled triangle ABC , a semicircle with radius r_a is constructed with its base on BC and tangent to the other two sides. r_b and r_c are defined similarly. If r is the radius of the incircle of triangle ABC then prove that $\frac{2}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}$.

Sol. As shown in the figure, semicircle with its base on BC touches AB and AC at E and D respectively.



Now, area of $\triangle ABC$ = Area of $\triangle ABI_1$ + Area of $\triangle ACI_1$

$$\therefore \Delta = \frac{1}{2}cr_a + \frac{1}{2}br_a$$

$$\therefore \frac{1}{r_a} = \frac{c+b}{2\Delta} \quad \dots(1)$$

$$\text{Similarly, } \frac{1}{r_b} = \frac{c+a}{2\Delta} \quad \dots(2)$$

$$\text{and } \frac{1}{r_c} = \frac{a+b}{2\Delta} \quad \dots(3)$$

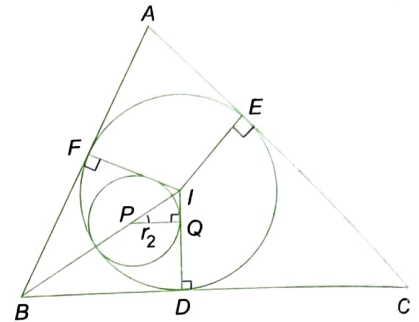
On adding Eqs. (1), (2) and (3), we get

$$\begin{aligned}\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} &= \frac{1}{2\Delta} (b+c + c+a + a+b) \\ &= \frac{a+b+c}{\Delta} \\ &= \frac{2s}{rs} \\ &= \frac{2}{r}\end{aligned}$$

ILLUSTRATION 5.62

Let the incircle with center I of $\triangle ABC$ touch sides BC , CA and AB at D , E and F , respectively. Let a circle is drawn touching ID , IF and incircle of $\triangle ABC$ having radius r_2 . Similarly r_1 and r_3 are defined. Prove that $\frac{r_1}{r-r_1} \cdot \frac{r_2}{r-r_2} \cdot \frac{r_3}{r-r_3} = \frac{a+b+c}{8R}$.

Sol.



From the figure, in $\triangle IQP$

$$\cos \frac{B}{2} = \frac{PQ}{IP}$$

Since $PQ = r_2$

and $IP = r - r_2$ (as two circles touching internally), we have

$$\cos \frac{B}{2} = \frac{r_2}{r - r_2}$$

Similarly, for other such circles, $\cos \frac{A}{2} = \frac{r_1}{r - r_1}$

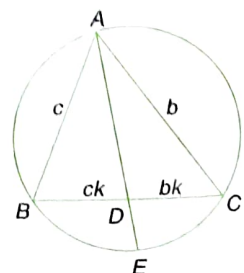
$$\text{and } \cos \frac{C}{2} = \frac{r_3}{r - r_3}$$

$$\begin{aligned}\frac{r_1}{r - r_1} \cdot \frac{r_2}{r - r_2} \cdot \frac{r_3}{r - r_3} &= \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \\ &= \frac{1}{4} (\sin A + \sin B + \sin C) \\ &= \frac{a+b+c}{8R}\end{aligned}$$

ILLUSTRATION 5.63

In $\triangle ABC$, the bisector of the angle A meets the side BC at D and the circumscribed circle at E . Prove that $DE = \frac{a^2 \sec \frac{A}{2}}{2(b+c)}$.

Sol. AD = length of angle bisector = $\frac{2bc \cos \frac{A}{2}}{b+c}$



Also, we have $ck + bk = a$

$$\Rightarrow k = \frac{a}{b+c}$$

From secant property of circle, we have

$$AD \times DE = BD \times DC$$

$$\Rightarrow AD \times DE = bc k^2$$

$$\Rightarrow \frac{2bc \cos \frac{A}{2}}{b+c} \times DE = \frac{bc a^2}{(b+c)^2}$$

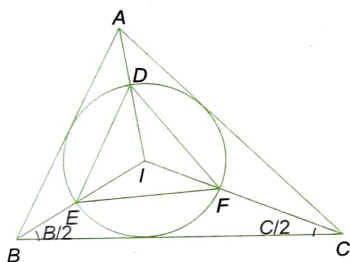
$$\text{Hence, } DE = \frac{a^2 \sec \frac{A}{2}}{2(b+c)}$$

ILLUSTRATION 5.64

Let I be the incentre of $\triangle ABC$ having inradius r . AI , BI and CI intersect incircle at D , E and F respectively.

Prove that area of $\triangle DEF$ is $\frac{r^2}{2} \left(\cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2} \right)$.

$$\text{Sol. } \angle BIC = \pi - \frac{B+C}{2} = \pi - \frac{\pi-A}{2} = \frac{\pi}{2} + \frac{A}{2}$$



$$\begin{aligned} \text{Now, area of } \triangle EIF &= \frac{1}{2} EI \times FI \times \sin \left(\frac{\pi}{2} + \frac{A}{2} \right) \\ &= \frac{1}{2} r^2 \cos \frac{A}{2} \end{aligned}$$

$$\begin{aligned} \therefore \text{Area of } \triangle DEF &= \triangle EIF + \triangle DIF + \triangle DIE \\ &= \frac{r^2}{2} \left(\cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2} \right) \end{aligned}$$

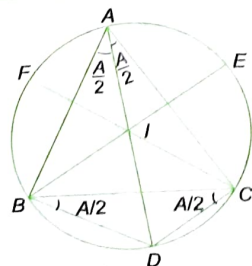
ILLUSTRATION 5.65

In $\triangle ABC$, the bisectors of the angles A , B and C are extended to intersect the circumcircle at D , E and F respectively. Prove that

$$AD \cos \frac{A}{2} + BE \cos \frac{B}{2} + CF \cos \frac{C}{2} = 2R (\sin A + \sin B + \sin C)$$

Sol. Using Sine law in $\triangle ABD$, we get

$$\frac{AD}{\sin \left(B + \frac{A}{2} \right)} = 2R$$



$$\therefore AD = 2R \sin \left(B + \frac{A}{2} \right)$$

$$\begin{aligned} \Rightarrow AD \cos \frac{A}{2} &= 2R \cos \frac{A}{2} \sin \left(B + \frac{A}{2} \right) \\ &= R (\sin (A+B) + \sin B) \\ &= R (\sin C + \sin B) \end{aligned}$$

$$\text{Similarly, } BE \cos \frac{B}{2} = R (\sin A + \sin C)$$

$$\text{And } CF \cos \frac{C}{2} = R (\sin A + \sin B)$$

$$\begin{aligned} \therefore AD \cos \frac{A}{2} + BE \cos \frac{B}{2} + CF \cos \frac{C}{2} &= 2R (\sin A + \sin B + \sin C) \end{aligned}$$

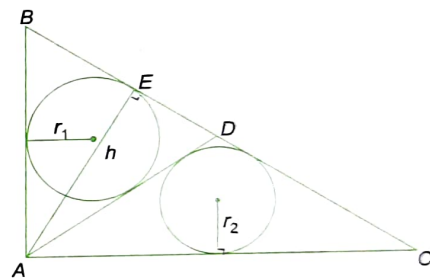
ILLUSTRATION 5.66

Given a right triangle ABC with $\angle A = 90^\circ$. Let D be the mid-point of BC . If the inradii of the triangles ABD and ACD are r_1 and r_2 then find the range of r_1/r_2 .

Sol. In the figure, D is mid-point of BC .

From the geometry, $AD = BD = CD = a/2$

$$\begin{aligned} \text{In } \triangle ABD, r_1 &= \frac{\Delta_1}{s_1} \\ \Rightarrow r_1 &= \frac{\frac{1}{2} \left(\frac{a}{2} \right) h}{\frac{\frac{a}{2} + \frac{a}{2} + c}{2}} \\ \Rightarrow r_1 &= \frac{ah}{2(a+c)} \end{aligned}$$



$$\begin{aligned} \text{In } \triangle ACD, r_2 &= \frac{\Delta_2}{s_2} \\ \Rightarrow r_2 &= \frac{\frac{1}{2} \left(\frac{a}{2} \right) h}{\frac{\frac{a}{2} + \frac{a}{2} + b}{2}} \end{aligned}$$

$$\Rightarrow r_2 = \frac{ah}{2(a+b)}$$

$$\begin{aligned} \Rightarrow \frac{r_1}{r_2} &= \frac{a+b}{a+c} \\ &= \frac{2R \sin A + 2R \sin B}{2R \sin A + 2R \sin C} \end{aligned}$$

(as $A = 90^\circ$)

$$\begin{aligned} &= \frac{1 + \sin B}{1 + \sin C} \\ &= \frac{1 + \sin B}{1 + \cos B} \end{aligned}$$

(as $C = 90^\circ - B$)

When B approaches to 90° , $\frac{r_1}{r_2}$ approaches to 2.

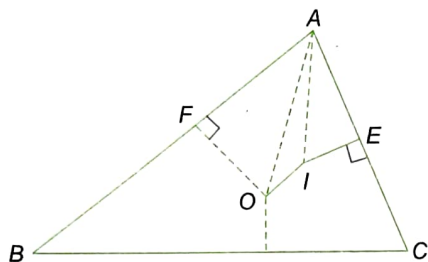
When B approaches to 0° , $\frac{r_1}{r_2}$ approaches to $\frac{1}{2}$.

$$\therefore \frac{r_1}{r_2} \in \left(\frac{1}{2}, 2\right)$$

ILLUSTRATION 5.67

Prove that the distance between the circumcenter and the incenter of triangle ABC is $\sqrt{R^2 - 2Rr}$.

Sol. Let O be the circumcenter and OF be the perpendicular to AB .



Let I be the incenter and IE be the perpendicular to AC . Then,

$$\angle OAF = 90^\circ - C.$$

$$\Rightarrow \angle OAI = \angle IAF - \angle OAF$$

$$= \frac{A}{2} - (90^\circ - C)$$

$$= \frac{A}{2} + C - \frac{A+B+C}{2} = \frac{C-B}{2}$$

Also,

$$AI = \frac{IE}{\sin \frac{A}{2}} = \frac{r}{\sin \frac{A}{2}} = 4R \sin \frac{B}{2} \sin \frac{C}{2}$$

Hence, in $\triangle OAI$, $OI^2 = OA^2 + AI^2 - 2OA \cdot AI \cos \angle OAI$

$$= R^2 + 16R^2 \sin^2 \frac{B}{2} \sin^2 \frac{C}{2} - 8R^2 \sin \frac{B}{2} \sin \frac{C}{2} \cos \frac{C-B}{2}$$

$$\Rightarrow \frac{OI^2}{R^2} = 1 + 16 \sin^2 \frac{B}{2} \sin^2 \frac{C}{2} - 8 \sin \frac{B}{2} \sin \frac{C}{2} \left(\cos \frac{B}{2} \cos \frac{C}{2} + \sin \frac{B}{2} \sin \frac{C}{2} \right)$$

$$= 1 - 8 \sin \frac{B}{2} \sin \frac{C}{2} \left(\cos \frac{B}{2} \cos \frac{C}{2} - \sin \frac{B}{2} \sin \frac{C}{2} \right)$$

$$= 1 - 8 \sin \frac{B}{2} \sin \frac{C}{2} \cos \frac{B+C}{2}$$

$$= 1 - 8 \sin \frac{B}{2} \sin \frac{C}{2} \sin \frac{A}{2}$$

$$\therefore OI = R \sqrt{1 - 8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} = \sqrt{R^2 - 2Rr}$$

ILLUSTRATION 5.68

Prove that $a \cos A + b \cos B + c \cos C \leq s$.

$$\begin{aligned} \text{Sol. } a \cos A + b \cos B + c \cos C &= R(2 \sin A \cos A + 2 \sin B \cos B + 2 \sin C \cos C) \\ &= R(\sin 2A + \sin 2B + \sin 2C) \\ &= 4R \sin A \sin B \sin C \\ &= \frac{2}{R} (2R^2 \sin A \sin B \sin C) \\ &= \frac{2}{R} \Delta \\ &= 2 \frac{rs}{R} \leq s \end{aligned}$$

[$\because R \geq 2r$]

ILLUSTRATION 5.69

If Δ is the area of a triangle with side lengths a , b , and c , then show that $\Delta \leq \frac{1}{4} \sqrt{(a+b+c)abc}$. Also show that the equality occurs in the above inequality if and only if $a=b=c$.

Sol. We have to prove that

$$\Delta \leq \frac{1}{4} \sqrt{(a+b+c)abc}$$

$$\text{or } \Delta \leq \frac{1}{4} \sqrt{2sabc}$$

$$\text{or } \Delta^2 \leq \frac{1}{16} 2sabc$$

$$\text{or } \Delta^2 \leq \frac{1}{16} 2s \Delta 4R$$

$$\text{or } rs \leq \frac{1}{2} sR$$

Hence, $R \geq 2r$

[which is always true in Δ]

Alternative Method:

In triangle, sum of two sides is greater than the third side.

So $a+b > c$, $b+c > a$ and $c+a > b$

Now consider quantities $a+b-c$, $b+c-a$, $c+a-b$.

Using A. M. $\geq$ G.M., we get

$$\frac{(a+b-c) + (b+c-a)}{2} \geq \sqrt{(a+b-c)(b+c-a)}$$

$$\text{or } b \geq \sqrt{(a+b-c)(b+c-a)}$$

$$\text{Similarly we get } c \geq \sqrt{(c+a-b)(b+c-a)}$$

$$\text{and } a \geq \sqrt{(a+b-c)(c+a-b)}$$

Multiplying we get

$$abc \geq (a+b-c)(b+c-a)(c+a-b)$$

$$\Rightarrow abc \geq (2s-2a)(2s-2b)(2s-2c)$$

$$\Rightarrow sabc \geq 8s(s-a)(s-b)(s-c)$$

$$\Rightarrow (a+b+c)abc \geq 16\Delta^2$$

$$\Rightarrow \Delta \leq \frac{1}{4} \sqrt{(a+b+c)abc}$$

CONCEPT APPLICATION EXERCISE 5.8

1. If the incircle of the triangle ABC passes through its circumcenter, then find the value of $4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$.
2. In $\triangle ABC$, $a = 10$, $A = \frac{2\pi}{3}$, and circle through B and C passes through the incenter. Find the radius of this circle.
3. Let ABC be a triangle with $\angle BAC = \frac{2\pi}{3}$ and $AB = x$ such that $(AB)(AC) = 1$. If x varies, then find the longest possible length of the angle bisector AD .
4. If the incircle of $\triangle ABC$ touches its sides, respectively, at L , M , and N and if x , y , z are the circumradii of the triangles MIN , NIL , and LIM , where I is the incenter, then prove that $xyz = \frac{1}{2} R r^2$.
5. In triangle ABC , CD is the bisector of the angle C . If $\cos \frac{C}{2} = \frac{1}{3}$ and $CD = 6$, then find the value of $\left(\frac{1}{a} + \frac{1}{b}\right)$.
6. In $\triangle ABC$, $\angle A = \frac{\pi}{3}$ and its inradius is 6 units. Find the radius of the circle touching the sides AB , AC internally and the incircle of $\triangle ABC$ externally.
7. In triangle ABC , prove that the maximum value of $\tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}$ is $\frac{R}{2s}$.

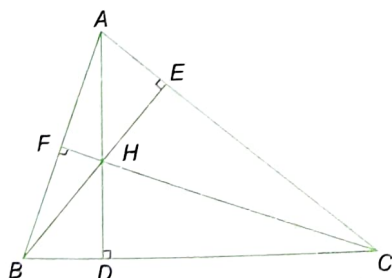
ANSWERS

1. $\sqrt{2} - 1$ 2. 10 3. $\frac{1}{2}$ unit 5. $\frac{1}{9}$ 6. 2 units

ORTHOCENTRE AND CENTROID

Orthocentre(H) of the triangle is the point where all the altitudes are concurrent.

Orthocentre of acute angled triangle lies inside the triangle.



In the figure, altitudes AD , BE and CF are concurrent at orthocentre H .

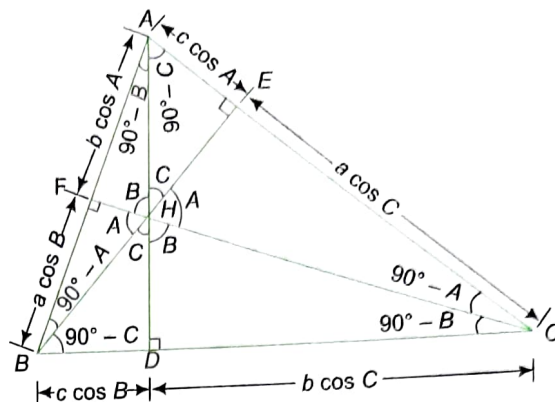
Also, in $\triangle HBC$, AD , AB and CA are altitudes which are concurrent at A .

So, A is orthocentre of $\triangle HBC$. Similarly, B and C are orthocentres of triangles HAC and HAB , respectively.

Since $\triangle HBC$ is obtuse at H , we can say that orthocentre of obtuse angled triangle lies outside the triangle.

Orthocentre of right angled triangle is at vertex where the angle is 90° .

DISTANCE OF ORTHOCENTRE FROM VERTICES AND SIDES



$$\text{In } \triangle AFH, \cos(90^\circ - B) = \frac{AF}{AH} = \frac{b \cos A}{AH}$$

$$\therefore \sin B = \frac{b \cos A}{AH}$$

$$\Rightarrow AH = \frac{b}{\sin B} \cos A$$

$$\therefore AH = 2R \cos A$$

Similarly, $BH = 2R \cos B$ and $CH = 2R \cos C$

$$\text{Also, in } \triangle AFH, \tan(90^\circ - B) = \frac{FH}{AF} = \frac{FH}{b \cos A}$$

$$\therefore FH = b \cos A \frac{\cos B}{\sin B}$$

$$\Rightarrow FH = \frac{b}{\sin B} \cos A \cos B$$

$$\Rightarrow FH = 2R \cos A \cos B$$

Similarly, $EH = 2R \cos A \cos C$ and $DH = 2R \cos B \cos C$

ILLUSTRATION 5.70

If in $\triangle ABC$, the distances of the vertices from the orthocenter

are x , y , and z , then prove that $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{xyz}$.

Sol. We know that distance of orthocenter (H) from vertex (A) is $2R \cos A$

$$\text{or } x = 2R \cos A$$

$$\text{Similarly, } y = 2R \cos B, z = 2R \cos C$$

$$\begin{aligned} \Rightarrow \frac{a}{x} + \frac{b}{y} + \frac{c}{z} &= \frac{2R \sin A}{2R \cos A} + \frac{2R \sin B}{2R \cos B} + \frac{2R \sin C}{2R \cos C} \\ &= \tan A + \tan B + \tan C \\ &= \tan A \tan B \tan C \end{aligned}$$

$$\begin{aligned} \text{Also, } \frac{abc}{xyz} &= \frac{(2R \sin A)(2R \sin B)(2R \sin C)}{(2R \cos A)(2R \cos B)(2R \cos C)} \\ &= \tan A \tan B \tan C \end{aligned}$$

$$\text{Hence, } \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{xyz} \quad (\because \tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C)$$

ILLUSTRATION 5.71

ABC is an acute angled triangle with circumcenter O and orthocenter H . If $AO = AH$, then find the angle A .

Sol. Given that $OA = HA$

$$\Rightarrow R = 2R \cos A$$

$$\Rightarrow \cos A = \frac{1}{2} \quad \text{or} \quad A = \frac{\pi}{3}$$

ILLUSTRATION 5.72

In an acute angled triangle ABC , point D , E , and F are the feet of the perpendiculars from A , B , and C onto BC , AC , and AB , respectively. H is orthocenter. If $\sin A = 3/5$ and $BC = 39$, then find the length of AH .

Sol. Given $\sin A = 3/5$

$$\Rightarrow \cos A = 4/5$$

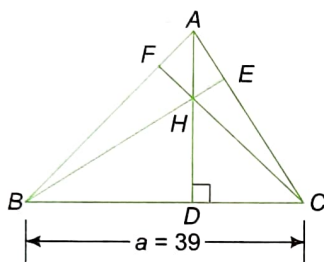
$$\text{Also, } a = 39$$

$$\therefore \frac{a}{\sin A} = 2R$$

$$\text{or } \frac{39 \times 5}{3} = 2R$$

$$\text{or } 2R = 65$$

$$\Rightarrow AH = 2R \cos A = 65 \times \frac{4}{5} = 52$$

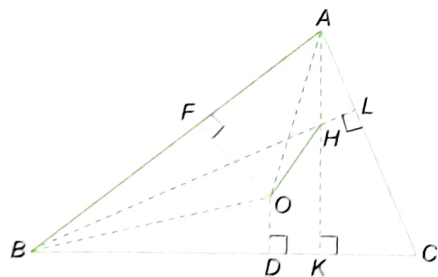
**ILLUSTRATION 5.73**

Prove that the distance between the circumcenter and the orthocenter of triangle ABC is $R\sqrt{1 - 8 \cos A \cos B \cos C}$.

Sol. Let O and H be the circumcenter and the orthocenter, respectively.

If OF is the perpendicular to AB , we have

$$\angle OAF = 90^\circ - \angle AOF = 90^\circ - C$$



$$\text{Also, } \angle HAL = 90^\circ - C$$

$$\text{Hence, } \angle OAH = A - \angle OAF - \angle HAL$$

$$= A - 2(90^\circ - C)$$

$$= A + 2C - 180^\circ$$

$$= A + 2C - (A + B + C) = C - B$$

$$\text{Also, } OA = R \text{ and } HA = 2R \cos A$$

Now in $\triangle AOH$,

$$\begin{aligned} OH^2 &= OA^2 + HA^2 - 2OA \cdot HA \cos(\angle OAH) \\ &= R^2 + 4R^2 \cos^2 A - 4R^2 \cos A \cos(C - B) \\ &= R^2 + 4R^2 \cos A [\cos A - \cos(C - B)] \end{aligned}$$

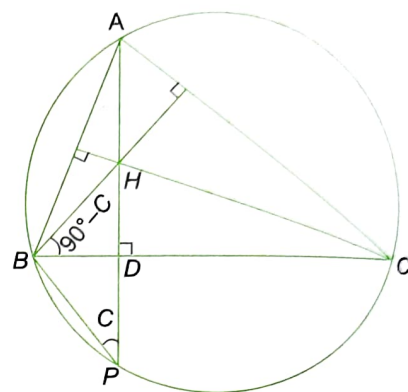
$$\begin{aligned} &= R^2 - 4R^2 \cos A [\cos(B + C) + \cos(C - B)] \\ &= R^2 - 8R^2 \cos A \cos B \cos C \end{aligned}$$

$$\text{Hence, } OH = R \sqrt{1 - 8 \cos A \cos B \cos C}$$

ILLUSTRATION 5.74

Let ABC be an acute angled triangle whose orthocentre is H . If altitude from A is produced to meet the circumcircle of triangle ABC at P , then prove that P is the image of H in BC and also prove that $HP = 4R \cos B \cos C$.

Sol.



In the figure, altitude AD meets BC at D and circumcircle at P . In circumcircle of triangle ABC , chord AB subtends same angle at points C and P .

$$\therefore \angle BPA = \angle BCA = C$$

$$\text{i.e. } \angle BPD = C$$

$$\text{Also } \angle HBD = 90^\circ - C \quad \therefore \angle BHD = C$$

Thus $\triangle BPD$ and $\triangle BHD$ are similar.

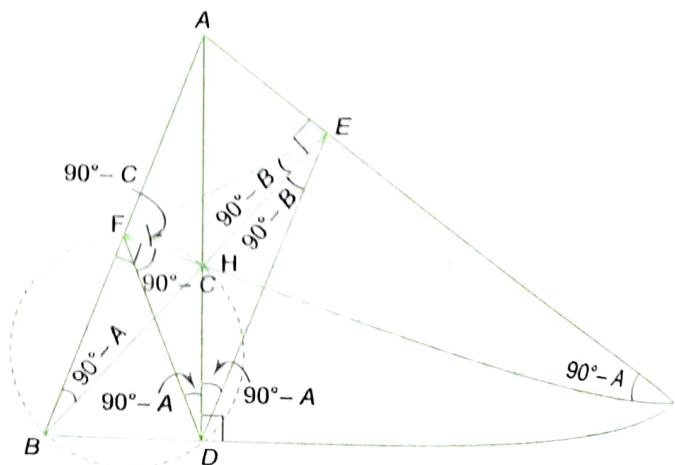
$$\therefore HD = DP$$

Therefore, P is image of H in BC .

$$\begin{aligned} \text{Also, } HP &= 2HD = 2(2R \cos B \cos C) \\ &= 4R \cos B \cos C \end{aligned}$$

PEDAL TRIANGLE AND ORTHIC TRIANGLE

Consider the triangle ABC , and any point P in the same plane. Let the feet of perpendicular from P on the sides BC , AC and AB are D , E and F , respectively. Then triangle DEF is called pedal triangle. If point P is orthocentre (H) then triangle DEF is called orthic triangle as shown in the figure.



In the figure, projections of orthocentre H on sides BC , CA and AB are D , E and F , respectively. So, triangle DEF is orthic triangle (pedal triangle w.r.t. orthocentre). For our convenience, we will call this a pedal triangle.

Clearly, points B , D , H , F are concyclic.

$$\angle FDH = \angle FBH = \angle ABE = 90^\circ - A$$

$\therefore$ points C , D , H , E are concyclic.

$$\angle EDH = \angle ECH = \angle ACF = 90^\circ - A$$

$\therefore$ Thus, altitude AD of triangle ABC is internal angle bisector of pedal triangle.

Similarly, we get other angles of pedal triangle as shown in the figure.

We observe that all the altitudes of triangle ABC are internal angle bisectors of pedal triangle.

Therefore, orthocentre of triangle ABC is incentre of its pedal triangle DEF .

Length of Sides of Pedal Triangle

In $\triangle AFE$, $AF = b \cos A$, $AE = c \cos A$

Using cosine rule in $\triangle AFE$,

$$EF^2 = AF^2 + AE^2 - 2AF \times AE \cos(\angle EAF)$$

$$\Rightarrow EF^2 = b^2 \cos^2 A + c^2 \cos^2 A - 2bc \cos^3 A$$

$$\Rightarrow EF^2 = \cos^2 A (b^2 + c^2 - 2bc \cos A)$$

$$= \cos^2 A (a^2)$$

$$\Rightarrow EF = a \cos A$$

Similarly, $FD = b \cos B$ and $DE = c \cos C$

Circumradius of Pedal Triangle

Let circumradius of pedal triangle be R' .

$$\therefore 2R' = \frac{EF}{\sin(\angle EDF)} = \frac{a \cos A}{\sin(180^\circ - 2A)}$$

$$= \frac{a \cos A}{2 \sin A \cos A}$$

$$= \frac{a}{2 \sin A} = R$$

$$\therefore R' = R/2$$

ILLUSTRATION 5.75

In $\triangle ABC$, let L , M , N be the feet of the altitudes. Then prove that $\sin(\angle MLN) + \sin(\angle LMN) + \sin(\angle MNL) = 4 \sin A \sin B \sin C$.

Sol. Using properties of pedal triangle, we have

$$\angle MLN = 180^\circ - 2A$$

$$\angle LMN = 180^\circ - 2B$$

$$\angle MNL = 180^\circ - 2C$$

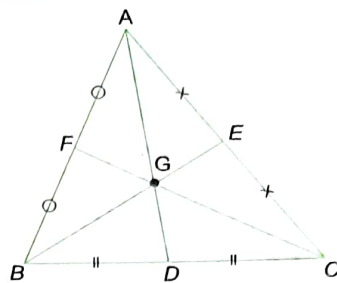
$$\Rightarrow \sin(\angle MLN) + \sin(\angle LMN) + \sin(\angle MNL)$$

$$= \sin 2A + \sin 2B + \sin 2C$$

$$= 4 \sin A \sin B \sin C$$

CENTROID OF TRIANGLE

Centroid of triangle, usually denoted by G , is the point of concurrency of medians of the triangle (median is the line joining any vertex and midpoint of opposite side).



Centroid G divides median AD in the ratio $AG : GD = 2 : 1$.

$$\therefore AG = \frac{2}{3} AD.$$

$$\text{Similarly, } BG = \frac{2}{3} BE \text{ and } CG = \frac{2}{3} CF$$

Medians divide the triangles into six parts having equal area.

In the figure, areas of $\triangle AGF$, $\triangle FGB$, $\triangle BGD$, $\triangle DGC$, $\triangle CGE$ and $\triangle EGA$ are equal.

Length of Medians

In the figure, using cosine rule in triangle ADC , we get

$$AD^2 = AC^2 + CD^2 - 2AC \times CD \times \cos C$$

$$\Rightarrow AD^2 = b^2 + \frac{a^2}{4} - ab \cos C$$

$$\Rightarrow AD^2 = b^2 + \frac{a^2}{4} - ab \left(\frac{a^2 + b^2 - c^2}{2ab} \right)$$

$$\Rightarrow AD^2 = \frac{2b^2 + 2c^2 - a^2}{4}$$

$$\Rightarrow AD = \frac{1}{2} \sqrt{2b^2 + 2c^2 - a^2}$$

$$\text{Similarly, } BE = \frac{1}{2} \sqrt{2c^2 + 2a^2 - b^2}$$

$$\text{and } CF = \frac{1}{2} \sqrt{2a^2 + 2b^2 - c^2}$$

Apollonius Theorem

Apollonius's theorem relates the length of a median of a triangle to the lengths of its side.

Mathematical statement of the theorem is:

In a triangle ABC , $AB^2 + AC^2 = 2(AD^2 + BD^2)$, where D is midpoint of BC .

Proof:

$$2(AD^2 + BD^2) = 2 \left[\frac{1}{4} (2b^2 + 2c^2 - a^2) + \frac{a^2}{4} \right]$$

$$= b^2 + c^2$$

$$= AB^2 + AC^2$$

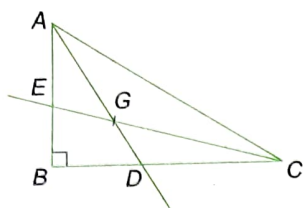
$$\text{Similarly, } AB^2 + BC^2 = 2(BE^2 + AE^2)$$

$$\text{and } AC^2 + BC^2 = 2(CF^2 + AF^2)$$

ILLUSTRATION 5.76

The lengths of the medians through acute angles of a right-angled triangle are 3 and 4. Find the area of the triangle.

Sol. $AD = 3$, $CE = 4$



Using Apollonius theorem for median AD , we have

$$c^2 + b^2 = 2\left(\frac{a^2}{4} + 9\right) \quad \dots(i)$$

Using Apollonius theorem for median CE , we have

$$b^2 + a^2 = 2\left(\frac{c^2}{4} + 16\right) \quad \dots(ii)$$

$$\text{Also } a^2 + c^2 = b^2 \quad \dots(iii)$$

Adding (i), (ii) and (iii), we get

$$3b^2 = 2\left(\frac{b^2}{4} + 25\right)$$

$$\text{or } b^2 = 20$$

$$\text{Solving (i) and (ii), we get } c = \frac{4}{\sqrt{3}} \text{ and } a = 2\sqrt{\frac{11}{3}}$$

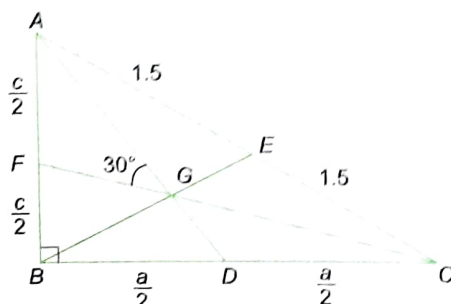
$$\begin{aligned} \text{Hence, Area of triangle} &= \frac{1}{2} \left(\frac{4}{\sqrt{3}} \right) \left(2\sqrt{\frac{11}{3}} \right) \\ &= \frac{4}{3} \sqrt{11} \end{aligned}$$

ILLUSTRATION 5.77

Two medians drawn from the acute angles of a right-angled triangle intersect at an angle of $\pi/6$. If the length of the hypotenuse of the triangle is 3 units, then find the area of the triangle (in sq. units).

Sol. In $\triangle ABD$, $AD^2 = c^2 + a^2/4$

In $\triangle FBC$, $CF^2 = c^2/4 + a^2$



In $\triangle AGF$, using cosine rule, we get

$$\cos 30^\circ = \frac{AG^2 + FG^2 - c^2}{2AG \cdot FG}$$

$$\therefore \frac{\sqrt{3}}{2} = \frac{\left(\frac{2}{3}AD\right)^2 + \left(\frac{1}{3}CF\right)^2 - \frac{c^2}{4}}{2AG \cdot FG}$$

$$\Rightarrow \sqrt{3}AG \cdot FG = \frac{4}{9} \left(c^2 + \frac{a^2}{4} \right) + \frac{1}{9} \left(a^2 + \frac{c^2}{4} \right) - \frac{c^2}{4}$$

$$\Rightarrow \sqrt{3}AG \cdot FG = \frac{2}{9} (a^2 + c^2) = 2 \quad (\text{As } a^2 + c^2 = b^2)$$

$$\Rightarrow \sqrt{3}AG \cdot FG = 2$$

$$\Rightarrow AG \cdot FG = \frac{2}{\sqrt{3}}$$

$$\therefore \text{Area of } \triangle ABC = 6 \times (\text{Area of } \triangle AGF)$$

$$= 6 \times \frac{1}{2} AG \cdot FG \cdot \sin 30^\circ$$

$$= 6 \times \frac{1}{2} \times \frac{2}{\sqrt{3}} \times \frac{1}{2} = \sqrt{3} \text{ sq. units}$$

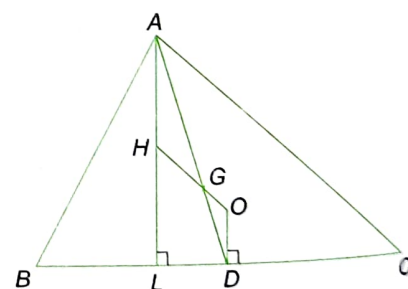
Division of Line Joining Orthocentre and Circumcentre by Centroid

In a triangle, which is not equilateral, orthocentre (H), centroid (G) and circumcentre (O) are collinear and $HG : GO = 2 : 1$.

The line joining these centres is called Euler's line.

Proof:

Consider acute angled triangle ABC .



In the figure, AL is altitude to BC . So, orthocentre (H) lies on AL .

Also, OD is perpendicular bisector of BC .

So, AD is median of triangle ABC .

Let the line HO meet the median AD at G .

Now, we shall prove that G is the centroid of $\triangle ABC$.

Clearly, $\triangle OGD$ and $\triangle HGA$ are similar.

$$\therefore \frac{OG}{HG} = \frac{GD}{GA} = \frac{OD}{HA} = \frac{R \cos A}{2R \cos A} = \frac{1}{2}$$

$$\therefore GD = \frac{1}{2} GA$$

Thus, G is centroid of $\triangle ABC$ and $OG : HG = 1 : 2$

CONCEPT APPLICATION EXERCISE 5.9

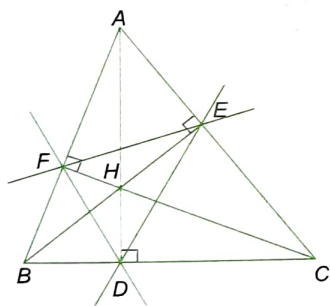
- Line joining vertex A of triangle ABC and orthocenter (H) meets the side BC in D . Then prove that
 - $BD : DC = \tan C : \tan B$
 - $AH : HD = (\tan B + \tan C) : \tan A$
- In a triangle ABC , $\angle A = 30^\circ$, $BC = 2 + \sqrt{5}$, then find the distance of the vertex A from the orthocenter.
- If the perimeter of the triangle formed by feet of altitudes of the triangle ABC is equal to four times the circumradius of $\triangle ABC$, then identify the type of $\triangle ABC$.
- AD , BE and CF are the medians of triangle ABC whose centroid is G . If the points A , F , G and E are concyclic, then prove that $2a^2 = b^2 + c^2$.
- In an acute angled triangle ABC ; AD , BE and CF are the altitudes. Prove that $\frac{FE}{a} + \frac{FD}{b} + \frac{DE}{c} \leq \frac{3}{2}$.

ANSWERS

- $(2 + \sqrt{5})\sqrt{3}$
- Triangle is not possible

EScribed CIRCLE AND EXCENTRE

Consider the triangle ABC with the triangle DEF (pedal triangle) formed by the feet of its altitudes.



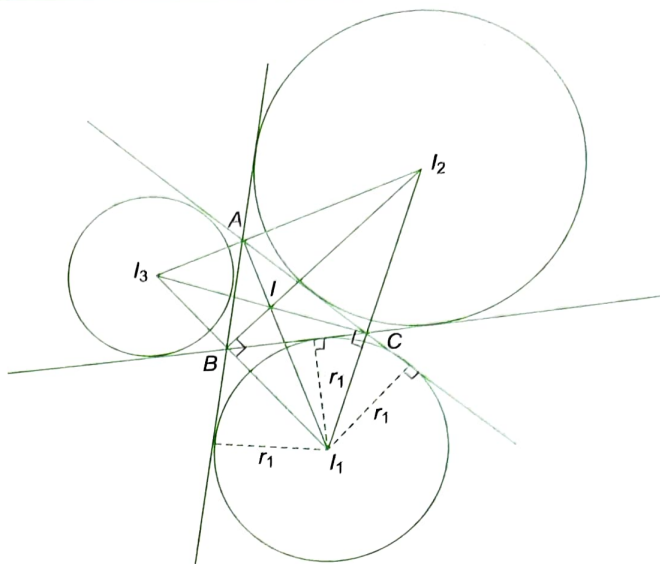
In the figure, H is orthocentre ABC . We have already learned that orthocentre of triangle is incentre of its pedal triangle.

In the figure, H is incentre of pedal triangle DEF . BE is one of the angle bisectors of the angle formed by lines DE and EF .

Since AC is perpendicular to BE , AC is another angle bisector of angle formed by lines DE and EF as two bisectors of angles formed by two lines are always perpendicular. Here, BE is internal angle bisector and AC is external angle bisector of $\angle DEF$ in triangle DEF .

Thus, at vertex A , two external angle bisectors AC and AB and one internal angle bisector AD of triangle DEF meet. Such point is called excentre. A is one of the excentres of triangle DEF , or we can say that A is centre the excircle of triangle DEF . This excircle touches the side EF and other two sides DE and DF produced. Similarly, other excircles of $\triangle DEF$ can be drawn.

Now, have a look at the following figure, where all three excentres I_1 , I_2 and I_3 of triangle ABC are shown.



Clearly, triangle ABC is pedal triangle of triangle $I_1I_2I_3$.

Incentre of triangle ABC lies on the line joining vertex A and excentre I_1 . AI_1 is an internal angle bisector of $\angle A$ of triangle ABC and also altitude of triangle $I_1I_2I_3$. Here, incentre of triangle ABC is orthocentre of triangle $I_1I_2I_3$.

RADII OF EXCIRCLES (EXRADII)

The radii of the excircles opposite to vertex A , B and C are denoted by r_1 , r_2 and r_3 , respectively. We have the following formulas for exradii:

$$(i) \quad r_1 = \frac{\Delta}{s-a}, r_2 = \frac{\Delta}{s-b}, r_3 = \frac{\Delta}{s-c}$$

$$(ii) \quad r_1 = s \tan \frac{A}{2}, r_2 = s \tan \frac{B}{2}, r_3 = s \tan \frac{C}{2}$$

$$(iii) \quad r_1 = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2},$$

$$r_2 = 4R \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2},$$

$$r_3 = 4R \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}$$

Proof:

- In the above figure, area of $\triangle ABC$

$$= \text{area of } \triangle I_1AC + \text{Area of } \triangle I_1AB - \text{Area of } \triangle I_1BC$$

$$\therefore \Delta = \frac{1}{2} r_1 b + \frac{1}{2} r_1 c - \frac{1}{2} r_1 a$$

$$\Rightarrow \Delta = \frac{1}{2} r_1 (b + c - a)$$

$$\Rightarrow \Delta = \frac{r_1}{2} (2s - 2a)$$

$$\Rightarrow r_1 = \frac{\Delta}{s-a}$$

Similarly, other formulas can be proved.

(ii) We know that $r = (s-a) \tan \frac{A}{2}$

$$\begin{aligned}\therefore r_1 &= \frac{\Delta}{s-a} = \frac{rs}{s-a} \\ &= \frac{s(s-a) \tan \frac{A}{2}}{s-a} \\ &= s \tan \frac{A}{2}\end{aligned}$$

Similarly, other formulas can be proved.

$$\begin{aligned}\text{(iii)} \quad 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \\ &= 4R \sqrt{\frac{(s-b)(s-c)}{bc}} \sqrt{\frac{s(s-b)}{ac}} \sqrt{\frac{s(s-c)}{ab}} \\ &= 4R \frac{s(s-a)(s-b)(s-c)}{abc(s-a)} \\ &= \frac{4R \Delta^2}{abc(s-a)} \\ &= \frac{\Delta}{s-a} = r_1\end{aligned}$$

Similarly, other formulas can be proved.

ILLUSTRATION 5.78

Prove that $r_1 + r_2 + r_3 - r = 4R$.

$$\begin{aligned}\text{Sol.} \quad r_1 + r_2 + r_3 - r &= \frac{\Delta}{s-a} + \frac{\Delta}{s-b} + \frac{\Delta}{s-c} - \frac{\Delta}{s} \\ &= \Delta \left[\frac{s-b+s-a}{(s-a)(s-b)} + \frac{(s-s+c)}{s(s-c)} \right] \\ &= \Delta \left[\frac{c}{(s-a)(s-b)} + \frac{c}{s(s-c)} \right] \\ &= \Delta c \left[\frac{s(s-c) + (s-a)(s-b)}{s(s-a)(s-b)(s-c)} \right] \\ &= \frac{\Delta c}{\Delta^2} [2s^2 - s(a+b+c) + ab] \\ &= \left(\frac{c}{\Delta} \right) [2s^2 - s(2s) + ab] \\ &= 4 \left(\frac{abc}{4\Delta} \right) = 4R\end{aligned}$$

ILLUSTRATION 5.79

If in a triangle $r_1 = r_2 + r_3 + r$, prove that the triangle is right angled.

Sol. We have $r_1 = r_2 + r_3 + r$

$$\text{or} \quad r_1 - r = r_2 + r_3$$

$$\Rightarrow \frac{\Delta}{s-a} - \frac{\Delta}{s} = \frac{\Delta}{s-b} + \frac{\Delta}{s-c}$$

$$\text{or} \quad \frac{\Delta a}{s(s-a)} = \frac{\Delta(2s-b-c)}{(s-b)(s-c)} = \frac{\Delta a}{(s-b)(s-c)}$$

$$\text{or} \quad s(s-a) = (s-b)(s-c)$$

$$\text{or} \quad s^2 - sa = s^2 - (b+c)s + bc$$

$$\text{or} \quad 2s(b+c-a) = 2bc$$

$$\text{or} \quad (a+b+c)(b+c-a) = 2bc$$

$$\text{or} \quad (b+c)^2 - a^2 = 2bc$$

$$\text{or} \quad b^2 + c^2 = a^2$$

Hence, the triangle is right angled.

ILLUSTRATION 5.80

Prove that $\frac{r_1 + r_2}{1 + \cos C} = 2R$.

$$\begin{aligned}\text{Sol.} \quad r_1 + r_2 &= 4R \left(\sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} + \sin \frac{B}{2} \cos \frac{A}{2} \cos \frac{C}{2} \right) \\ &= 4R \cos \frac{C}{2} \left(\sin \frac{A}{2} \cos \frac{B}{2} + \sin \frac{B}{2} \cos \frac{A}{2} \right) \\ &= 4R \cos \frac{C}{2} \sin \left(\frac{A+B}{2} \right) \\ &= 4R \left(\cos^2 \frac{C}{2} \right) = 2R(1 + \cos C)\end{aligned}$$

$$\text{or} \quad \frac{r_1 + r_2}{1 + \cos C} = 2R$$

ILLUSTRATION 5.81

Prove that $(r + r_1) \tan \left(\frac{B-C}{2} \right) + (r + r_2) \tan \left(\frac{C-A}{2} \right) + (r + r_3) \tan \left(\frac{A-B}{2} \right) = 0$.

$$\begin{aligned}\text{Sol.} \quad (r + r_1) \tan \left(\frac{B-C}{2} \right) \\ &= \left[4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} + 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \right] \\ &\quad \times \tan \left(\frac{B-C}{2} \right) \\ &= 4R \sin \frac{A}{2} \cos \left(\frac{B-C}{2} \right) \tan \left(\frac{B-C}{2} \right) \\ &= 4R \sin \frac{A}{2} \sin \left(\frac{B-C}{2} \right) \\ &= 4R \cos \left(\frac{B+C}{2} \right) \sin \left(\frac{B-C}{2} \right) \\ &= 2R (\sin B - \sin C)\end{aligned}$$

Similarly,

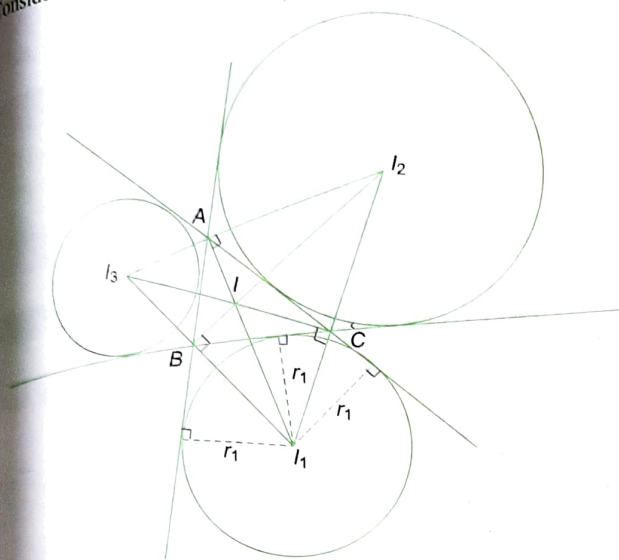
$$(r + r_2) \tan \left(\frac{C-A}{2} \right) = 2R (\sin C - \sin A)$$

$$(r + r_3) \tan \left(\frac{A-B}{2} \right) = 2R (\sin A - \sin B)$$

On adding Eqs. (i), (ii) and (iii), we get the result.

DISTANCE BETWEEN INCENTRE AND EXCENTRE

Consider the given figure.



$$\text{In } \triangle BI_1, \cos \angle BII_1 = \frac{BI}{II_1}$$

Points B, I, C and I_1 are concyclic.

$$\therefore \angle BII_1 = \angle BCI_1 = \frac{\pi - C}{2}$$

$$\text{Also, } BI = \frac{r}{\sin \frac{B}{2}}$$

$\therefore$ From (1), we have

$$II_1 = \frac{r}{\sin \frac{B}{2} \cos \left(\frac{\pi - C}{2} \right)}$$

$$\text{or } II_1 = \frac{r}{\sin \frac{B}{2} \sin \frac{C}{2}} = 4R \sin \frac{A}{2}$$

$$\text{Similarly, } II_2 = \frac{r}{\sin \frac{A}{2} \sin \frac{C}{2}} \text{ and } II_3 = \frac{r}{\sin \frac{A}{2} \sin \frac{B}{2}}$$

DISTANCE BETWEEN EXCENTRES

Let us find the distance I_1I_2 .

Points B, I, C and I_1 are concyclic.

$$\therefore \angle I_1IC = \angle IBC = B/2$$

Similarly, points A, I, C and I_2 are concyclic.

$$\therefore \angle I_2IC = \angle IAC = A/2$$

$$\text{So, in } \triangle I_1I_2, \angle I_1II_2 = \pi - \frac{A+B}{2}$$

Using sine rule in $\triangle I_1I_2$, we get

$$\frac{I_1I_2}{\sin \left(\pi - \frac{A+B}{2} \right)} = \frac{II_1}{\sin \frac{A}{2}}$$

$$\Rightarrow \frac{I_1I_2}{\cos \frac{C}{2}} = \frac{4R \sin \frac{A}{2}}{\sin \frac{A}{2}}$$

$$\Rightarrow I_1I_2 = 4R \cos \frac{C}{2}$$

$$\text{Similarly, } I_2I_3 = 4R \cos \frac{A}{2} \text{ and } I_1I_3 = 4R \cos \frac{B}{2}$$

ILLUSTRATION 5.82

If the distance between incentre and one of the excentre of an equilateral triangle is 4 units, then find the inradius of the triangle.

Sol. Distance between excentre I_1 and incentre I is

$$II_1 = 4R \sin \frac{A}{2} = 4 \quad (\text{given})$$

Since triangle is equilateral, we have

$$4 = 4R \sin \frac{\pi}{6}$$

$$\text{or } R = 2$$

$$\therefore R = 2r = 2$$

$$\text{or } r = 1$$

ILLUSTRATION 5.83

If I_1, I_2, I_3 are the centers of escribed circles of $\triangle ABC$, show that the area of $\triangle I_1I_2I_3$ is $\frac{abc}{2r}$.

$$\text{Sol. Area} = \frac{I_1I_2 \times I_2I_3 \times I_1I_3}{4R'}$$

where R' = Circumradius of $\triangle I_1I_2I_3$

$$= \frac{\left(4R \cos \frac{A}{2} \right) \left(4R \cos \frac{B}{2} \right) \left(4R \cos \frac{C}{2} \right)}{8R}$$

($\because \triangle ABC$ is pedal triangle for $\triangle I_1, I_2, I_3$)

$$= 8R^2 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$$

$$= \frac{R^2 \sin A \sin B \sin C}{\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}$$

$$= \frac{R^2 abc}{8R^3 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}$$

$$= \frac{abc}{2 \left(4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right)}$$

$$= \frac{abc}{2r}$$

CONCEPT APPLICATION EXERCISE 5.10

1. In $\triangle ABC$, if $r_1 < r_2 < r_3$, then find the order of lengths of the sides.
2. The exradii r_1, r_2 , and r_3 of $\triangle ABC$ are in H.P. Show that its sides a, b , and c are in A.P.
3. If in $\triangle ABC$, $(a-b)(s-c) = (b-c)(s-a)$, prove that r_1, r_2, r_3 are in A.P.
4. Prove that $2R \cos A = 2R + r - r_1$.
5. If the lengths of the perpendiculars from the vertices of a triangle ABC on the opposite sides are p_1, p_2, p_3 then prove that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$.
6. Prove that $r_1 r_2 + r_2 r_3 + r_3 r_1 = \frac{1}{4}(a+b+c)^2$.
7. In any triangle ABC , find the least value of $\frac{r_1 + r_2 + r_3}{r}$.
8. Prove that $\frac{r_1 - r}{a} + \frac{r_2 - r}{b} = \frac{c}{r_3}$.

ANSWERS

1. $a < b < c$

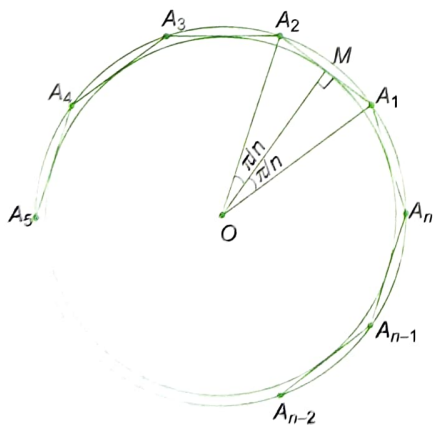
7. 9

REGULAR POLYGON

A polygon having all sides equal is called regular polygon. The sum of internal angles of any polygon of n sides is $(n-2)\pi$. So, in regular polygon measure of each internal angle is $\frac{(n-2)\pi}{n}$.

RADI OF THE INSCRIBED AND THE CIRCUMSCRIBED CIRCLES OF A REGULAR POLYGON

Consider a regular polygon of n sides having length of each side 'a'.



In the figure, R is the radius of circumscribed circle and r is the radius of the inscribed circle. Centres of both the circles coincide at O . Each side of polygon subtends same angle at the centre O .

$$\text{i.e., } \angle A_1 O A_2 = \frac{2\pi}{n}.$$

Point M is the foot of perpendicular from centre upon side $A_1 A_2$.

$$OA_1 = OA_2 = R \text{ and } OM = r.$$

$$\text{Also, } \angle A_1 O M = \angle A_2 O M = \frac{\pi}{n}$$

$$\text{In } \triangle O M A_1, \sin \frac{\pi}{n} = \frac{A_1 M}{O A_1} = \frac{a/2}{R}$$

$$\therefore R = \frac{a}{2} \operatorname{cosec} \frac{\pi}{n}$$

$$\text{Also, } \tan \frac{\pi}{n} = \frac{A_1 M}{O M} = \frac{a/2}{r}$$

$$\therefore r = \frac{a}{2} \cot \frac{\pi}{n}$$

AREA OF REGULAR POLYGON

Area in Terms of Side Length

Area of regular polygon,

$$\Delta = n \times (\text{Area of } \triangle O A_1 A_2)$$

$$= n \times \frac{1}{2} \times O M \times A_1 A_2$$

$$= n \times \frac{1}{2} \times \frac{a}{2} \cot \frac{\pi}{n} \times a$$

$$\therefore \Delta = \frac{n a^2}{4} \cot \frac{\pi}{n}$$

Area in Terms of Radius of Inscribed Circle

$$\Delta = n \times (\text{Area of the } \triangle O A_1 A_2)$$

$$= n \times \frac{1}{2} \times O M \times A_1 A_2$$

$$= n \times \frac{1}{2} \times r \times 2 M A_1$$

$$= n \times \frac{1}{2} \times r \times 2 r \tan \frac{\pi}{n}$$

$$\therefore \Delta = n r^2 \tan \frac{\pi}{n}$$

Area in Terms of Radius of Circumscribed Circle

$$\Delta = n \times (\text{Area of the } \triangle O A_1 A_2)$$

$$= n \times \frac{1}{2} R \times R \times \sin \frac{2\pi}{n}$$

$$\therefore \Delta = \frac{n R^2}{2} \sin \frac{2\pi}{n}$$

We know that circle is regular polygon having infinite number of sides.

So, from above formula, we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n R^2}{2} \sin \frac{2\pi}{n} &= \lim_{n \rightarrow \infty} \pi R^2 \frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}} \\ &= \pi R^2 \lim_{n \rightarrow \infty} \frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}} \end{aligned}$$

$$= \pi R^2 \times 1 \quad \left(\because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \right)$$

$$= \pi R^2$$

Thus, area of circle having radius R is πR^2 .

ILLUSTRATION 5.84

Prove that the sum of the radii of the circles, which are, respectively, inscribed and circumscribed about a polygon of n sides, whose side length is a , is $\frac{1}{2} a \cot \frac{\pi}{2n}$.

Sol. Radius of the circumscribed circle $= R = \frac{a}{2} \operatorname{cosec} \frac{\pi}{n}$

And, radius of the inscribed circle $= r = \frac{1}{2} a \cot \left(\frac{\pi}{n} \right)$

$$\Rightarrow R + r = \frac{a}{2 \sin(\pi/n)} + \frac{a \cos(\pi/n)}{2 \sin(\pi/n)}$$

$$= \frac{a[1 + \cos(\pi/n)]}{2 \times 2 \sin(\pi/2n) \cos(\pi/2n)}$$

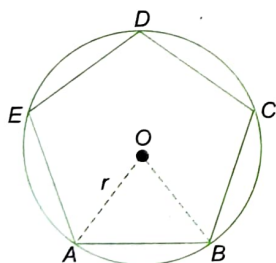
$$= \frac{2a \cos^2 \frac{\pi}{2n}}{4 \sin \frac{\pi}{2n} \cos \frac{\pi}{2n}}$$

$$= \frac{1}{2} a \cot \left(\frac{\pi}{2n} \right)$$

ILLUSTRATION 5.85

If the area of the circle is A_1 and the area of the regular pentagon inscribed in the circle is A_2 , then find the ratio A_1/A_2 .

Sol.



In $\triangle OAB$, $OA = OB = r$

and $\angle AOB = \frac{360^\circ}{5} = 72^\circ$

$$\therefore \text{Area of } \triangle AOB = \frac{1}{2} r \times r \sin 72^\circ$$

$$= \frac{1}{2} r^2 \cos 18^\circ$$

$$\therefore \text{Area of pentagon } (A_2) = 5 (\text{Area of } \triangle AOB)$$

$$= 5 \left(\frac{1}{2} r^2 \cos 18^\circ \right) \quad (i)$$

Also, Area of the circle $(A_1) = \pi r^2 \quad (ii)$

Hence, $\frac{A_1}{A_2} = \frac{\pi r^2}{\frac{5}{2} r^2 \cos 18^\circ} = \frac{2\pi}{5} \sec \left(\frac{\pi}{10} \right)$

[from Eqs. (i) and (ii)]

ILLUSTRATION 5.86

Prove that the area of a regular polygon having $2n$ sides, inscribed in a circle, is the geometric mean of the areas of the inscribed and circumscribed polygons of n sides.

Sol. Let a be the radius of the circle. Then,

S_1 = Area of regular polygon of n sides inscribed in the circle

$$= \frac{1}{2} n a^2 \sin \left(\frac{2\pi}{n} \right) = n a^2 \sin \frac{\pi}{n} \cos \frac{\pi}{n}$$

S_2 = Area of regular polygon of n sides circumscribing the circle

$$= n a^2 \tan \left(\frac{\pi}{n} \right)$$

S_3 = Area of regular polygon of $2n$ sides inscribed in the circle

$$= n a^2 \sin \left(\frac{\pi}{n} \right) \quad [\text{replacing } n \text{ by } 2n \text{ in } S_1]$$

$$\therefore \text{Geometric mean of } S_1 \text{ and } S_2 = \sqrt{(S_1 S_2)}$$

$$= n a^2 \sin \left(\frac{\pi}{n} \right)$$

$$= S_3$$

CONCEPT APPLICATION EXERCISE 5.11

- Regular pentagons are inscribed in two circles of radii 5 units and 2 units. Find the ratio of their areas.
- Let A be a point inside a regular polygon of 10 sides. Let $p_1, p_2, \dots, p_{10}$ be the distances of A from the sides of the polygon. If each side is of length 2 units, then find the value of $p_1 + p_2 + \dots + p_{10}$.
- Let $A_1, A_2, \dots, A_n$ be the vertices of an n -sided regular polygon such that $\frac{1}{A_1 A_2} = \frac{1}{A_1 A_3} + \frac{1}{A_1 A_4}$. Find the value of n .
- If I_n is the area of n -sided regular polygon inscribed in a circle of unit radius and O_n be the area of the polygon circumscribing the given circle, prove that

$$I_n = \frac{O_n}{2} \left(1 + \sqrt{1 - \left(\frac{2I_n}{n} \right)^2} \right).$$

ANSWERS

- 25 : 4
- $\frac{10}{\tan \frac{\pi}{10}}$
- $n = 7$

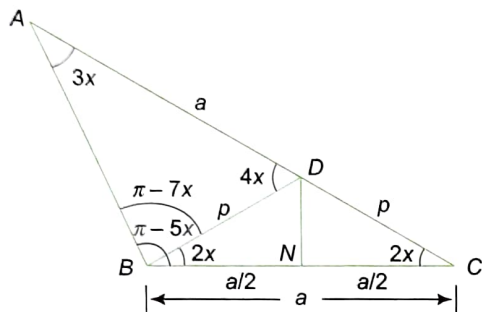
Solved Examples

EXAMPLE 5.1

In triangle ABC , D is on AC such that $AD = BC$, $BD = DC$, $\angle DBC = 2x$, and $\angle BAD = 3x$, all angles are in degrees, then find the value of x .

Sol. In $\triangle ABC$, $\frac{AC}{\sin 5x} = \frac{BC}{\sin 3x}$

$$\Rightarrow \frac{a+p}{\sin 5x} = \frac{a}{\sin 3x} \quad \dots(i)$$



$$\text{In } \triangle BDN, \cos 2x = \frac{a}{2p}$$

$$\text{or } a = 2p \cos 2x$$

$$\text{From Eq. (i), } \frac{2p \cos 2x + p}{\sin 5x} = \frac{2p \cos 2x}{\sin 3x}$$

$$\text{or } 2 \sin 3x \cos 2x + \sin 3x = 2 \sin 5x \cos 2x$$

$$\text{or } \sin 5x + \sin x + \sin 3x = \sin 7x + \sin 3x$$

$$\text{or } \sin 7x - \sin 5x = \sin x$$

$$\text{or } 2 \cos 6x \sin x = \sin x$$

$$\text{or } \cos 6x = \frac{1}{2}$$

$$\Rightarrow x = 10^\circ$$

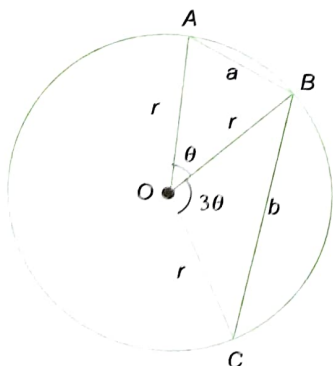
EXAMPLE 5.2

In a circle of radius r , chords of lengths a and b cm subtend angles θ and 3θ , respectively, at the center.

$$\text{Show that } r = a \sqrt{\frac{a}{3a-b}} \text{ cm.}$$

Sol. Applying cosine rule in $\triangle OAB$, we get

$$\cos \theta = \frac{2r^2 - a^2}{2r^2}$$



$$\text{or } a^2 = 2r^2 (1 - \cos \theta) \quad \text{or } a = 2r \sin \frac{\theta}{2}$$

Applying cosine rule in $\triangle OBC$, we get

$$\cos 3\theta = \frac{2r^2 - b^2}{2r^2}$$

$$\Rightarrow b = 2r \sin \left(\frac{3\theta}{2} \right) = 2r \left[3 \sin \frac{\theta}{2} - 4 \sin^3 \frac{\theta}{2} \right]$$

$$= 2r \left[\frac{3a}{2r} - \frac{4a^3}{8r^3} \right] = 3a - \frac{a^3}{r^2}$$

$$\text{or } r^2 = \frac{a^3}{3a-b}$$

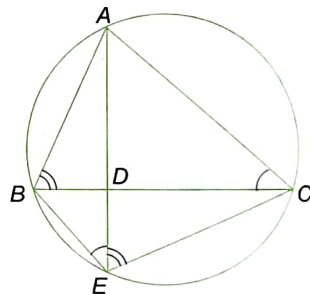
$$\text{or } r = a \sqrt{\frac{a}{3a-b}} \text{ cm}$$

EXAMPLE 5.3

Perpendiculars are drawn from the angles A , B , and C of an acute-angled triangle on the opposite sides, and produced to meet the circumscribing circle. If these produced parts are α , β , γ , respectively, then show that

$$\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 2(\tan A + \tan B + \tan C).$$

Sol. Let AD be the perpendicular from A on BC . When AD is produced, it meets the circumscribing circle at E . From question, $DE = \alpha$.



Since angles in the same segment are equal, we have

$$\angle AEB = \angle ACB = C$$

$$\text{and } \angle AEC = \angle ABC = B$$

From the right-angled triangle BDE ,

$$\tan C = \frac{BD}{DE}$$

From the right-angled triangle CDE ,

$$\tan B = \frac{CD}{DE}$$

Adding Eqs. (i) and (ii), we get

$$\tan B + \tan C = \frac{BD + CD}{DE} = \frac{BC}{DE} = \frac{a}{\alpha}$$

$$\text{Similarly, } \tan C + \tan A = \frac{b}{\beta}$$

$$\text{and } \tan A + \tan B = \frac{c}{\gamma}$$

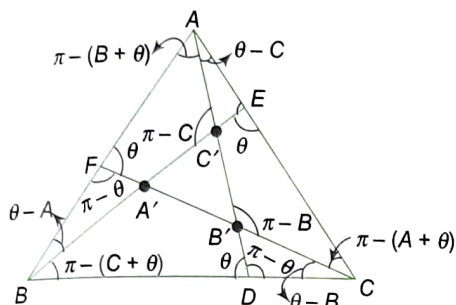
Adding Eqs. (iii), (iv), and (v), we get

$$\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 2(\tan A + \tan B + \tan C)$$

EXAMPLE 5.4

D, E, F are three points on the sides BC, CA, AB , respectively, such that $\angle ADB = \angle BEC = \angle CFA = \theta$. A', B', C' are the points of intersections of the lines AD, BE, CF inside the triangle. Show that area of $\Delta A'B'C' = 4\Delta \cos^2 \theta$, where Δ is the area of ΔABC .

Sol. From $\Delta AB'C'$, $\frac{AB'}{\sin(\pi - (A + \theta))} = \frac{AC}{\sin(\pi - B)}$



$$\Rightarrow AB' = 2R \sin(A + \theta)$$

From $\Delta AC'B$, $\frac{AC'}{\sin(\theta - A)} = \frac{AB}{\sin(\pi - C)}$

$$\Rightarrow AC' = 2R \sin(\theta - A)$$

$$\therefore B'C' = 2R(\sin(A + \theta) - \sin(\theta - A))$$

$$= 4R \cos \theta \sin A = 2a \cos \theta$$

Similarly, $C'A' = 2b \cos \theta$

$$\therefore \text{Area of } \Delta A'B'C' = \frac{1}{2}(B'C')(A'C') \sin \angle B'C'A'$$

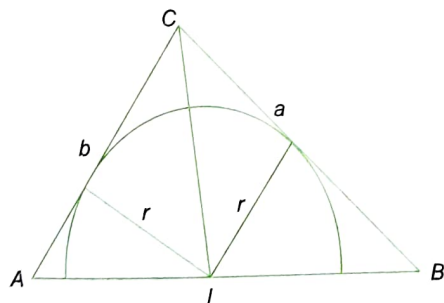
$$= \frac{1}{2}(2a \cos \theta)(2b \cos \theta) \sin C$$

$$= 4\cos^2 \theta \times \Delta$$

EXAMPLE 5.5

In ΔABC , a semicircle is inscribed, whose base lies on the side c . If x is the length of the angle bisector through angle C , then prove that the radius of the semicircle is $x \sin(C/2)$.

Sol.



From the figure

$$\left(\frac{1}{2}\right)ra + \left(\frac{1}{2}\right)rb = \left(\frac{1}{2}\right)ab \sin C$$

or $r(a + b) = 2\Delta$

or $r = \frac{2\Delta}{a + b} = \frac{ab \sin C}{a + b}$... (i)

Also $x = \frac{2ab}{a + b} \cos \frac{C}{2}$

[Length of angle bisector]

From Eq. (i), $r = \frac{2 \times (1/2) ab \sin C}{a + b}$

$$= \frac{2ab \sin \frac{C}{2} \cos \frac{C}{2}}{a + b}$$

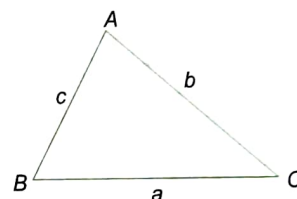
$$= \frac{2ab \cos \frac{C}{2}}{a + b} \cdot \sin \frac{C}{2}$$

$$= x \sin \frac{C}{2}$$

EXAMPLE 5.6

Given the base of a triangle, the opposite angle A , and the product k^2 of the other two sides, show that it is not possible for a to be less than $2k \sin \frac{A}{2}$.

Sol. Given $bc = k^2$



Now $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

or $2k^2 \cos A = b^2 + \left(\frac{k^2}{b}\right)^2 - a^2$

or $b^4 - (a^2 + 2k^2 \cos A) \cdot b^2 + k^4 = 0$

Since b^2 is real, $D \geq 0$ or $(a^2 + 2k^2 \cos A)^2 - 4k^4 \geq 0$

or $(a^2 + 2k^2 \cos A + 2k^2)(a^2 + 2k^2 \cos A - 2k^2) \geq 0$

or $\left(a^2 + 2k^2 \cdot 2 \cos^2 \frac{A}{2}\right) \left(a^2 - 2k^2 \cdot 2 \sin^2 \frac{A}{2}\right) \geq 0$

or $\left(a^2 + 4k^2 \cos^2 \frac{A}{2}\right) \left(a^2 - 4k^2 \sin^2 \frac{A}{2}\right) \geq 0$

or $a^2 - 4k^2 \sin^2 \frac{A}{2} \geq 0$

[since $a^2 \frac{A}{2} + 4k^2 \cos^2 A$ is always positive]

or $\left(a + 2k \sin \frac{A}{2}\right) \left(a - 2k \sin \frac{A}{2}\right) \geq 0$

or $a \leq -2k \sin \frac{A}{2}$ or $a \geq 2k \sin \frac{A}{2}$

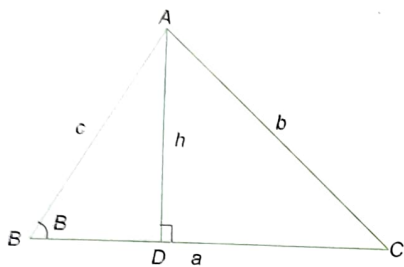
But a must be positive, which means $a \leq -2k \sin(A/2)$ is rejected.

Hence, $a \geq 2k \sin \frac{A}{2}$.

EXAMPLE 5.7

In a triangle of base a , the ratio of the other two sides is $r (< 1)$. Show that the altitude of the triangle is less than or equal to $\frac{ar}{1-r^2}$.

Sol. Given that in $\triangle ABC$, base = a and $\frac{c}{b} = r$. We have to find the altitude h .



We have, in $\triangle ABD$,

$$\begin{aligned} h &= c \sin B = \frac{c a \sin B}{a} \\ &= \frac{c \sin A \sin B}{\sin(B+C)} \\ &= \frac{c \sin A \sin B \sin(B-C)}{\sin(B+C) \sin(B-C)} \\ &= \frac{c \sin A \sin B \sin(B-C)}{\sin^2 B - \sin^2 C} \end{aligned}$$

$$= \frac{c \frac{a}{k} \frac{b}{k} \sin(B-C)}{\frac{b^2}{k^2} - \frac{c^2}{k^2}}$$

$$= \frac{abc \sin(B-C)}{b^2 - c^2}$$

$$= \frac{a \left(\frac{c}{b} \right) \sin(B-C)}{1 - \left(\frac{c}{b} \right)^2}$$

$$= \frac{ar \sin(B-C)}{1-r^2}$$

$$\leq \frac{ar}{1-r^2}$$

$$[\because \sin(B-C) \leq 1]$$

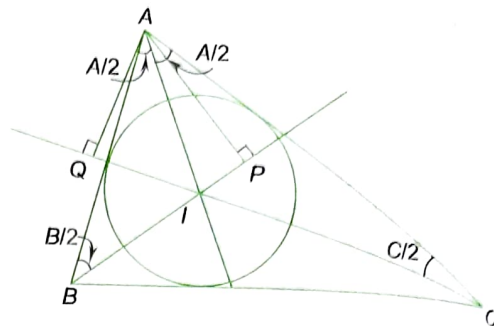
$$\therefore h \leq \frac{ar}{1-r^2}$$

EXAMPLE 5.8

Let ABC be a triangle with incentre I . If P and Q are the feet of the perpendiculars from A to BI and CI , respectively, then

prove that $\frac{AP}{BI} + \frac{AQ}{CI} = \cot \frac{A}{2}$.

Sol. In $\triangle APB$, $\sin \frac{B}{2} = \frac{AP}{AB}$



In $\triangle AQC$, $\sin \frac{C}{2} = \frac{AQ}{AC}$

Now in $\triangle ABI$, using sine rule, we get

$$\frac{BI}{\sin \frac{A}{2}} = \frac{AB}{\cos \frac{C}{2}}$$

And in $\triangle ACI$, using sine rule, we get

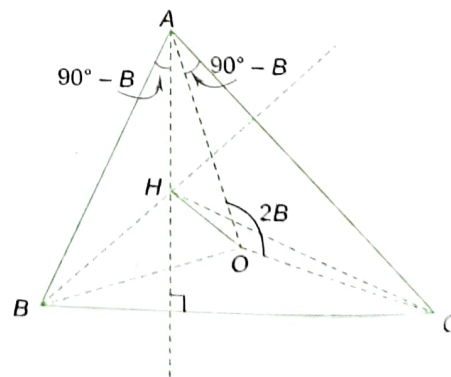
$$\frac{CI}{\sin \frac{A}{2}} = \frac{AC}{\cos \frac{B}{2}}$$

$$\begin{aligned} \therefore \frac{AP}{BI} + \frac{AQ}{CI} &= \frac{AB \sin \frac{B}{2} \cos \frac{C}{2}}{AB \sin \frac{A}{2}} + \frac{AC \sin \frac{C}{2} \cos \frac{B}{2}}{AC \sin \frac{A}{2}} \\ &= \frac{\sin \left(\frac{B+C}{2} \right)}{\sin \frac{A}{2}} = \cot \frac{A}{2} \end{aligned}$$

EXAMPLE 5.9

Let O be the circumcentre and H be the orthocentre of an acute angled triangle ABC . If $A > B > C$, then show that $Ar(\triangle BOH) = Ar(\triangle AOH) + Ar(\triangle COH)$.

Sol.



From the figure, we have

$$\angle OAH = A - 2(90^\circ - B) = B - C$$

Similarly, $\angle OBH = A - C$

and $\angle OCH = A - B$

Also, $AH = 2R \cos A$, $BH = 2R \cos B$, $CH = 2R \cos C$

$$\begin{aligned} \text{Ar}(\triangle AOH) &= \frac{1}{2}(R)(2R \cos A) \sin(B-C) \\ &= R^2 \cos(B+C) \sin(C-B) \\ &= \frac{R^2}{2} (\sin 2C - \sin 2B) \end{aligned}$$

$$\text{Similarly, } \text{Ar}(\triangle BOH) = \frac{R^2}{2} (\sin 2C - \sin 2A)$$

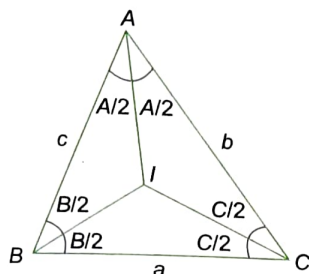
$$\text{and } \text{Ar}(\triangle COH) = \frac{R^2}{2} (\sin 2B - \sin 2A)$$

$$\text{Clearly, } \text{Ar}(\triangle AOH) + \text{Ar}(\triangle COH) = \text{Ar}(\triangle BOH)$$

EXAMPLE 5.10

If I is the incenter of $\triangle ABC$ and R_1, R_2 , and R_3 are, respectively, the radii of the circumcircles of the triangles IBC, ICA , and IAB , then prove that $R_1 R_2 R_3 = 2rR^2$.

Sol.



Let I be the incenter of $\triangle ABC$.

$$\text{In } \triangle IBC, \angle BIC = \pi - \frac{B+C}{2} = \pi - \frac{\pi-A}{2} = \frac{\pi+A}{2}$$

Now, the radius of circumcircle of $\triangle IBC$, by sine rule, is

$$\begin{aligned} R_1 &= \frac{BC}{2 \sin(\angle BIC)} = \frac{a}{2 \sin\left(\frac{\pi+A}{2}\right)} \\ &= \frac{2R \sin A}{2 \cos \frac{A}{2}} \\ &= 2R \sin \frac{A}{2} \end{aligned}$$

Similarly, the radii of circumcircle of $\triangle ICA$ and $\triangle IAB$ are given by

$$R_2 = 2R \sin \frac{B}{2} \text{ and } R_3 = 2R \sin \frac{C}{2}$$

$$\Rightarrow R_1 R_2 R_3 = 8R^3 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = 2rR^2$$

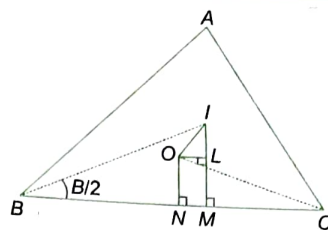
EXAMPLE 5.11

Show that the line joining the incenter to the circumcenter of triangle ABC is inclined to the side BC at an angle

$$\tan^{-1} \left(\frac{\cos B + \cos C - 1}{\sin C - \sin B} \right).$$

Sol.

Let I be the incenter and O be the circumcenter of the triangle ABC .



Let OL be parallel to BC .

Let $\angle IOL = \theta$

Now, $IM = r, OC = R, \angle NOC = A$. Then

$$\begin{aligned} \tan \theta &= \frac{IL}{OL} = \frac{IM - LM}{BM - BN} \\ &= \frac{IM - ON}{BM - NC} \\ &= \frac{r - R \cos A}{r \cot \frac{B}{2} - R \sin A} \\ &= \frac{4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} - R \cos A}{4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \cdot \cot \frac{B}{2} - R \sin A} \\ &= \frac{\cos A + \cos B + \cos C - 1 - \cos A}{\sin A + \sin C - \sin B - \sin A} \\ &= \frac{\cos B + \cos C - 1}{\sin C - \sin B} \end{aligned}$$

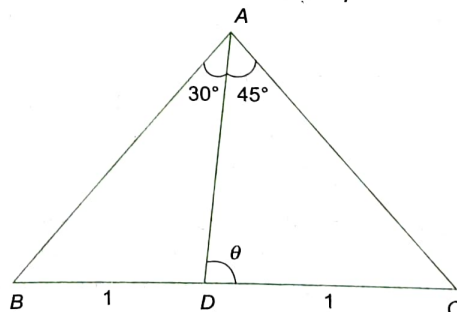
$$\text{or } \theta = \tan^{-1} \left[\frac{\cos B + \cos C - 1}{\sin C - \sin B} \right]$$

EXAMPLE 5.12

In triangle ABC , the median to the side BC is of length $\frac{1}{\sqrt{11-6\sqrt{3}}}$ and it divides the angle A into angles 30° and 45° .

Find the length of the side BC .

Sol. We know that if in $\triangle ABC$, AD divides BC in the ratio $m:n$ with $\angle ADC = \theta, \angle BAD = \alpha$ and $\angle CAD = \beta$.



The $m-n$ theorem states that

$$(m+n) \cot \theta = m \cot \alpha - n \cot \beta$$

By applying Eq. (i) in $\triangle ABC$, we get

$$(1+1) \cot \theta = 1 \cot 30^\circ - 1 \cot 45^\circ$$

$$\text{or } 2 \cot \theta = \sqrt{3} - 1$$

(i)

$$\text{or } \cot \theta = \frac{\sqrt{3}-1}{2}$$

$$\text{or } \tan \theta = \frac{2}{\sqrt{3}-1}$$

But $\theta = B + 30^\circ$, we get

$$\tan(B + 30^\circ) = \frac{2}{\sqrt{3}-1} = \frac{2(\sqrt{3}+1)}{3-1} = \sqrt{3} + 1$$

$$\text{or } \frac{\tan B + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}} \tan B} = \sqrt{3} + 1$$

$$\text{or } \frac{\sqrt{3} \tan B + 1}{\sqrt{3} - \tan B} = \sqrt{3} + 1$$

$$\text{or } \sqrt{3} \tan B + 1 = 3 + \sqrt{3} - (\sqrt{3} + 1) \tan B$$

$$\text{or } (2\sqrt{3} + 1) \tan B = 2 + \sqrt{3}$$

$$\text{or } \tan B = \frac{2 + \sqrt{3}}{2\sqrt{3} + 1}$$

$$\text{or } \cot B = \frac{2\sqrt{3} + 1}{2 + \sqrt{3}} \times \frac{2 - \sqrt{3}}{2 - \sqrt{3}} = \frac{4\sqrt{3} - 6 + 2 - \sqrt{3}}{4 - 3} = 3\sqrt{3} - 4$$

$$\text{or } \operatorname{cosec}^2 B = 1 + 27 + 16 - 24\sqrt{3} = 44 - 24\sqrt{3}$$

$$\text{or } \sin^2 B = \frac{1}{4(11 - 6\sqrt{3})}$$

$$\text{or } \sin B = \frac{1}{2\sqrt{11 - 6\sqrt{3}}}$$

Now in $\triangle ABD$, by applying the sine law, we get

$$\frac{BD}{\sin 30^\circ} = \frac{AD}{\sin B}$$

$$\text{or } BD = \frac{AD}{\sin B} \times \sin 30^\circ = \frac{\frac{1}{2\sqrt{11 - 6\sqrt{3}}}}{\frac{1}{2}} \times \frac{1}{2} = 1$$

$$\therefore BC = 2BD = 2 \text{ units}$$

EXAMPLE 5.13

Three circles touch one another externally. The tangents at their points of contact meet at a point whose distance from a point of contact is 4. Find the ratio of the product of the radii to the sum of the radii of the circles.

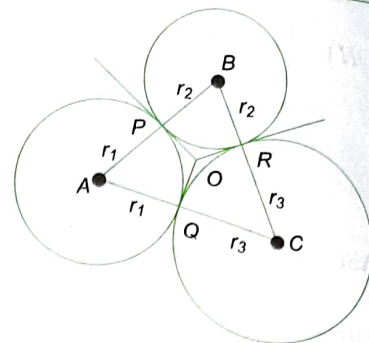
Sol. Let us consider three circles with centers at A , B , and C with radii r_1 , r_2 , and r_3 , respectively, which touch each other externally at P , Q , and R . Let the common tangents at P , Q , and R meet each other at O . Then

$$OP = OQ = OR = 4 \text{ (given)}$$

(lengths of tangents from a point to a circle are equal)

Also, $OP \perp AB$, $OQ \perp AC$, $OR \perp BC$

Therefore, O is the incenter of $\triangle ABC$.



Thus, for $\triangle ABC$,

$$s = \frac{(r_1 + r_2) + (r_2 + r_3) + (r_3 + r_1)}{2} = r_1 + r_2 + r_3$$

$$\Rightarrow \Delta = \sqrt{(r_1 + r_2 + r_3) \cdot r_1 \cdot r_2 \cdot r_3}$$

(Heron's formula)

$$\text{Now, } r = \frac{\Delta}{s}$$

$$\text{or } 4 = \frac{\sqrt{(r_1 + r_2 + r_3) r_1 r_2 r_3}}{r_1 + r_2 + r_3}$$

$$= \frac{\sqrt{r_1 r_2 r_3}}{\sqrt{r_1 + r_2 + r_3}}$$

$$\text{or } \frac{r_1 r_2 r_3}{r_1 + r_2 + r_3} = \frac{16}{1}$$

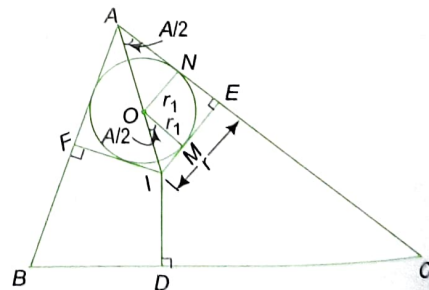
$$\text{or } r_1 r_2 r_3 : (r_1 + r_2 + r_3) = 16:1$$

EXAMPLE 5.14

Let ABC be a triangle with incenter I and inradius r . Let D , E , and F be the feet of the perpendiculars from I to the sides BC , CA , and AB , respectively. If r_1 , r_2 , and r_3 are the radii of the circles inscribed in the quadrilaterals $AFIE$, $BDIF$, and $CEDI$ respectively, prove that

$$\frac{r_1}{r - r_1} + \frac{r_2}{r - r_2} + \frac{r_3}{r - r_3} = \frac{r_1 r_2 r_3}{(r - r_1)(r - r_2)(r - r_3)}$$

Sol.



I is incenter of $\triangle ABC$.

r_1 is the radius of the circle inscribed in quadrilateral $AFIE$.

From the figure, AF and AE are tangents to the circle.

Also IE and IF are tangent to the circle.

So AI is passing through the center of the circle by symmetry.

Now ON is parallel to IE .

So $\triangle ANO$ and $\triangle OMI$ are similar.

In $\triangle OMI$, we have $\cot \frac{A}{2} = \frac{OM}{IM} = \frac{r_1}{r - r_1}$

Similarly from other circles we get

$$\cot \frac{B}{2} = \frac{r_2}{r - r_2} \text{ and } \cot \frac{C}{2} = \frac{r_3}{r - r_3}$$

Now in $\triangle ABC$, we know that

$$\cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2} = \cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2}$$

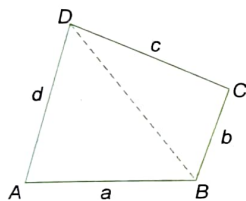
$$\Rightarrow \frac{r_1}{r - r_1} \cdot \frac{r_2}{r - r_2} \cdot \frac{r_3}{r - r_3} = \frac{r_1}{r - r_1} + \frac{r_2}{r - r_2} + \frac{r_3}{r - r_3}$$

EXAMPLE 5.15

In convex quadrilateral $ABCD$, $AB = a$, $BC = b$, $CD = c$, $DA = d$. This quadrilateral is such that a circle can be inscribed in it and a circle can also be circumscribed about it. Prove that

$$\tan^2 \frac{A}{2} = \frac{bc}{ad}$$

Sol. Since a circle can be inscribed in the quadrilateral, we have $a + c = b + d$.



Further, it is also concyclic, thus $\angle A + \angle C = \pi$

$$\text{In } \triangle BAD, \quad BD^2 = a^2 + d^2 - 2ad \cos A$$

$$\text{In } \triangle BCD, \quad BD^2 = b^2 + c^2 - 2bc \cos C$$

$$= b^2 + c^2 + 2bc \cos A$$

$$\therefore 2 \cos A (bc + ad) = a^2 + d^2 - b^2 - c^2$$

$$\text{or } 2 \cos A = \frac{a^2 + d^2 - b^2 - c^2}{bc + ad}$$

$$\text{Since } a + c = b + d, \quad a - d = b - c$$

$$\Rightarrow a^2 + d^2 - 2ad = b^2 + c^2 - 2bc$$

$$\Rightarrow a^2 + d^2 - b^2 - c^2 = 2(ad - bc)$$

$$\therefore \cos A = \frac{ad - bc}{bc + ad} = \frac{1 - \tan^2 \frac{A}{2}}{1 + \tan^2 \frac{A}{2}}$$

$$\Rightarrow \tan^2 \frac{A}{2} = \frac{bc}{ad}$$

Exercises

Single Correct Answer Type

- In $\triangle ABC$, $\frac{\sin A (a - b \cos C)}{\sin C (c - b \cos A)} =$
 (1) -2 (2) -1 (3) 0 (4) 1
- If in a triangle ABC , $\frac{1 + \cos A}{a} + \frac{1 + \cos B}{b} + \frac{1 + \cos C}{c} = \frac{k^2 (1 + \cos A)(1 + \cos B)(1 + \cos C)}{abc}$, then k is equal to
 (1) $\frac{1}{2\sqrt{2}R}$ (2) $2R$
 (3) $\frac{1}{R}$ (4) none of these
- In triangle ABC , $2ac \sin\left(\frac{1}{2}(A - B + C)\right)$ is equal to
 (1) $a^2 + b^2 - c^2$ (2) $c^2 + a^2 - b^2$
 (3) $b^2 - c^2 - a^2$ (4) $c^2 - a^2 - b^2$
- If the angles of a triangle are in the ratio $4:1:1$, then the ratio of the longest side to the perimeter is
 (1) $\sqrt{3}:(2 + \sqrt{3})$ (2) $1:6$
 (3) $1:2 + \sqrt{3}$ (4) $2:3$
- Which of the following pieces of data does NOT uniquely determine an acute-angled triangle ABC (R being the radius of the circumcircle)?
 (1) $a, \sin A, \sin B$ (2) a, b, c
 (3) $a, \sin B, R$ (4) $a, \sin A, R$
- The side of a triangle are in the ratio $1:\sqrt{3}:2$, then the angles of the triangle are in the ratio
 (1) $1:3:5$ (2) $2:3:4$ (3) $3:2:1$ (4) $1:2:3$
- In $\triangle ABC$, $a = 5$, $b = 12$, $C = 90^\circ$ and D is a point on AB so that $\angle BCD = 45^\circ$. Then which of the following is not true?
 (1) $CD = \frac{60\sqrt{2}}{17}$ (2) $BD = \frac{65}{17}$
 (3) $AD = \frac{60\sqrt{2}}{17}$ (4) none of these
- In $\triangle ABC$, $(a + b + c)(b + c - a) = kbc$ if
 (1) $k < 0$ (2) $k > 0$
 (3) $0 < k < 4$ (4) $k > 4$
- Let D be the middle point of the side BC of a triangle ABC . If the triangle ADC is equilateral, then $a^2 : b^2 : c^2$ is equal to
 (1) $1:4:3$ (2) $4:1:3$
 (3) $4:3:1$ (4) $3:4:1$
- In a triangle ABC , the altitude from A is not less than BC and the altitude from B is not less than AC . The triangle is
 (1) right angled (2) isosceles
 (3) obtuse angled (4) equilateral
- In $\triangle ABC$, if $\frac{\sin A}{c \sin B} + \frac{\sin B}{c} + \frac{\sin C}{b} = \frac{c}{ab} + \frac{b}{ac} + \frac{a}{bc}$, then the value of angle A is
 (1) 120° (2) 90° (3) 60° (4) 30°
- If in $\triangle ABC$, sides a, b, c are in A.P. then
 (1) $B > 60^\circ$ (2) $B < 60^\circ$
 (3) $B \leq 60^\circ$ (4) $B = |A - C|$
- In a triangle ABC , AD is the altitude from A . If $b > c$, $\angle C = 23^\circ$ and $AD = \frac{abc}{b^2 - c^2}$, then $\angle B =$
 (1) 83° (2) 97° (3) 113° (4) 127°
- If the sides a, b, c of a triangle ABC form successive terms of G.P. with common ratio $r (> 1)$ then which of the following is correct?
 (1) $A > \pi/3$ (2) $B \geq \pi/3$
 (3) $C < \pi/3$ (4) $A < B < \pi/3$
- In triangle ABC , $b^2 \sin 2C + c^2 \sin 2B = 2bc$ where $b = 20$, $c = 21$, then inradius =
 (1) 4 (2) 6 (3) 8 (4) 9
- In $\triangle ABC$, if $AB = x$, $BC = x + 1$, $\angle C = \frac{\pi}{3}$, then the least integer value of x is
 (1) 6 (2) 7
 (3) 8 (4) none of these
- If one side of a triangle is double the other, and the angles on opposite sides differ by 60° , then the triangle is
 (1) equilateral (2) obtuse angled
 (3) right angled (4) acute angled
- If the hypotenuse of a right-angled triangle is four times the length of the perpendicular drawn from the opposite vertex to it, then the difference of the two acute angles will be
 (1) 60° (2) 15° (3) 75° (4) 30°
- If P is a point on the altitude AD of the triangle ABC such that $\angle CBP = B/3$, then AP is equal to
 (1) $2a \sin \frac{C}{3}$ (2) $2b \sin \frac{C}{3}$
 (3) $2c \sin \frac{B}{3}$ (4) $2c \sin \frac{C}{3}$
- With usual notations, in triangle ABC , $a \cos(B - C) + b \cos(C - A) + c \cos(A - B)$ is equal to
 (1) $\frac{abc}{R^2}$ (2) $\frac{abc}{4R^2}$ (3) $\frac{4abc}{R^2}$ (4) $\frac{abc}{2R^2}$

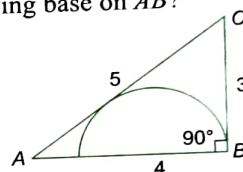
21. If in $\triangle ABC$, $8R^2 = a^2 + b^2 + c^2$, then the triangle ABC is
 (1) right angled (2) isosceles
 (3) equilateral (4) none of these
22. Let ABC be a triangle with $\angle A = 45^\circ$. Let P be a point on side BC with $PB = 3$ and $PC = 5$. If O is circumcenter of triangle ABC , then length OP is
 (1) $\sqrt{18}$ (2) $\sqrt{17}$ (3) $\sqrt{19}$ (4) $\sqrt{15}$
23. In any triangle ABC , $\frac{a^2 + b^2 + c^2}{R^2}$ has the maximum value of
 (1) 3 (2) 6
 (3) 9 (4) none of these
24. In triangle ABC , $R(b + c) = a\sqrt{bc}$, where R is the circumradius of the triangle. Then the triangle is
 (1) isosceles but not right (2) right but not isosceles
 (3) right isosceles (4) equilateral
25. In $\triangle ABC$, if $b^2 + c^2 = 2a^2$, then value of $\frac{\cot A}{\cot B + \cot C}$ is
 (1) $\frac{1}{2}$ (2) $\frac{3}{2}$
 (3) $\frac{5}{2}$ (4) $\frac{5}{3}$
26. If $\sin \theta$ and $-\cos \theta$ are the roots of the equation $ax^2 - bx - c = 0$, where a, b , and c are the sides of a triangle ABC , then $\cos B$ is equal to
 (1) $1 - \frac{c}{2a}$ (2) $1 - \frac{c}{a}$
 (3) $1 + \frac{c}{2a}$ (4) $1 + \frac{c}{3a}$
27. If D is the mid-point of the side BC of triangle ABC and AD is perpendicular to AC , then
 (1) $3b^2 = a^2 - c^2$ (2) $3a^2 = b^2 - 3c^2$
 (3) $b^2 = a^2 - c^2$ (4) $a^2 + b^2 = 5c^2$
28. In a $\triangle ABC$, if $\cot A : \cot B : \cot C = 30 : 19 : 6$ then the sides a, b, c are
 (1) in A.P. (2) in G.P.
 (3) in H.P. (4) none of these
29. In $\triangle ABC$, P is an interior point such that $\angle PAB = 10^\circ$, $\angle PBA = 20^\circ$, $\angle PCA = 30^\circ$, $\angle PAC = 40^\circ$ then $\triangle ABC$ is
 (1) isosceles (2) right angled
 (3) equilateral (4) obtuse angled
30. In $\triangle ABC$, if $AB = c$ is fixed, and $\cos A + \cos B + 2\cos C = 2$ then the locus of vertex C is
 (1) ellipse (2) hyperbola
 (3) circle (4) parabola
31. If in $\triangle ABC$, $A = \frac{\pi}{7}$, $B = \frac{2\pi}{7}$, $C = \frac{4\pi}{7}$ then $a^2 + b^2 + c^2$ must be
 (1) R^2 (2) $3R^2$ (3) $4R^2$ (4) $7R^2$

32. In $\triangle ABC$, $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2}$ is equal to
 (1) $\frac{\Delta}{r^2}$ (2) $\frac{(a+b+c)^2}{abc} 2R$
 (3) $\frac{\Delta}{r}$ (4) $\frac{\Delta}{Rr}$

33. In $\triangle ABC$, $\left(\cot \frac{A}{2} + \cot \frac{B}{2} \right) \left(a \sin^2 \frac{B}{2} + b \sin^2 \frac{A}{2} \right) =$
 (1) $\cot C$ (2) $c \cot C$
 (3) $\cot \frac{C}{2}$ (4) $c \cot \frac{C}{2}$

34. In a right-angled isosceles triangle, the ratio of the circumradius and inradius is
 (1) $2(\sqrt{2} + 1):1$ (2) $(\sqrt{2} + 1):1$
 (3) $2:1$ (4) $\sqrt{2}:1$

35. In the given figure, what is the radius of the inscribed semicircle having base on AB ?

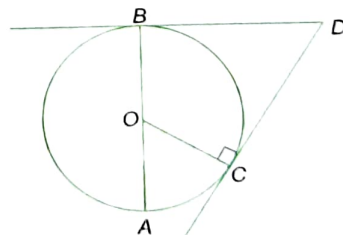


- (1) $3/2$ (2) $5/2$
 (3) $7/5$ (4) none of these

36. In $\triangle ABC$, $A = \frac{2\pi}{3}$, $b - c = 3\sqrt{3}$ cm and area of $\triangle ABC = \frac{9\sqrt{3}}{2}$ cm², then $BC =$
 (1) $6\sqrt{3}$ cm (2) 9 cm (3) 18 cm (4) 27 cm

37. In triangle ABC , let $\angle C = \pi/2$. If r is the inradius and R is circumradius of the triangle, then $2(r + R)$ is equal to
 (1) $a + b$ (2) $b + c$
 (3) $c + a$ (4) $a + b + c$

38. In the given figure, AB is the diameter of the circle, centered at O . If $\angle COA = 60^\circ$, $AB = 2r$, $AC = d$, and $CD = l$, then l is equal to



- (1) $d\sqrt{3}$ (2) $d/\sqrt{3}$ (3) $3d$ (4) $\sqrt{3}d/2$

39. In triangle ABC , if P, Q, R divides sides BC, AC , and AB , respectively, in the ratio $k : 1$ (in order). If the ratio $\left(\frac{\text{area } \triangle PQR}{\text{area } \triangle ABC} \right)$ is $\frac{1}{3}$, then k is equal to
 (1) $1/3$ (2) 2
 (3) 3 (4) none of these

40. If the angles of a triangle are 30° and 45° and the included side is $(\sqrt{3} + 1)$ cm, then the area of the triangle is
- (1) $\frac{\sqrt{3} + 1}{2}$ sq. units (2) $(\sqrt{3} + 1)$ sq. units
 (3) $2(\sqrt{3} - 1)$ sq. units (4) $\frac{2\sqrt{3} - 1}{2}$ sq. units
41. In triangle ABC , base BC and area of triangle are fixed. The locus of the centroid of triangle ABC is a straight line that is
- (1) parallel to side BC
 (2) right bisector of side BC
 (3) perpendicular to BC
 (4) inclined at an angle $\sin^{-1}(\sqrt{\Delta}/BC)$ to side BC
42. Let the area of triangle ABC be $(\sqrt{3} - 1)/2$, $b = 2$, and $c = (\sqrt{3} - 1)$, and $\angle A$ be acute. The measure of the angle C is
- (1) 15° (2) 30° (3) 60° (4) 75°
43. In $\triangle ABC$, $\Delta = 6$, $abc = 60$, $r = 1$. Then the value of $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ is nearly
- (1) 0.5 (2) 0.6 (3) 0.4 (4) 0.8
44. Triangle ABC is isosceles with $AB = AC$ and $BC = 65$ cm. P is a point on BC such that the perpendicular distances from P to AB and AC are 24 cm and 36 cm, respectively. The area of triangle ABC (in sq. cm is)
- (1) 1254 (2) 1950 (3) 2535 (4) 5070
45. In an equilateral triangle, the inradius, circumradius, and one of the exradii are in the ratio
- (1) 2:4:5 (2) 1:2:3 (3) 1:2:4 (4) 2:4:3
46. In triangle ABC , if $\cos A + \cos B + \cos C = \frac{7}{4}$, then $\frac{R}{r}$ is equal to
- (1) $\frac{3}{4}$ (2) $\frac{4}{3}$ (3) $\frac{2}{3}$ (4) $\frac{3}{2}$
47. If two sides of a triangle are roots of the equation $x^2 - 7x + 8 = 0$ and the angle between these sides is 60° then the product of inradius and circumradius of the triangle is
- (1) $\frac{8}{7}$ (2) $\frac{5}{3}$
 (3) $\frac{5\sqrt{2}}{3}$ (4) 8
48. Given $b = 2$, $c = \sqrt{3}$, $\angle A = 30^\circ$, then inradius of $\triangle ABC$ is
- (1) $\frac{\sqrt{3} - 1}{2}$ (2) $\frac{\sqrt{3} + 1}{2}$
 (3) $\frac{\sqrt{3} - 1}{4}$ (4) none of these
49. In triangle ABC , if $A - B = 120^\circ$ and $R = 8r$, where R and r have their usual meanings, then $\cos C$ equals
- (1) $3/4$ (2) $2/3$ (3) $5/6$ (4) $7/8$
50. ABC is an equilateral triangle of side 4 cm. If R , r , and h are the circumradius, inradius, and altitude, respectively, then $\frac{R + r}{h}$ is equal to
- (1) 4 (2) 2 (3) 1 (4) 3
51. A circle is inscribed in a triangle ABC touching the side AB at D such that $AD = 5$, $BD = 3$. If $\angle A = 60^\circ$ then length BC equals
- (1) 9 (2) $\frac{120}{13}$ (3) 13 (4) 12
52. In triangle ABC , medians AD and CE are drawn. If $AD = 5$, $\angle DAC = \pi/8$, and $\angle ACE = \pi/4$, then the area of the triangle ABC is equal to
- (1) $\frac{25}{9}$ (2) $\frac{25}{3}$ (3) $\frac{25}{18}$ (4) $\frac{10}{3}$
53. Let AD be a median of the $\triangle ABC$. If AE and AF are medians of the triangle ABD and ADC , respectively, and $AD = m_1$, $AE = m_2$, $AF = m_3$, then $a^2/8$ is equal to
- (1) $m_2^2 + m_3^2 - 2m_1^2$ (2) $m_1^2 + m_2^2 - 2m_3^2$
 (3) $m_1^2 + m_3^2 - 2m_2^2$ (4) none of these
54. For triangle ABC , $R = 5/2$ and $r = 1$. Let I be the incentre of the triangle and D , E , and F be the feet of the perpendiculars from I to BC , CA and AB , respectively. The value of $\frac{ID \times IE \times IF}{IA \times IB \times IC}$ is equal to
- (1) $\frac{5}{2}$ (2) $\frac{5}{4}$ (3) $\frac{1}{10}$ (4) $\frac{1}{5}$
55. In triangle ABC , $\angle A = 60^\circ$, $\angle B = 40^\circ$, and $\angle C = 80^\circ$. If P is the center of the circumcircle of triangle ABC with radius unity, then the radius of the circumcircle of triangle BPC is
- (1) 1 (2) $\sqrt{3}$ (3) 2 (4) $\sqrt{3}/2$
56. If H is the orthocenter of an acute angled triangle ABC whose circumcircle is $x^2 + y^2 = 16$, then circumdiameter of the triangle HBC is
- (1) 1 (2) 2 (3) 4 (4) 8
57. In triangle ABC , the line joining the circumcenter and incentre is parallel to side AC , then $\cos A + \cos C$ is equal to
- (1) $\frac{1}{2}$ (2) 1 (3) $\sqrt{3}$ (4) 2
58. In triangle ABC , line joining the circumcenter and orthocenter is parallel to side AC , then the value of $\tan A \tan C$ is equal to
- (1) $\sqrt{3}$ (2) 3
 (3) $3\sqrt{3}$ (4) none of these
59. In triangle ABC , $\angle C = 2\pi/3$ and CD is the internal angle bisector of $\angle C$, meeting the side AB at D . If Length CD is 1, then H.M. of a and b is equal to
- (1) 1 (2) 2 (3) 3 (4) 4
60. $\triangle ABC$ is equilateral triangle of side a . P lies on AB such that A is midpoint of PB . If r_1 is inradius of PAC and r_2 is ex radius of PBC opposite to P , then $r_1 + r_2 =$
- (1) $\frac{1}{2}$ (2) $\frac{3}{2}a$ (3) $\frac{\sqrt{3}}{2}a$ (4) $a\sqrt{2}$

61. A variable triangle ABC is circumscribed about a fixed circle of unit radius. Side BC always touches the circle at D and has fixed direction. If B and C vary in such a way that $(BD)(CD) = 2$, then locus of vertex A will be a straight line

- (1) parallel to side BC
- (2) perpendicular to side BC
- (3) making an angle $(\pi/6)$ with BC
- (4) making an angle $\sin^{-1}(2/3)$ with BC

62. In ΔABC , if $a = 10$ and $b \cot B + c \cot C = 2(r + R)$ then the maximum area of ΔABC will be

- (1) 50
- (2) $\sqrt{50}$
- (3) 25
- (4) 5

63. Let C be incircle of ΔABC . If the tangents of lengths t_1, t_2 and t_3 are drawn inside the given triangle parallel to sides a, b and c , respectively, then $\frac{t_1}{a} + \frac{t_2}{b} + \frac{t_3}{c}$ is equal to

- (1) 0
- (2) 1
- (3) 2
- (4) 3

64. In ΔABC with fixed length of AB , the internal bisector of angle C meets the side AB at D and the circumcircle at E . The maximum value of $CD \times DE$ is

- (1) c^2
- (2) $\frac{c^2}{2}$
- (3) $\frac{c^2}{4}$
- (4) none of these

65. In triangle ABC , if $r_1 = 2r_2 = 3r_3$, then $a:b$ is equal to

- (1) $\frac{5}{4}$
- (2) $\frac{4}{5}$
- (3) $\frac{7}{4}$
- (4) $\frac{4}{7}$

66. If in a triangle, $\left(1 - \frac{r_1}{r_2}\right)\left(1 - \frac{r_1}{r_3}\right) = 2$, then the triangle is

- (1) right angled
- (2) isosceles
- (3) equilateral
- (4) none of these

67. If in a triangle $\frac{r}{r_1} = \frac{r_2}{r_3}$, then

- (1) $A = 90^\circ$
- (2) $B = 90^\circ$
- (3) $C = 90^\circ$
- (4) none of these

68. In ΔABC , I is the incentre. Area of ΔIBC , ΔIAC and ΔIAB are, respectively, Δ_1, Δ_2 and Δ_3 . If the values of Δ_1, Δ_2 and Δ_3 are in A.P., then the altitudes of the ΔABC are in

- (1) A.P.
- (2) G.P.
- (3) H.P.
- (4) None of these

69. In an acute angled triangle ABC , $r + r_1 = r_2 + r_3$ and $\angle B > \frac{\pi}{3}$, then

- (1) $b + 2c < 2a < 2b + 2c$
- (2) $b + 4c < 4a < 2b + 4c$
- (3) $b + 4c < 4a < 4b + 4c$
- (4) $b + 3c < 3a < 3b + 3c$

70. If in triangle ABC , $\sum \sin \frac{A}{2} = \frac{6}{5}$ and $\sum II_1 = 9$ (where I_1, I_2 , and I_3 are excenters and I is incentre, then circumradius R is equal to

- (1) $\frac{15}{8}$
- (2) $\frac{15}{4}$
- (3) $\frac{15}{2}$
- (4) $\frac{4}{12}$

71. The radii r_1, r_2, r_3 of the escribed circles of the triangle ABC are in H.P. If the area of the triangle is 24 cm^2 and its perimeter is 24 cm , then the length of its largest side is

- (1) 10
- (2) 9
- (3) 8
- (4) none of these

72. In ΔABC with usual notations, if $r = 1, r_1 = 7$ and $R = 3$, the ΔABC is

- (1) equilateral
- (2) acute angled which is not equilateral
- (3) obtuse angled
- (4) right angled

73. Which of the following expresses the circumference of a circle inscribed in a sector OAB with radius R and $AB = 2a$?

- (1) $2\pi \frac{Ra}{R+a}$
- (2) $\frac{2\pi R^2}{a}$
- (3) $2\pi (R-a)^2$
- (4) $2\pi \frac{R}{R-a}$

74. In ΔABC , the median AD divides $\angle BAC$ such that $\angle BAD : \angle CAD = 2:1$. Then $\cos(A/3)$ is equal to

- (1) $\frac{\sin B}{2 \sin C}$
- (2) $\frac{\sin C}{2 \sin B}$
- (3) $\frac{2 \sin B}{\sin C}$
- (4) none of these

75. The area of the circle and the area of a regular polygon of n sides and of perimeter equal to that of the circle are in the ratio of

- (1) $\tan\left(\frac{\pi}{n}\right) : \frac{\pi}{n}$
- (2) $\cos\left(\frac{\pi}{n}\right) : \frac{\pi}{n}$
- (3) $\sin\left(\frac{\pi}{n}\right) : \frac{\pi}{n}$
- (4) $\cot\left(\frac{\pi}{n}\right) : \frac{\pi}{n}$

76. The ratio of the area of a regular polygon of n sides inscribed in a circle to that of the polygon of same number of sides circumscribing the same circle is $3:4$. Then the value of n is

- (1) 6
- (2) 4
- (3) 8
- (4) 12

77. In any triangle, the minimum value of $r_1 r_2 r_3 / r^3$ is equal to

- (1) 1
- (2) 9
- (3) 27
- (4) none of these

78. If R_1 is the circumradius of the pedal triangle of a given triangle ABC , and R_2 is the circumradius of the pedal triangle of the pedal triangle formed, and so on $R_3, R_4, \dots$,

then the value of $\sum_{i=1}^{\infty} R_i$, where R (circumradius) of ΔABC is 5 is

- (1) 8
- (2) 10
- (3) 12
- (4) 15

79. A sector $OABO$ of central angle θ is constructed in a circle with centre O and of radius 6. The radius of the circle that is circumscribed about the triangle OAB , is

(1) $6 \cos \frac{\theta}{2}$ (2) $6 \sec \frac{\theta}{2}$
 (3) $3 \sec \frac{\theta}{2}$ (4) $3 \left(\cos \frac{\theta}{2} + 2 \right)$

80. There is a point P inside an equilateral $\triangle ABC$ of side a whose distances from vertices A, B and C are 3, 4, and 5, respectively. Rotate the triangle and P through 60° about C . Let A go to A' and P to P' . Then the area of $\triangle PAP'$ (in sq. units) is

(1) 8 (2) 12 (3) 16 (4) 6

Multiple Correct Answers Type

1. The sides of $\triangle ABC$ satisfy the equation $2a^2 + 4b^2 + c^2 = 4ab + 2ac$. Then
 (1) the triangle is isosceles (2) the triangle is obtuse
 (3) $B = \cos^{-1}(7/8)$ (4) $A = \cos^{-1}(1/4)$
2. If sides of triangle ABC are a, b , and c such that $2b = a + c$, then
 (1) $\frac{b}{c} > \frac{2}{3}$ (2) $\frac{b}{c} > \frac{1}{3}$
 (3) $\frac{b}{c} < 2$ (4) $\frac{b}{c} < \frac{3}{2}$
3. If the sines of the angles A and B of a triangle ABC satisfy the equation $c^2x^2 - c(a+b)x + ab = 0$, then the triangle
 (1) is acute angled
 (2) is right angled
 (3) is obtuse angled
 (4) satisfies the equation $\sin A + \cos A = \frac{(a+b)}{c}$
4. There exists triangle ABC satisfying
 (1) $\tan A + \tan B + \tan C = 0$
 (2) $\frac{\sin A}{2} = \frac{\sin B}{3} = \frac{\sin C}{7}$
 (3) $(a+b)^2 = c^2 + ab$ and $\sqrt{2}(\sin A + \cos A) = \sqrt{3}$
 (4) $\sin A + \sin B = \frac{\sqrt{3}+1}{2}$,
 $\cos A \cos B = \frac{\sqrt{3}}{4} = \sin A \sin B$
5. In triangle, ABC if $2a^2b^2 + 2b^2c^2 = a^4 + b^4 + c^4$, then angle B is equal to
 (1) 45° (2) 135° (3) 120° (4) 60°
6. If in triangle ABC , a, c , and angle A are given and $c \sin A < a < c$, then (b_1 and b_2 are values of b)
 (1) $b_1 + b_2 = 2c \cos A$ (2) $b_1 + b_2 = c \cos A$
 (3) $b_1b_2 = c^2 - a^2$ (4) $b_1b_2 = c^2 + a^2$
7. If area of $\triangle ABC$ (Δ) and angle C are given and if the side c opposite to given angle is minimum, then
 (1) $a = \sqrt{\frac{2\Delta}{\sin C}}$ (2) $b = \sqrt{\frac{2\Delta}{\sin C}}$
 (3) $a = \frac{4\Delta}{\sin C}$ (4) $b = \frac{4\Delta}{\sin^2 C}$
8. If Δ represents the area of acute angled triangle ABC , then
 $\sqrt{a^2b^2 - 4\Delta^2} + \sqrt{b^2c^2 - 4\Delta^2} + \sqrt{c^2a^2 - 4\Delta^2} =$
 (1) $a^2 + b^2 + c^2$
 (2) $\frac{a^2 + b^2 + c^2}{2}$
 (3) $ab \cos C + bc \cos A + ca \cos B$
 (4) $ab \sin C + bc \sin A + ca \sin B$
9. Sides of $\triangle ABC$ are in A.P. If $a < \min\{b, c\}$, then $\cos A$ may be equal to
 (1) $\frac{4b-3c}{2b}$ (2) $\frac{3c-4b}{2c}$
 (3) $\frac{4c-3b}{2b}$ (4) $\frac{4c-3b}{2c}$
10. If the angles of a triangle are 30° and 45° , and the included side is $(\sqrt{3} + 1)$ cm, then
 (1) area of the triangle is $\frac{1}{2}(\sqrt{3} + 1)$ sq. units
 (2) area of the triangle is $\frac{1}{2}(\sqrt{3} - 1)$ sq. units
 (3) ratio of greater side to smaller side is $\frac{\sqrt{3}+1}{\sqrt{2}}$
 (4) ratio of greater side to smaller side is $\frac{1}{4\sqrt{3}}$
11. Lengths of the tangents from A, B , and C to the incircle are in A.P., then
 (1) r_1, r_2, r_3 are in H.P. (2) r_1, r_2, r_3 are in A.P.
 (3) a, b, c are in A.P. (4) $\cos A = \frac{4c-3b}{2c}$
12. CF is the internal bisector of angle C of $\triangle ABC$, then CF is equal to
 (1) $\frac{2ab}{a+b} \cos \frac{C}{2}$ (2) $\frac{a+b}{2ab} \cos \frac{C}{2}$
 (3) $\frac{b \sin A}{\sin\left(B + \frac{C}{2}\right)}$ (4) none of these
13. The incircle of $\triangle ABC$ touches side BC at D . The difference between BD and CD (R is circumradius of $\triangle ABC$) is
 (1) $\left| 4R \sin \frac{A}{2} \sin \frac{B-C}{2} \right|$ (2) $\left| 4R \cos \frac{A}{2} \sin \frac{B-C}{2} \right|$
 (3) $|b-c|$ (4) $\left| \frac{b-c}{2} \right|$

14. A circle of radius 4 cm is inscribed in $\triangle ABC$, which touches the side BC at D . If $BD = 6$ cm, $DC = 8$ cm then

(1) the triangle is necessarily acute angled

(2) $\tan \frac{A}{2} = \frac{4}{7}$

(3) perimeter of the triangle ABC is 42 cm

(4) area of $\triangle ABC$ is 84 cm^2

15. If H is the orthocentre of triangle ABC , $R =$ circumradius and $P = AH + BH + CH$, then

(1) $P = 2(R + r)$ (2) max. of P is $3R$

(3) min. of P is $3R$ (4) $P = 2(R - r)$

16. Let ABC be an isosceles triangle with base BC . If r is the radius of the circle inscribed in $\triangle ABC$ and r_1 is the radius of the circle escribed opposite to the angle A , then the product $r_1 r$ can be equal to (where R is the radius of the circumcircle of $\triangle ABC$)

(1) $R^2 \sin^2 A$ (2) $R^2 \sin^2 2B$

(3) $\frac{1}{2} a^2$ (4) $\frac{a^2}{4}$

17. If inside a big circle exactly $n (n \geq 3)$ small circles, each of radius r , can be drawn in such a way that each small circle touches the big circle and also touches both its adjacent small circles, then the radius of bigger circle is

(1) $r \left(1 + \operatorname{cosec} \frac{\pi}{n} \right)$ (2) $r \left(\frac{1 + \tan \pi/n}{\cos \pi/n} \right)$

(3) $r \left[1 + \operatorname{cosec} \frac{2\pi}{n} \right]$ (4) $\frac{r \left[\sin \frac{\pi}{2n} + \cos \frac{2\pi}{n} \right]^2}{\sin \pi/n}$

18. The area of a regular polygon of n sides is (where r is inradius, R is circumradius, and a is side of the triangle)

(1) $\frac{nR^2}{2} \sin \left(\frac{2\pi}{n} \right)$ (2) $nr^2 \tan \left(\frac{\pi}{n} \right)$

(3) $\frac{na^2}{4} \cot \frac{\pi}{n}$ (4) $nR^2 \tan \left(\frac{\pi}{n} \right)$

19. In acute angled triangle ABC , AD is the altitude. Circle drawn with AD as its diameter cuts AB and AC at P and Q , respectively. Length of PQ is equal to

(1) $\frac{\Delta}{2R}$ (2) $\frac{abc}{4R^2}$

(3) $2R \sin A \sin B \sin C$ (4) $\frac{\Delta}{R}$

20. If A is the area and $2s$ is the sum of the sides of a triangle, then

(1) $A \leq \frac{s^2}{4}$ (2) $A \leq \frac{s^2}{3\sqrt{3}}$

(3) $A < \frac{s^2}{\sqrt{3}}$

(4) none of these

21. In $\triangle ABC$, internal angle bisector of $\angle A$ meet side BC in D . $DE \perp AD$ meet AC in E and AB in F . Then

(1) AE is H.M. of b and c

(2) $AD = \frac{2bc}{b+c} \cos \frac{A}{2}$

(3) $EF = \frac{4bc}{b+c} \sin \frac{A}{2}$

(4) $\triangle AEF$ is isosceles

22. In a triangle ABC , $AB = 5$, $BC = 7$, $AC = 6$. A point P is in the plane such that it is at distance '2' units from AB and 3 units from AC then its distance from BC

(1) is $\frac{12\sqrt{6} - 28}{7}$ when P is inside the triangle

(2) may be $\frac{12\sqrt{6} - 8}{7}$ when P is outside the triangle

(3) may be $\frac{12\sqrt{6} + 8}{7}$ when P is inside the triangle

(4) may be $\frac{12\sqrt{6} + 14}{7}$ when P is outside the triangle

23. The base BC of $\triangle ABC$ is fixed and the vertex A moves, satisfying the condition $\cot \frac{B}{2} + \cot \frac{C}{2} = 2 \cot \frac{A}{2}$, then

(1) $b + c = a$

(2) $b + c = 2a$

(3) vertex A moves along a straight line

(4) vertex A moves along an ellipse

24. D, E and F are the middle points of the sides of the triangle ABC , then

(1) centroid of the triangle DEF is the same as that of ABC

(2) orthocenter of the triangle DEF is the circumcentre of ABC

(3) orthocenter of the triangle DEF is the incentre of ABC

(4) centroid of the triangle DEF is not the same as that of ABC

Linked Comprehension Type

For Problems 1-3

Given that $\Delta = 6$, $r_1 = 2$, $r_2 = 3$, $r_3 = 6$.

1. Circumradius R is equal to

(1) 2.5

(2) 3.5

(3) 1.5

(4) none of these

2. Inradius is equal to

(1) 2

(2) 1

(3) 1.5

(4) 2.5

3. Difference between the greatest and the least angles is

- (1) $\cos^{-1} \frac{4}{5}$ (2) $\tan^{-1} \frac{3}{4}$
 (3) $\cos^{-1} \frac{3}{5}$ (4) none of these

For Problems 4–6

Let $a = 6$, $b = 3$, and $\cos(A - B) = \frac{4}{5}$.

4. Area (in sq. units) of the triangle is equal to
 (1) 9 (2) 12 (3) 11 (4) 10
5. Angle C is equal to
 (1) $\frac{3\pi}{4}$ (2) $\frac{\pi}{4}$
 (3) $\frac{\pi}{2}$ (4) none of these
6. Value of $\sin A$ is equal to
 (1) $\frac{1}{2\sqrt{5}}$ (2) $\frac{1}{\sqrt{3}}$ (3) $\frac{1}{\sqrt{5}}$ (4) $\frac{2}{\sqrt{5}}$

For Problems 7–9

Let ABC be an acute angled triangle with orthocenter H . D , E , and F are the feet of perpendicular from A , B , and C , respectively, on opposite sides. Also, let R be the circumradius of $\triangle ABC$. Given $AH \cdot BH \cdot CH = 3$ and $(AH)^2 + (BH)^2 + (CH)^2 = 7$.

Then answer the following:

7. Value of $\frac{\cos A \cdot \cos B \cdot \cos C}{\cos^2 A + \cos^2 B + \cos^2 C}$ is
 (1) $\frac{3}{14R}$ (2) $\frac{3}{7R}$ (3) $\frac{7}{3R}$ (4) $\frac{14}{3R}$
8. Value of R is
 (1) 1 (2) $\frac{3}{2}$ (3) $\frac{5}{2}$ (4) None
9. Value of $HD \cdot HE \cdot HF$ is
 (1) $\frac{9}{64R^3}$ (2) $\frac{9}{8R^3}$ (3) $\frac{8}{9R^3}$ (4) $\frac{64}{9R^3}$

For Problems 10–12

Let O be a point inside $\triangle ABC$ such that $\angle OAB = \angle OBC = \angle OCA = \theta$.

10. $\cot A + \cot B + \cot C$ is equal to
 (1) $\tan^2 \theta$ (2) $\cot^2 \theta$ (3) $\tan \theta$ (4) $\cot \theta$
11. $\operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C$ is equal to
 (1) $\cot^2 \theta$ (2) $\operatorname{cosec}^2 \theta$ (3) $\tan^2 \theta$ (4) $\sec^2 \theta$
12. Area of $\triangle ABC$ is equal to
 (1) $\left(\frac{a^2 + b^2 + c^2}{4} \right) \tan \theta$ (2) $\left(\frac{a^2 + b^2 + c^2}{4} \right) \cot \theta$
 (3) $\left(\frac{a^2 + b^2 + c^2}{2} \right) \tan \theta$ (4) $\left(\frac{a^2 + b^2 + c^2}{2} \right) \cot \theta$

For Problems 13–15

Given an isosceles triangle with equal side of length b and base angle $\alpha < \pi/4$, then

13. the circumradius R is given by
 (1) $\frac{1}{2} b \operatorname{cosec} \alpha$ (2) $b \operatorname{cosec} \alpha$
 (3) $2b$ (4) none of these
14. the inradius r is given by
 (1) $\frac{b \sin 2\alpha}{2(1 - \cos \alpha)}$ (2) $\frac{b \sin 2\alpha}{2(1 + \cos \alpha)}$
 (3) $\frac{b \sin \alpha}{2}$ (4) $\frac{b \sin \alpha}{2(1 + \sin \alpha)}$
15. the distance between circumcenter O and incenter I is
 (1) $\left| \frac{b \cos(3\alpha/2)}{2 \sin \alpha \cos(\alpha/2)} \right|$ (2) $\left| \frac{b \cos 3\alpha}{\sin 2\alpha} \right|$
 (3) $\left| \frac{b \cos 3\alpha}{\cos \alpha \sin(\alpha/2)} \right|$ (4) $\left| \frac{b}{\sin \alpha \cos \alpha/2} \right|$

For Problems 16–18

Incircle of $\triangle ABC$ touches the sides BC , AC , and AB at D , E , and F respectively. Then answer the following questions:

16. $\angle DEF$ is equal to
 (1) $\frac{\pi - B}{2}$ (2) $\pi - 2B$
 (3) $A - C$ (4) none of these
17. Area of $\triangle DEF$ is
 (1) $2r^2 \sin(2A) \sin(2B) \sin(2C)$
 (2) $2r^2 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$
 (3) $2r^2 \sin(A - B) \sin(B - C) \sin(C - A)$
 (4) none of these
18. The length of side EF is
 (1) $r \sin \frac{A}{2}$ (2) $2r \sin \frac{A}{2}$
 (3) $r \cos \frac{A}{2}$ (4) $2r \cos \frac{A}{2}$

For Problems 19–21

Internal bisectors of $\triangle ABC$ meet the circumcircle at points D , E , and F .

19. Length of side EF is
 (1) $2R \cos \frac{A}{2}$ (2) $2R \sin \left(\frac{A}{2} \right)$
 (3) $R \cos \left(\frac{A}{2} \right)$ (4) $2R \cos \left(\frac{B}{2} \right) \cos \left(\frac{C}{2} \right)$
20. Area of $\triangle DEF$ is
 (1) $2R^2 \cos^2 \left(\frac{A}{2} \right) \cos^2 \left(\frac{B}{2} \right) \cos^2 \left(\frac{C}{2} \right)$

$$(2) 2R^2 \sin\left(\frac{A}{2}\right) \sin\left(\frac{B}{2}\right) \sin\left(\frac{C}{2}\right)$$

$$(3) 2R^2 \sin^2\left(\frac{A}{2}\right) \sin^2\left(\frac{B}{2}\right) \sin^2\left(\frac{C}{2}\right)$$

$$(4) 2R^2 \cos\left(\frac{A}{2}\right) \cos\left(\frac{B}{2}\right) \cos\left(\frac{C}{2}\right)$$

21. Ratio of area of triangle ABC and triangle DEF is
 (1) ≥ 1 (2) ≤ 1 (3) $\geq 1/2$ (4) $\leq 1/2$

For Problems 22–24

The area of any cyclic quadrilateral $ABCD$ is given by $A^2 = (s-a)(s-b)(s-c)(s-d)$, where $2s = a + b + c + d$, a, b, c , and d are the sides of the quadrilateral.

Now consider a cyclic quadrilateral $ABCD$ of area 1 sq. unit and answer the following questions:

22. The minimum perimeter of the quadrilateral is
 (1) 4 (2) 2
 (3) 1 (4) None of these
23. The minimum value of the sum of the lengths of diagonals is
 (1) $2\sqrt{2}$ (2) 2
 (3) $\sqrt{2}$ (4) None of these
24. When the perimeter is minimum, the quadrilateral is necessarily
 (1) a square
 (2) a rectangle but not a square
 (3) a rhombus but not a square
 (4) none of these

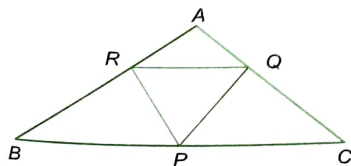
For Problems 25–27

In $\triangle ABC$, R, r, r_1, r_2, r_3 denote the circumradius, inradius, the radii opposite to the vertices A, B, C , respectively. Given that $r_1:r_2:r_3 = 1:2:3$.

25. The sides of the triangle are in the ratio
 (1) 1 : 2 : 3 (2) 3 : 5 : 7
 (3) 1 : 5 : 9 (4) 5 : 8 : 9
26. The value of $R : r$ is
 (1) 5 : 2 (2) 5 : 4 (3) 5 : 3 (4) 3 : 2
27. The greatest angle of the triangle is given by
 (1) $\cos^{-1}\left(\frac{1}{30}\right)$ (2) $\cos^{-1}\left(\frac{1}{3}\right)$
 (3) $\cos^{-1}\left(\frac{1}{10}\right)$ (4) $\cos^{-1}\left(\frac{1}{5}\right)$

For Problems 28 and 29

In $\triangle ABC$, P, Q, R are the feet of angle bisectors from the vertices to their opposite sides as shown in the figure. $\triangle PQR$ is constructed.



28. If $\angle BAC = 120^\circ$, then measure of $\angle RPQ$ will be
 (1) 60° (2) 90° (3) 120° (4) 150°
29. If $AB = 7$ units, $BC = 8$ units, $AC = 5$ units, then the side PQ will be
 (1) $\frac{\sqrt{28}}{3}$ units (2) $\frac{\sqrt{88}}{3}$ units
 (3) $\frac{\sqrt{78}}{3}$ units (4) $\frac{\sqrt{84}}{3}$ units

For Problems 30–32

Let G be the centroid of triangle ABC and the circumcircle of triangle AGC touches the side AB at A .

30. If $BC = 6$, $AC = 8$, then the length of side AB is equal to
 (1) $\frac{1}{2}$ (2) $\frac{2}{\sqrt{3}}$
 (3) $5\sqrt{2}$ (4) none of these
31. If $\angle GAC = \frac{\pi}{3}$ and $a = 3b$, then $\sin C$ is equal to
 (1) $\frac{3}{4}$ (2) $\frac{1}{2}$
 (3) $\frac{2}{\sqrt{3}}$ (4) None of these
32. If $AC = 1$, then the length of the median of triangle ABC through the vertex A is equal to
 (1) $\frac{\sqrt{3}}{2}$ (2) $\frac{1}{2}$
 (3) $\frac{2}{\sqrt{3}}$ (4) $\frac{5}{\sqrt{2}}$

For Problems 33 and 34

The inradius in a right angled triangle with integer sides is r .

33. If $r = 4$, the greatest perimeter (in units) is.
 (1) 96 (2) 90
 (3) 60 (4) 48
34. If $r = 5$, the greatest area (in sq. units) is
 (1) 150 (2) 210
 (3) 330 (4) 450

Matrix Match Type

1.

List I	List II
a. $b > c \sin B$, $b < c$, and B is an acute angle	p. 0
b. $b > c \sin B$, $c < b$, and B is an acute angle	q. 2
c. $b > c \sin B$, $c < b$, and B is an obtuse angle	r. data is insufficient
d. $b > c \sin B$, $c > b$, and B is an obtuse angle	s. 1

2. In acute-angled triangle ABC

List I	List II
a. $\cos A, \cos B, \cos C$ are in A.P.	p. Distances of orthocenter from vertices of triangle are in A.P.
b. $\sin(A/2), \sin(B/2), \sin(C/2)$ are in A.P.	q. Distances of orthocenter from sides of triangle are in H.P.
c. Distances of circumcenter from the sides of the triangle ABC are in A.P.	r. Distances of incenter from vertices of triangle are in H.P.
d. Circumradii of triangles OBC, OAC , and OAB are in H.P. (where O is the circumcenter of triangle ABC)	s. Distances of incenter from excenters of triangle are in A.P.

3.

List I	List II
a. If the sines of the angles A and B of a triangle ABC satisfy the equation $c^2x^2 - c(a+b)x + ab = 0$, the triangle can be	p. right angled
b. If one angle of a triangle is 30° and the lengths of the sides adjacent to it are 40 and $40\sqrt{3}$, the triangle can be	q. isosceles
c. If two angles of a triangle ABC satisfy the equation $81^{\sin^2 x} + 81^{\cos^2 x} = 30$, then the triangle can be ($x \in (0, \pi/2)$)	r. equilateral
d. In triangle ABC , $\cos A \cos B + \sin A \sin B \sin C = 1$, then the triangle can be	s. obtuse angled

4. Let O be the circumcenter, H be the orthocenter, I be the incenter, and I_1, I_2, I_3 be the excenters of acute-angled $\triangle ABC$.

List I	List II
a. Angle subtended by OI at vertex A	p. $ B - C $
b. Angle subtended by HI at vertex A	q. $\frac{ B - C }{2}$
c. Angle subtended by OH at vertex A	r. $\frac{B + C}{2}$
d. Angle subtended by I_2I_3 at I_1	s. $\frac{B}{2} - C$

5. In triangle ABC , AD is perpendicular to BC and DE is perpendicular to AB .

List I	List II
a. Area of $\triangle ADB$	p. $\frac{b^2}{4} \sin 2C$
b. Area of $\triangle ADC$	q. $\frac{c^2}{4} \cos^2 B \sin 2B$
c. Area of $\triangle ADE$	r. $\frac{c^2}{4} \sin 2B$
d. Area of $\triangle BDE$	s. $\frac{c^2}{4} \sin^2 B \sin 2B$

Codes

	a	b	c	d
(1)	p	r	q	q
(2)	q	r	p	s
(3)	s	p	q	r
(4)	r	p	s	q

6. In a triangle ABC , $a = 7$, $b = 8$, $c = 9$, BD is the median and BE the altitude from the vertex B . Match the following lists.

List I	List II
a. $BD =$	p. 2
b. $BE =$	q. 7
c. $ED =$	r. $\sqrt{45}$
d. $AE =$	s. 6

Codes

	a	b	c	d
(1)	p	r	q	q
(2)	r	q	s	p
(3)	q	r	p	s
(4)	s	p	q	r

Numerical Value Type

- Suppose α, β, γ , and δ are the interior angles of regular pentagon, hexagon, decagon, and dodecagon, respectively then the value of $|\cos \alpha \sec \beta \cos \gamma \operatorname{cosec} \delta|$ is _____.
- Let $ABCDEFGHIJKL$ be a regular dodecagon. Then the value of $\frac{AB}{AF} + \frac{AF}{AB}$ is equal to _____.
- In a $\triangle ABC$, $b = 12$ units, $c = 5$ units and $\Delta = 30$ sq. units. If d is the distance between vertex A and incentre of the triangle then the value of d^2 is _____.
- In $\triangle ABC$, if $r = 1$, $R = 3$, and $s = 5$, then the value of $a^2 + b^2 + c^2$ is _____.
- Consider a $\triangle ABC$ in which the sides are $a = (n + 1)$, $b = (n + 2)$, $c = n$ with $\tan C = 4/3$, then the value of Δ is _____.
- In $\triangle AEX$, T is the midpoint of XE and P is the midpoint of ET . If $\triangle APE$ is equilateral of side length equal to unity, then the value of $(AX)^2$ is _____.
- In $\triangle ABC$, the incircle touches the sides BC, CA , and AB respectively, at D, E , and F . If the radius of the incircle is 4 units and BD, CE , and AF are consecutive integers, then the value of s , where s is a semi-perimeter of triangle, is _____.
- The altitudes from the angular points A, B , and C on the opposite sides BC, CA , and AB of $\triangle ABC$ are 210, 195, and 182, respectively. Then the value of a is _____.
- In $\triangle ABC$, if $\angle C = 3\angle A$, $BC = 27$, and $AB = 48$. Then the value of AC is _____.
- The area of a right triangle is 6864 sq. units. If the ratio of its legs is 143 : 24, then the value of r is _____.

11. In $\triangle ABC$, if $\cos A + \sin A - \frac{2}{\cos B + \sin B} = 0$, then the value of $\left(\frac{a+b}{c}\right)^4$ is _____.

12. In $\triangle ABC$, $\angle C = 2\angle A$, and $AC = 2BC$, then the value of $\frac{a^2 + b^2 + c^2}{R^2}$ (where R is circumradius of triangle) is _____.

13. In $\triangle ABC$, if $b(b+c) = a^2$ and $c(c+a) = b^2$, then $|\cos A \cdot \cos B \cdot \cos C|$ is _____.

14. The sides of triangle ABC satisfy the relations $a + b - c = 2$ and $2ab - c^2 = 4$, then the square of the area of triangle is _____.

15. The lengths of the tangents drawn from the vertices A, B , and C to the incircle of $\triangle ABC$ are 5, 3, and 2, respectively. If the lengths of the parts of tangents within the triangle which are drawn parallel to the sides BC, CA , and AB of the triangle to the incircle are α, β , and γ , respectively, then the value of $[\alpha + \beta + \gamma]$ (where $[\cdot]$ represents the greatest integer function) is _____.

16. If a, b , and c represent the lengths of sides of a triangle, then the possible integral value of $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$ is _____.

17. In triangle ABC , $\sin A \sin B + \sin B \sin C + \sin C \sin A = 9/4$ and $a = 2$, then the value of $\sqrt{3}\Delta$, where Δ is the area of triangle, is _____.

18. In $\triangle ABC$, $AB = 52, BC = 56, CA = 60$. Let D be the foot of the altitude from A and E be the intersection of the internal angle bisector of $\angle BAC$ with BC . The length DE is _____.

19. Points D, E are taken on the side BC of an acute angled triangle ABC , such that $BD = DE = EC$. If $\angle BAD = x, \angle DAE = y$ and $\angle EAC = z$ then the value of $\frac{\sin(x+y)\sin(y+z)}{\sin x \sin z}$ is _____.

20. For a triangle $ABC, R = \frac{5}{2}$ and $r = 1$. Let D, E and F be the feet of the perpendiculars from incentre I to BC, CA and AB , respectively. Then the value of $\frac{(IA)(IB)(IC)}{(ID)(IE)(IF)}$ is equal to _____.

21. Circumradius of $\triangle ABC$ is 3 cm and its area is 6 cm^2 . If DEF is the triangle formed by feet of the perpendiculars drawn from A, B and C on the sides BC, CA and AB , respectively, then the perimeter of $\triangle DEF$ (in cm) is _____.

22. The distances of incentre of the right-angled triangle ABC (right angled at A) from B and C are $\sqrt{10}$ and $\sqrt{5}$, respectively. The perimeter of the triangle is _____.

Archives

JEE MAIN

Single Correct Answer Type

1. For a regular polygon, let r and R be the radii of the inscribed and the circumscribed circles, respectively. A false statement among the following is

(1) There is a regular polygon with $\frac{r}{R} = \frac{\sqrt{3}}{2}$

(2) There is a regular polygon with $\frac{r}{R} = \frac{1}{2}$

(3) There is a regular polygon with $\frac{r}{R} = \frac{1}{\sqrt{2}}$

(4) There is a regular polygon with $\frac{r}{R} = \frac{2}{3}$

(AIEEE 2010)

2. $ABCD$ is a trapezium such that AB and CD are parallel and $BC \perp CD$. If $\angle ADB = \theta, BC = p$ and $CD = q$, then AB is equal to:

(1) $\frac{(p^2 + q^2) \sin \theta}{p \cos \theta + q \sin \theta}$

(2) $\frac{(p^2 + q^2) \cos \theta}{p \cos \theta + q \sin \theta}$

(3) $\frac{p^2 + q^2}{p^2 \cos \theta + q^2 \sin \theta}$

(4) $\frac{(p^2 + q^2) \sin \theta}{(p \cos \theta + q \sin \theta)}$

(JEE Main 2013)

JEE ADVANCED

Single Correct Answer Type

1. Let ABC be a triangle such that $\angle ACB = \pi/6$ and let a, b , and c denote the lengths of the side opposite to A, B , and C , respectively. The value(s) of x for which $a = x^2 + x + 1, b = x^2 - 1$, and $c = 2x + 1$ is (are)

(1) $-(2 + \sqrt{3})$

(2) $1 + \sqrt{3}$

(3) $2 + \sqrt{3}$

(4) $4\sqrt{3}$

(IIT-JEE 2010)

2. If the angles A, B and C of a triangle are in an arithmetic progression and if a, b and c denote the lengths of the sides opposite to A, B and C respectively, then the value of the expression $\frac{a}{c} \sin 2C + \frac{c}{a} \sin 2A$ is

(1) $\frac{1}{2}$

(2) $\frac{\sqrt{3}}{2}$

(3) 1

(4) $\sqrt{3}$

(IIT-JEE 2010)

3. Let PQR be a triangle of area Δ with $a = 2$, $b = 7/2$, and $c = 5/2$, where a , b , and c are the lengths of the sides of the triangle opposite to the angles at P , Q , and R , respectively.

Then $\frac{2\sin P - \sin 2P}{2\sin P + \sin 2P}$ equals

- (1) $\frac{3}{4\Delta}$ (2) $\frac{45}{4\Delta}$ (3) $\left(\frac{3}{4\Delta}\right)^2$ (4) $\left(\frac{45}{4\Delta}\right)^2$

(IIT-JEE 2012)

Multiple Correct Answers Type

1. In a triangle ABC with fixed base BC , the vertex A moves such that $\cos B + \cos C = 4 \sin^2 A/2$.

If a , b , and c denote the lengths of the sides of the triangle opposite to the angles A , B , and C , respectively, then

- (1) $b + c = 4a$
 (2) $b + c = 2a$
 (3) locus of point A is an ellipse
 (4) locus of point A is a pair of straight lines

(IIT-JEE 2009)

2. In a triangle PQR , P is the largest angle and $\cos P = 1/3$. Further the incircle of the triangle touches the sides PQ , QR , and PR at N , L , and M , respectively, such that the length of PN , QL , and RM are consecutive even integers. Then possible length(s) of the side(s) of the triangle is (are)

- (1) 16 (2) 18 (3) 24 (4) 22

(JEE Advanced 2013)

3. In a triangle XYZ , let x , y , z be the lengths of sides opposite to the angles X , Y , Z , respectively, and $2s = x + y + z$. If

$$\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2} \text{ and area of incircle of the triangle}$$

XYZ is $\frac{8\pi}{3}$, then

- (1) area of the triangle XYZ is $6\sqrt{6}$
 (2) the radius of circumcircle of the triangle XYZ is $\frac{35}{6}\sqrt{6}$
 (3) $\sin \frac{X}{2} \sin \frac{Y}{2} \sin \frac{Z}{2} = \frac{4}{35}$
 (4) $\sin^2 \left(\frac{X+Y}{2} \right) = \frac{3}{5}$

(JEE Advanced 2016)

4. In a triangle PQR , let $\angle PQR = 30^\circ$ and the sides PQ and QR have lengths $10\sqrt{3}$ and 10, respectively. Then, which of the following statement(s) is (are) TRUE?

- (1) $\angle QPR = 45^\circ$
 (2) The area of the triangle PQR is $25\sqrt{3}$ and $\angle QRP = 120^\circ$
 (3) The radius of the incircle of the triangle PQR is $10\sqrt{3} - 15$
 (4) The area of the circumcircle of the triangle PQR is 100π

(JEE Advanced 2018)

Matrix Match Type

1. Match the statements/expressions in List I with the statements/expressions in List II.

List I	List II
a. In a triangle ΔXYZ , let a , b and c be the lengths of the sides opposite to the angles X , Y and Z respectively. If $2(a^2 - b^2) = c^2$ and $\lambda = \frac{\sin(X-Y)}{\sin Z}$ then possible values of n for which $\cos(n\pi\lambda) = 0$ is (are)	p. 1
b. In a triangle ΔXYZ , let a , b and c be the lengths of the sides opposite to the angles X , Y and Z , respectively. If $1 + \cos 2X - 2\cos 2Y = 2\sin X \sin Y$, then possible value(s) of $\frac{a}{b}$ is (are)	q. 2
c. In R^2 , let $\sqrt{3}\hat{i} + \hat{j}$, $\hat{i} + \sqrt{3}\hat{j}$ and $\beta\hat{i} + (1-\beta)\hat{j}$ be the position vectors of X , Y and Z with respect of the origin O , respectively. If the distance of Z from the bisector of the acute angle of $\overrightarrow{OX}$ and $\overrightarrow{OY}$ is $\frac{3}{\sqrt{2}}$, then possible value(s) of $ \beta $ is (are)	r. 3
d. Suppose that $F(\alpha)$ denotes the area of the region bounded by $x = 0$, $x = 2$, $y^2 = 4x$ and $y = \alpha x - 1 + \alpha x - 2 + \alpha x$, where $\alpha \in \{0, 1\}$. Then the value(s) of $F(\alpha) + \frac{8}{3}\sqrt{2}$, when $\alpha = 0$ and $\alpha = 1$, is (are)	s. 5
	t. 6

(JEE Advanced 2015)

Numerical Value Type

1. Let ABC and ABC' be two non-congruent triangles with sides $AB = 4$, $AC = AC' = 2\sqrt{2}$ and angle $B = 30^\circ$. The absolute value of the difference between the areas of these triangles is _____. (IIT-JEE 2009)

2. Two parallel chords of a circle of radius 2 are at a distance $\sqrt{3} + 1$ apart. If the chord subtend angles $\frac{\pi}{k}$ and $\frac{2\pi}{k}$ at the center, where $k > 0$, then the value of $[k]$ is _____. (IIT-JEE 2010)

- (Note: $[k]$ denotes the largest integer less than or equal to k)
3. Consider a triangle ABC and let a , b and c denote the lengths of the sides opposite to vertices A , B , and C , respectively. Suppose $a = 6$, $b = 10$, and the area of triangle is $15\sqrt{3}$. If $\angle ACB$ is obtuse and if r denotes the radius of the incircle of the triangle, then the value of r^2 is _____. (IIT-JEE 2010)

Answers Key

EXERCISES

Single Correct Answer Type

- | | | | | |
|---------|---------|---------|---------|---------|
| 1. (4) | 2. (2) | 3. (2) | 4. (1) | 5. (4) |
| 6. (4) | 7. (3) | 8. (3) | 9. (2) | 10. (1) |
| 11. (2) | 12. (3) | 13. (3) | 14. (4) | 15. (2) |
| 16. (2) | 17. (3) | 18. (1) | 19. (3) | 20. (1) |
| 21. (1) | 22. (2) | 23. (3) | 24. (3) | 25. (1) |
| 26. (3) | 27. (1) | 28. (1) | 29. (1) | 30. (1) |
| 31. (4) | 32. (1) | 33. (4) | 34. (2) | 35. (1) |
| 36. (2) | 37. (1) | 38. (1) | 39. (2) | 40. (1) |
| 41. (1) | 42. (1) | 43. (4) | 44. (3) | 45. (2) |
| 46. (2) | 47. (2) | 48. (1) | 49. (4) | 50. (3) |
| 51. (3) | 52. (2) | 53. (1) | 54. (3) | 55. (1) |
| 56. (4) | 57. (2) | 58. (2) | 59. (2) | 60. (3) |
| 61. (1) | 62. (3) | 63. (2) | 64. (3) | 65. (1) |
| 66. (1) | 67. (3) | 68. (3) | 69. (4) | 70. (1) |
| 71. (1) | 72. (4) | 73. (1) | 74. (1) | 75. (1) |
| 76. (1) | 77. (3) | 78. (2) | 79. (3) | 80. (4) |

Multiple Correct Answers Type

- | | |
|------------------------|------------------------|
| 1. (1), (3), (4) | 2. (1), (3) |
| 3. (2), (4) | 4. (3), (4) |
| 5. (1), (2) | 6. (1), (3) |
| 7. (1), (2) | 8. (2), (3) |
| 9. (1), (4) | 10. (1), (3) |
| 11. (1), (3), (4) | 12. (1), (3) |
| 13. (1), (3) | 14. (1), (2), (3), (4) |
| 15. (1), (2) | 16. (1), (2), (4) |
| 17. (1), (4) | 18. (1), (2), (3) |
| 19. (3), (4) | 20. (1), (2) |
| 21. (1), (2), (3), (4) | 22. (1), (2), (3) |
| 23. (2), (4) | 24. (1), (2) |

Linked Comprehension Type

- | | | | | |
|---------|---------|---------|---------|---------|
| 1. (1) | 2. (2) | 3. (3) | 4. (1) | 5. (3) |
| 6. (4) | 7. (1) | 8. (2) | 9. (2) | 10. (4) |
| 11. (2) | 12. (1) | 13. (1) | 14. (2) | 15. (1) |

- | | | | | |
|---------|---------|---------|---------|---------|
| 16. (1) | 17. (2) | 18. (4) | 19. (1) | 20. (4) |
| 21. (2) | 22. (1) | 23. (1) | 24. (1) | 25. (4) |
| 26. (1) | 27. (3) | 28. (2) | 29. (4) | 30. (3) |
| 31. (2) | 32. (1) | 33. (2) | 34. (3) | |

Matrix Match Type

- $a \rightarrow q, b \rightarrow s, c \rightarrow s, d \rightarrow p.$
- $a \rightarrow p, q; b \rightarrow r, s; c \rightarrow p, q; d \rightarrow p, q.$
- $a \rightarrow p; b \rightarrow q, s; c \rightarrow p; d \rightarrow p, q.$
- $a \rightarrow q, b \rightarrow q, c \rightarrow p, d \rightarrow r.$
- (4)
- (3)

Numerical Value Type

- | | | | | |
|----------|----------|-------------|--------------|---------|
| 1. (1) | 2. (4) | 3. (8) | 4. (24) | 5. (84) |
| 6. (13) | 7. (21) | 8. (211.25) | | 9. (35) |
| 10. (22) | 11. (4) | 12. (8) | 13. (-0.125) | |
| 14. (3) | 15. (6) | 16. (2) | 17. (3) | 18. (6) |
| 19. (4) | 20. (10) | 21. (4) | 22. (12) | |

ARCHIVES

JEE MAIN

Single Correct Answer Type

1. (4) 2. (1)

JEE ADVANCED

Single Correct Answer Type

1. (2) 2. (4) 3. (3)

Multiple Correct Answers Type

- | | |
|------------------|------------------|
| 1. (2), (3) | 2. (2), (4) |
| 3. (1), (3), (4) | 4. (2), (3), (4) |

Matrix Match Type

1. $(a) \rightarrow (p), (r), (s); (b) \rightarrow (p)$

Numerical Value Type

1. (4) 2. (3) 3. (3)

6

Height and Distance

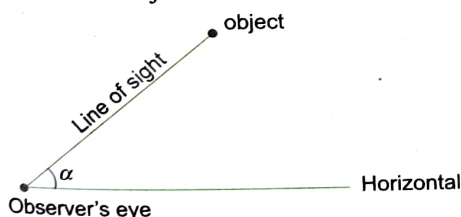
INTRODUCTION

We have studied about trigonometric ratios and their applications in solving triangles involved in different cases. These ratios are also useful to solve problems regarding heights and distances around us in real life. This is why scholars around the world have been studying trigonometry for ages. It is not possible to measure every distance using measuring tape, for instance, altitude of an aeroplane at a certain time, distance of a ship from a light house, height of a hill (distance between its foot and summit), distance between two celestial objects etc.. To measure such distances, scientists developed the method of trigonometric ratios. When dealing with heights and depths, we have to measure two types of angles (above and below the observer's eye-level). The instruments called theodolite and sextant are used to measure these angles and then the method of solution of triangles is used to find the required height or distance.

ANGLE OF ELEVATION AND DEPRESSION

ANGLE OF ELEVATION

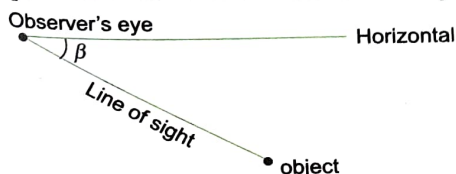
If the object is above the observer's eye-level, the angle between the horizontal line and the line of sight (line from the observer's eye to the object) is called the angle of elevation of the object. Here, the horizontal line is taken in the same vertical plane as that of the observer and the object.



In the figure, α is the angle of elevation.

ANGLE OF DEPRESSION

If the object is below the observer's eye-level, then the angle between the horizontal and the observer's line of sight is called the angle of depression. Here, the horizontal line is taken in the same vertical plane as that of the observer and the object.



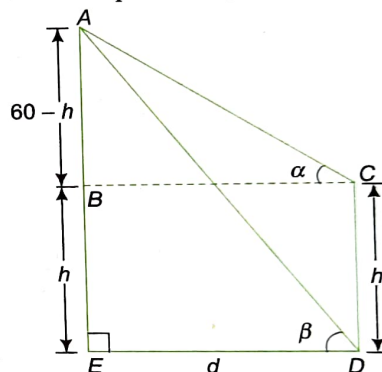
In the figure, β is the angle of depression.

Solved Examples

EXAMPLE 6.1

From the top of a tower, 60 meters high, the angles of depression of the top and bottom of a pole are α and β , respectively. Find the height of the pole.

Sol. Let AE be the tower, CD be the pole and d be the distance between the tower and the pole. Also, let h be the height of the pole.



In triangle ABC , $d = (60 - h) \cot \alpha$

In triangle AED , $d = 60 \cot \beta$

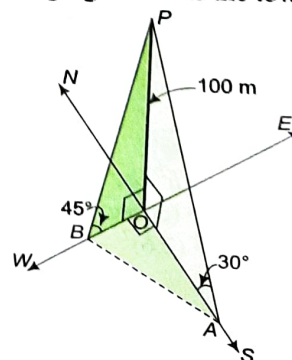
$$\therefore 60 \cot \beta = (60 - h) \cot \alpha$$

$$\Rightarrow h = \frac{60(\cot \alpha - \cot \beta)}{\cot \alpha}$$

EXAMPLE 6.2

The angle of elevation of the top of a tower from a point A due south of it is 30° and from a point B due west of it is 45° . If the height of the tower is 100 meters, then find the distance AB .

Sol. In the following figure OP is the tower.



$$OP = 100 \text{ m}$$

In right angled triangle POA ,

$$\tan 30^\circ = \frac{100}{OA}$$

$$\therefore OA = 100 \cot 30^\circ = 100\sqrt{3}$$

In right angled triangle POB ,

$$\tan 45^\circ = \frac{100}{OB}$$

$$\therefore OB = 100 \cot 45^\circ = 100$$

In right angled triangle AOB , using Pythagoras theorem, we get

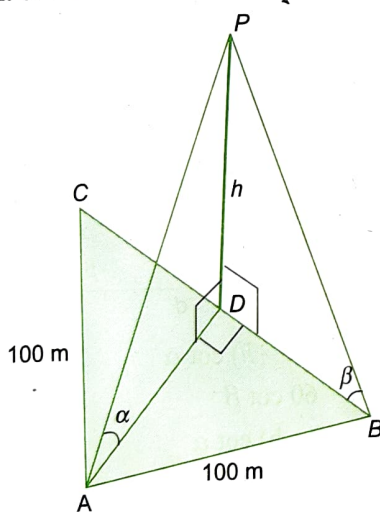
$$\begin{aligned} AB^2 &= OA^2 + OB^2 \\ &= 3 \times 100^2 + 100^2 \\ &= 4 \times 100^2 \end{aligned}$$

$$\therefore AB = 2 \times 100 = 200 \text{ m}$$

EXAMPLE 6.3

ABC is a triangular park with $AB = AC = 100 \text{ m}$. A clock tower is situated at the midpoint of BC . The angles of elevation α and β of the top of the tower at A and B , respectively, are such that $\cot \alpha = 3.2$ and $\operatorname{cosec} \beta = 2.6$. Find the height of the tower.

Sol. In the figure DP is a clock tower of height ' h ' standing at D , the midpoint of BC .



Since $AB = AC$, AD is perpendicular to BC .

In triangle ADP ,

$$\tan \alpha = \frac{h}{AD}$$

$$\Rightarrow AD = h \cot \alpha$$

In triangle PDB ,

$$\tan \beta = \frac{h}{BD}$$

$$\Rightarrow BD = h \cot \beta$$

Now, in right triangle ADB ,

$$AB^2 = AD^2 + BD^2$$

$$\Rightarrow 100^2 = h^2 \cot^2 \alpha + h^2 \cot^2 \beta$$

$$\Rightarrow 100^2 = h^2 [\cot^2 \alpha + (\operatorname{cosec}^2 \beta - 1)]$$

$$\Rightarrow 100^2 = h^2 [(3.2)^2 + (2.6)^2 - 1]$$

$$\Rightarrow 100^2 = 16 h^2$$

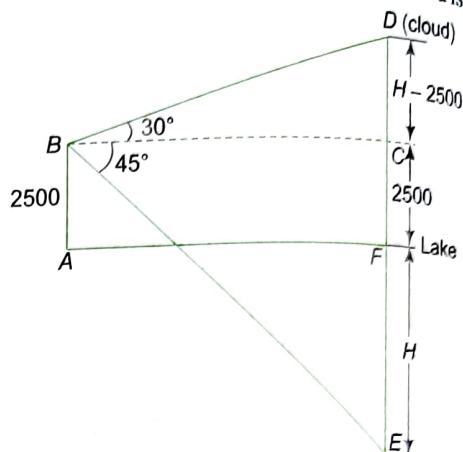
$$\Rightarrow h^2 = 625$$

$$\Rightarrow h = 25 \text{ m}$$

EXAMPLE 6.4

The angle of elevation of a stationary cloud from a point 2500 feet above a lake is 30° and the angle of depression of its reflection in the lake is 45° . Find the height of cloud above the lake water surface.

Sol. In the figure, observer is at point B and cloud is at point D .



E is the reflection of cloud in the lake.

$$\therefore FD = FE = H$$

In triangle BCD ,

$$\tan 30^\circ = \frac{H - 2500}{BC}$$

$$\Rightarrow H - 2500 = BC \tan 30^\circ$$

In triangle BCE ,

$$\tan 45^\circ = \frac{H + 2500}{BC}$$

$$\Rightarrow H + 2500 = BC \tan 45^\circ$$

$$\therefore \frac{H - 2500}{H + 2500} = \frac{BC \tan 30^\circ}{BC \tan 45^\circ} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \sqrt{3} (H - 2500) = H + 2500$$

$$\Rightarrow (\sqrt{3} - 1) H = 2500 (\sqrt{3} + 1)$$

$$\Rightarrow H = 2500 \times \frac{\sqrt{3} + 1}{\sqrt{3} - 1} \text{ feet}$$

EXAMPLE 6.5

Some portion of a 20 metres long tree is broken by the wind and its top struck the ground at an angle of 30° . Find the height of the point where the tree is broken.

Sol. Let AD be the tree before it was broken.

$$AD = 20 \text{ m}$$

Tree is broken at point B and the broken portion is touching the ground at point C .

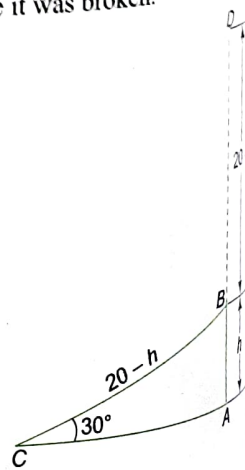
In triangle BAC ,

$$\sin 30^\circ = \frac{h}{20 - h}$$

$$\Rightarrow \frac{1}{2} = \frac{h}{20 - h}$$

$$\Rightarrow 20 - h = 2h$$

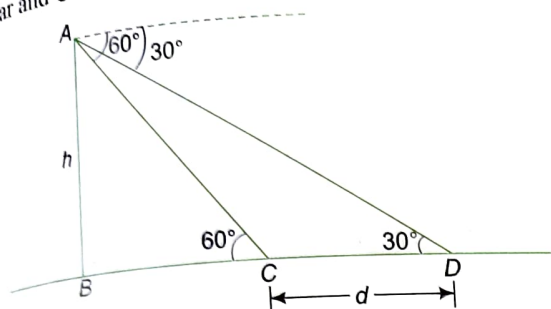
$$\Rightarrow h = \frac{20}{3} \text{ m}$$



EXAMPLE 6.6

An observer on the top of a tree, finds the angle of depression of a car moving towards the tree to be 30° . After 3 minutes this angle becomes 60° . After how much more time, the car will reach the tree?

Sol. Let the observer be at point A . Also, let D be initial position of the car and C the position after 3 minutes.



$$\text{In } \triangle ABD, BD = h \cot 30^\circ$$

$$\text{In } \triangle ABC, BC = h \cot 60^\circ$$

$$CD = BD - BC$$

$$d = h \cot 30^\circ - h \cot 60^\circ$$

$$\text{Speed of the car} = \frac{h(\cot 30^\circ - \cot 60^\circ)}{3}$$

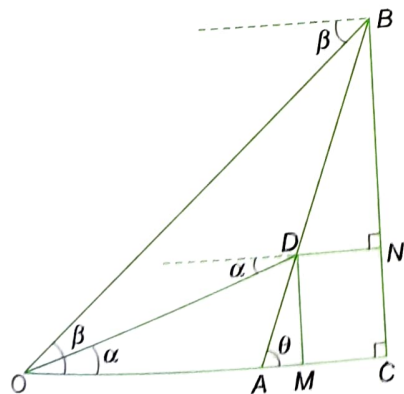
$$\text{Time required to travel distance } BC$$

$$= \frac{h \cot 60^\circ \times 3}{h(\cot 30^\circ - \cot 60^\circ)} = 1.5 \text{ minute}$$

EXAMPLE 6.7.

A man observes that when he has climbed up $\frac{1}{3}$ of the length of an inclined ladder, placed against a wall, the angular depression of an object on the floor is α and that after he reached the top of the ladder, the angular depression is β . If the inclination of the ladder to the floor is θ , then prove that $\cot \theta = \frac{3 \cot \beta - \cot \alpha}{2}$.

Sol. Let AB be the ladder and the object be at point O .



Also, let l be the length of the ladder and the man be at point D initially.

$$\text{In triangle } AMD, DM = AD \sin \theta = \frac{1}{3} l \sin \theta$$

$$\text{In triangle } ACB, BC = l \sin \theta$$

$$\text{In triangle } OMD, OM = DM \cot \alpha = \frac{1}{3} l \sin \theta \cot \alpha$$

$$\text{In triangle } OCB, OC = BC \cot \beta = l \sin \theta \cot \beta$$

$$\text{Now, } MC = OC - OM = l \sin \theta \cot \beta - \frac{1}{3} l \sin \theta \cot \alpha \quad \dots(1)$$

$$\text{Also, } MC = DN = \frac{2}{3} l \cos \theta \quad (\text{From triangle } BND)$$

$$\therefore l \sin \theta \cot \beta - \frac{1}{3} l \sin \theta \cot \alpha = \frac{2}{3} l \cos \theta$$

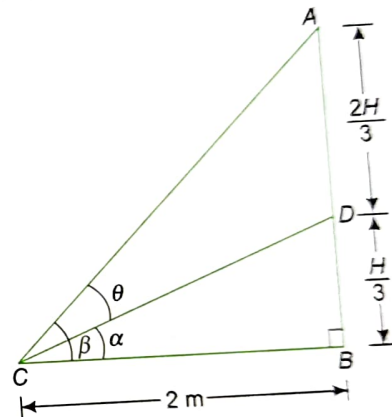
$$\Rightarrow (3 \cot \beta - \cot \alpha) \sin \theta = 2 \cos \theta$$

$$\Rightarrow \cot \theta = \frac{3 \cot \beta - \cot \alpha}{2}$$

EXAMPLE 6.8

A vertical pole consists of two parts, the lower part being one third of the whole. At a point in the horizontal plane through the base of the pole and distance 2 m from it, the upper part of the pole subtends an angle whose tangent is $1/2$. Find the possible height of the pole.

Sol. Let AB be the pole and C be the point 2 m apart from B , the base of the pole.



In the above figure, $\tan \theta = 1/2$.

In triangle DBC ,

$$BC = BD \cot \alpha$$

$$\therefore 2 = \frac{H}{3} \cot \alpha$$

$$\text{or } \tan \alpha = \frac{H}{6}$$

In triangle ABC ,

$$BC = AB \cot \beta$$

$$\therefore 2 = H \cot \beta$$

$$\text{or } \tan \beta = \frac{H}{2}$$

$$\text{Now, } \tan \theta = \tan(\beta - \alpha)$$

$$\therefore \frac{1}{2} = \frac{\tan \beta - \tan \alpha}{1 + \tan \beta \tan \alpha} = \frac{\frac{H}{2} - \frac{H}{6}}{1 + \frac{H^2}{12}}$$

$$\Rightarrow 1 + \frac{H^2}{12} = \frac{2H}{3}$$

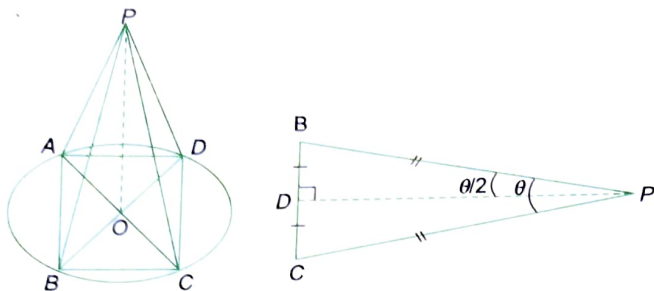
$$\Rightarrow H^2 - 8H + 12 = 0$$

$$\Rightarrow H = 2 \text{ m or } 6 \text{ m}$$

EXAMPLE 6.9

A circular ring of radius 3 cm hangs horizontally from a point 4 cm vertically above its centre by 4 strings attached at equal intervals to its circumference. If the angle between two consecutive strings is θ , then find the value of $\cos \theta$.

Sol. Let P be the point from which the ring is hanging. Also, let the strings be attached to the points A, B, C and D at the ring.



In the figure,

$$OP = 4$$

$$OA = OB = OC = OD = 3$$

$$AP = BP = CP = DP = \sqrt{3^2 + 4^2} = 5$$

$ABCD$ is a square of side length $3\sqrt{2}$ cm.

Now, in triangle PDB ,

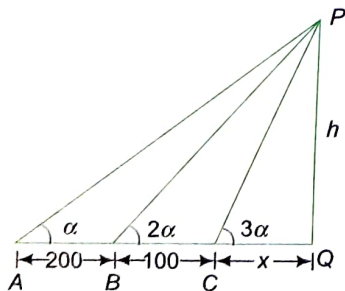
$$\sin \frac{\theta}{2} = \frac{\frac{1}{2}BC}{BP} = \frac{\frac{3}{\sqrt{2}}}{5} = \frac{3}{5\sqrt{2}}$$

$$\therefore \cos \theta = 1 - 2\sin^2 \frac{\theta}{2} = \frac{16}{25}$$

EXAMPLE 6.10

A balloon is observed simultaneously from three points A, B and C on a straight road directly under it. The angular elevation at B is twice and at C is thrice that of A . If the distance between A and B is 200 meters and the distance between B and C is 100 meters, then find the altitude of balloon from the road.

Sol. Let P be the balloon at height h above the road.



From the above figure,

$$x = h \cot 3\alpha \quad \dots(1)$$

$$(x + 100) = h \cot 2\alpha \quad \dots(2)$$

$$(x + 300) = h \cot \alpha \quad \dots(3)$$

Subtracting (1) from (2), we get

$$100 = h (\cot 2\alpha - \cot 3\alpha)$$

$$\therefore 100 = h \left(\frac{\sin \alpha}{\sin 3\alpha \sin 2\alpha} \right) \quad \dots(4)$$

Subtracting (2) from (3), we get

$$200 = h (\cot \alpha - \cot 2\alpha)$$

$$\therefore 200 = h \left(\frac{\sin \alpha}{\sin 2\alpha \sin \alpha} \right) \quad \dots(5)$$

Dividing (5) by (4), we get

$$\frac{\sin 3\alpha}{\sin \alpha} = \frac{200}{100} = 2$$

$$\Rightarrow 3 \sin \alpha - 4 \sin^3 \alpha - 2 \sin \alpha = 0$$

$$\Rightarrow \sin^2 \alpha = \frac{1}{4} = \sin^2 \frac{\pi}{6}$$

$$\Rightarrow \alpha = \frac{\pi}{6}$$

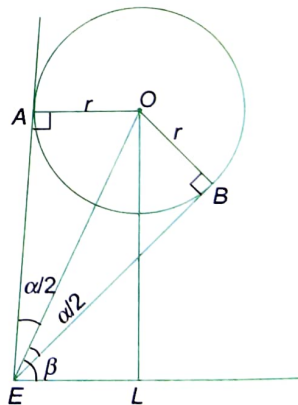
Using this value in (5), we get

$$\therefore h = 200 \sin \frac{\pi}{3} = 200 \frac{\sqrt{3}}{2} = 100\sqrt{3}$$

EXAMPLE 6.11

A balloon of radius r subtends an angle α at the eye of an observer and the elevation of the centre of the balloon from the eye is β . Find the height of the centre of the balloon from the eye of the observer.

Sol. Let O be the centre of the balloon of radius r which subtends an angle α at E , the eye of the observer.



EA and EB are the tangents to the balloon.

$$\therefore \angle OEA = \angle OEB = \alpha/2$$

Let the height OL of the centre of the balloon be h .

In $\triangle OLE$,

$$h = OE \sin \beta$$

$$= r \operatorname{cosec} (\alpha/2) \sin \beta$$

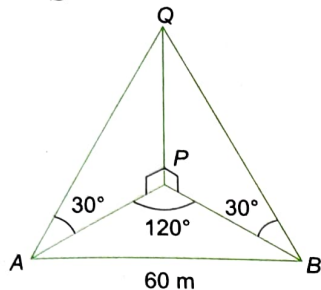
$$= \frac{r \sin \beta}{\sin (\alpha/2)}$$

(in triangle OEB)

EXAMPLE 6.12

A vertical tower PQ subtends the same angle of 30° at each of two points A and B , 60 m apart on the ground. If AB subtends an angle of 120° at P , the foot of the tower, then find the height of the tower.

Sol. In triangle APQ , we have



$$AP = PQ \cot 30^\circ = \sqrt{3} PQ$$

In triangle BPQ , we have

$$BP = PQ \cot 30^\circ = \sqrt{3} PQ$$

In $\triangle APB$, using cosine rule, we get

$$\cos 120^\circ = \frac{AP^2 + BP^2 - AB^2}{2AP \cdot BP}$$

$$\Rightarrow -\frac{1}{2} = \frac{6PQ^2 - 60^2}{2 \cdot 3PQ^2}$$

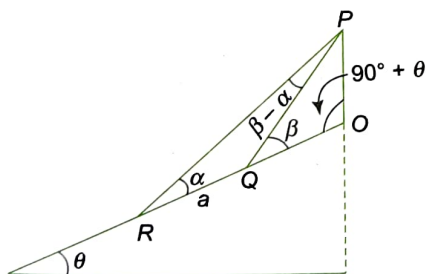
$$\Rightarrow 9PQ^2 = 60^2$$

$$\Rightarrow PQ = 20 \text{ m}$$

EXAMPLE 6.13

From a point on a hillside of constant inclination, the angle of elevation of the top of a flagstaff on its summit is observed to be α , and a metres nearer the top of the hill, it is β . If h is the height of the flagstaff, find the inclination of the hill to the horizon.

Sol. Let OP be the flagstaff of height h and θ the inclination of the hill.



Then $\angle ORP = \alpha$, $\angle OQP = \beta$ and $RQ = a$.

Also, $\angle RPQ = \beta - \alpha$.

Now, in triangle PQR , using sine rule, we get

$$\frac{PR}{\sin(180^\circ - \beta)} = \frac{RQ}{\sin(\beta - \alpha)}$$

$$\Rightarrow PR = \frac{a \sin \beta}{\sin(\beta - \alpha)} \quad \dots(1)$$

$$\text{Also, } \angle POR = \frac{\pi}{2} + \theta.$$

Now, in triangle POR , using sine rule, we get

$$\frac{PO}{\sin \alpha} = \frac{PR}{\sin\left(\frac{\pi}{2} + \theta\right)}$$

$$\Rightarrow PR = \frac{h \cos \theta}{\sin \alpha} = \frac{a \sin \beta}{\sin(\beta - \alpha)} \quad (\text{Using (1)})$$

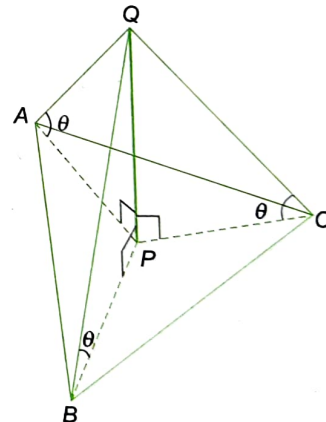
$$\Rightarrow \theta = \cos^{-1} \left[\frac{a \sin \alpha \sin \beta}{h \sin(\beta - \alpha)} \right]$$

EXAMPLE 6.14

PQ is a vertical tower and A, B, C are three points in the horizontal plane through the foot of the tower P . The angles of elevation of Q from A, B and C are equal and each measures θ . The sides of the triangle ABC are a, b, c and the area of the triangle ABC is Δ . Prove that the height of the tower is

$$\frac{abc}{4\Delta} \tan \theta.$$

Sol. Let the height of the tower PQ be h .



$$AP = BP = CP = h \cot \theta$$

Therefore, P is the circumcentre of $\triangle ABC$.

$$\therefore AP = BP = CP = R = \frac{abc}{4\Delta}$$

$$\therefore h = \frac{abc}{4\Delta} \tan \theta$$

Exercises

Single Correct Answer Type

- The tops of two poles of height 20 m and 14 m are connected by a wire. If the wire makes an angle 30° with the horizontal, then the length of the wire is
 (1) 8 m (2) 12 m (3) 10 m (4) 3 m
- The angle of elevation of the top of an unfinished tower at a point distant 120 m from its base is 45° . If the elevation of the proposed top at the same point is to be 60° , the tower must be raised to a height
 (1) $120(\sqrt{3} + 1)$ m (2) $120(\sqrt{3} - 1)$ m
 (3) $120\sqrt{3}$ m (4) 120 m
- A tower of height b subtends an angle at a point O on the ground level through the foot of the tower and at a distance a from the foot of the tower. A pole mounted on the top of the tower also subtends an equal angle at O . The height of the pole is
 (1) $a \left(\frac{a^2 - b^2}{a^2 + b^2} \right)$ (2) $a \left(\frac{a^2 + b^2}{a^2 - b^2} \right)$
 (3) $b \left(\frac{a^2 - b^2}{a^2 + b^2} \right)$ (4) $b \left(\frac{a^2 + b^2}{a^2 - b^2} \right)$
- A ladder rests against a wall making an angle α with the horizontal. The foot of the ladder is pulled away from the wall through a distance x , so that it slides a distance y down the wall making an angle β with the horizontal. The correct relation is
 (1) $y = x \tan \frac{\alpha + \beta}{2}$ (2) $x = y \tan \frac{\alpha + \beta}{2}$
 (3) $x = y \tan(\alpha + \beta)$ (4) $y = x \tan(\alpha + \beta)$
- Two flagstuffs stand on a horizontal plane. A and B are two points on the line joining their feet and between them. The angles of elevation of the tops of the flagstuffs as seen from A are 30° and 60° and as seen from B are 60° and 45° . If AB is 30 m, then the distance between the flagstuffs is
 (1) $30 + 15\sqrt{3}$ (2) $45 + 15\sqrt{3}$
 (3) $60 - 15\sqrt{3}$ (4) $60 + 15\sqrt{3}$
- A snake observes an eagle perching on the top of a pole 20 m high. Its elevation from snake's eye is 45° before it flies off horizontally straight away from the snake and after one second the elevation of the eagle reduces to 30° . The speed of the eagle is
 (1) 14.64 m/s (2) 17.71 m/s
 (3) 12 m/s (4) None of these
- For a man, the angle of elevation of the highest point of a tower situated west to him is 60° . On walking 240 metres to north, the angle of elevation reduces to 30° . The height of the tower is
 (1) $50\sqrt{3}$ m (2) $30\sqrt{6}$ m
 (3) $60\sqrt{6}$ m (4) 60 m
- A flagstaff stands in the centre of a rectangular field whose diagonal is 120 m. It subtends angles of 15° and 45° at the midpoints of the sides of the field. The height of the flagstaff is
 (1) 20 m (2) $30\sqrt{2 + \sqrt{3}}$ m
 (3) $30\sqrt{2 - \sqrt{3}}$ m (4) 40 m
- AB is a vertical pole resting at the end A on the level ground. P is a point on the level ground such that $AP = 3AB$ and C is the midpoint of AB . If AC and CB subtend angles α and β , respectively, at P , then the value of $\tan \beta$ is
 (1) $18/19$ (2) $3/19$ (3) $1/6$ (4) $1/3$
- From the bottom of a pole of height h , the angle of elevation of the top of a tower is α and the pole subtends angle β at the top of the tower. The height of the tower is
 (1) $\frac{h \cot(\alpha - \beta)}{\cot(\alpha - \beta) - \cot \alpha}$ (2) $\frac{h \tan(\alpha - \beta)}{\tan(\alpha - \beta) - \tan \alpha}$
 (3) $\frac{\cot(\alpha - \beta)}{\cot(\alpha - \beta) - \cot \alpha}$ (4) None of these
- A tower subtends an angle α at a point on the same level as the foot of the tower. At a second point, b meters above the first, the angle of depression of the foot of the tower is β . The height of the tower is
 (1) $b \cot \alpha \tan \beta$ (2) $b \tan \alpha \tan \beta$
 (3) $b \tan \alpha \cot \beta$ (4) $b \cot \alpha \cot \beta$
- A man standing on a level plane observes the elevation of the top of a pole to be θ . He then walks a distance equal to double the height of the pole and then finds that the elevation is now 2θ . The value of $\cot \theta$ is
 (1) $\sqrt{2} + 1$ (2) $2 - \sqrt{3}$
 (3) $\sqrt{2} - 1$ (4) $2 + \sqrt{3}$
- A 5 m high pole stands on a building of height 25 m. The pole and the building subtend equal angles at an antenna placed at a height of 30 m. The distance of the antenna from the top of the pole is
 (1) $5\sqrt{\frac{2}{3}}$ (2) $\frac{5\sqrt{3}}{2}$
 (3) $5\sqrt{\frac{3}{2}}$ (4) $5\sqrt{6}$
- A vertical tower stands on a declivity which is inclined at 15° to the horizontal. From the foot of the tower, a man ascends the declivity for 80 feet and then finds that the tower subtends an angle of 30° . The height of the tower is
 (1) $40(\sqrt{6} + \sqrt{2})$ (2) $20(\sqrt{6} - \sqrt{2})$
 (3) $40(\sqrt{6} - \sqrt{2})$ (4) $80(\sqrt{6} - \sqrt{2})$
- The length of the shadow of a pole inclined at 10° to the vertical towards the sun is 2.05 metres, when the elevation of the sun is 38° . The length of the pole is

- (1) $\frac{2.05 \sin 42^\circ}{\sin 38^\circ}$ (2) $\frac{2.05 \cos 38^\circ}{\cos 42^\circ}$
 (3) $\frac{2.05 \sin 38^\circ}{\sin 42^\circ}$ (4) None of these
16. A tower subtends angles α , 2α , and 3α , respectively, at points A , B and C all lying on a horizontal line through the foot of the tower. Then $\frac{AB}{BC} =$

- (1) $\frac{3 \sin \alpha}{\sin 2\alpha}$ (2) $1 + 2 \cos 2\alpha$
 (3) $2 + \cos 3\alpha$ (4) $\frac{\sin 2\alpha}{\sin \alpha}$

17. A harbour lies in a direction 60° south-west from a fort and at a distance 30 km from it. A ship sets out from the harbour at noon and sails due east at 10 km/hour. The ship will be 70 km from the fort at

- (1) 7 p.m. (2) 8 p.m. (3) 5 p.m. (4) 10 p.m.

18. A tower AB leans towards west making an angle α with the vertical. The angular elevation of B , the topmost point of the tower, is β , as observed from a point C due east of A at a distance d from A . If the angular elevation of B from a point D at a distance $2d$ due east of C is γ , then

- (1) $2 \tan \alpha = 2 \cot \beta - \cot \gamma$
 (2) $2 \tan \alpha = 3 \cot \beta - \cot \gamma$
 (3) $\tan \alpha = \cot \beta - \cot \gamma$
 (4) none of these

Archives

JEE MAIN

Single Correct Answer Type

1. A bird is sitting on the top of a vertical pole 20 m high and its elevation from a point O on the ground is 45° . It flies off horizontally straight away from the point O . After one second, the elevation of the bird from O is reduced to 30° . Then the speed (in m/s) of the bird is

- (1) $40(\sqrt{2} - 1)$ (2) $40(\sqrt{3} - \sqrt{2})$
 (3) $20\sqrt{2}$ (4) $20(\sqrt{3} - 1)$

(JEE Main 2014)

2. If the angles of elevation of the top of a tower from three collinear points A , B and C , on a line leading to the foot of the tower, are 30° , 45° and 60° , respectively, then the ratio, $AB : BC$, is

- (1) $\sqrt{3} : 1$ (2) $\sqrt{3} : \sqrt{2}$ (3) $1 : \sqrt{3}$ (4) $2 : 3$

(JEE Main 2014)

3. PQR is a triangular park with $PQ = PR = 200$ m. A TV tower stands at the mid-point of QR . If the angles of elevation of the top of the tower at P , Q and R are, respectively, 45° , 30° and 30° , then the height of the tower (in m) is

- (1) $50\sqrt{2}$ (2) 100 (3) 50 (4) $100\sqrt{3}$

(JEE Main 2018)

Answers Key

EXERCISES

Single Correct Answer Type

- | | | | | |
|---------|---------|---------|---------|---------|
| 1. (2) | 2. (2) | 3. (4) | 4. (2) | 5. (4) |
| 6. (1) | 7. (3) | 8. (3) | 9. (2) | 10. (1) |
| 11. (3) | 12. (4) | 13. (3) | 14. (3) | 15. (3) |
| 16. (2) | 17. (2) | 18. (2) | | |

ARCHIVES

JEE MAIN

Single Correct Answer Type

1. (4) 2. (1) 3. (2)p

7

Inverse Trigonometric Functions

INTRODUCTION

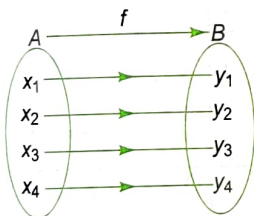
For function $f: A \rightarrow B$; $y = f(x)$ we can define inverse function or reverse function $f^{-1}(x)$ from set B to set A , if function $y = f(x)$ follows certain rules. For that we need to understand different types of mappings (functions). A function or mapping from a set A to another set B defines a relation f that for every element $a \in A$, there exists one and only one element $b \in B$ such that $(a, b) \in f$. In other words, a function f is a relation from a non-empty set A to a non-empty set B such that the domain of f is A and no two distinct ordered pairs in f have the same first element.

If f is a function from A to B and $(a, b) \in f$, then $f(a) = b$, where b is called the image of a under f and a is called the preimage of b under f . Here, set B is called codomain of the function.

The set of all possible images of the preimages is called the range of function.

Following are different cases of mappings:

Case I:

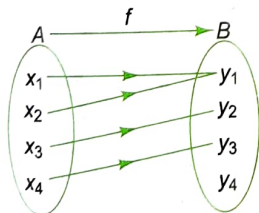


We observe that each image has one and only one preimage. Such function is called one-one or injective function.

Here, range of the function is $\{y_1, y_2, y_3, y_4\}$ which is same as codomain. Such function is called onto or surjective function.

Function which is both injective and surjective is called bijective function.

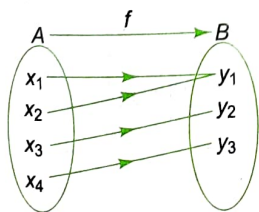
Case II:



Here, image y_1 has two pre-images, i.e., $f(x_1) = f(x_2) = y_1$. If at least one image has more than one pre-images, then function is called many-one.

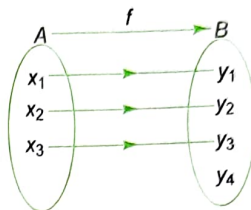
Also, range is $\{y_1, y_2, y_3\}$ which is subset of codomain. Such function is called into.

Case III:



Here, function is many-one and surjective.

Case IV:



Here, function is one-one and into.

Now, $f^{-1}(x)$ or the inverse of the function f will be a function from B to A such that the domain of f is B and no two distinct ordered pairs in f have the same first element.

In which of the above cases, $f^{-1}(x)$ is defined from set B to set A ?

Clearly, $f^{-1}(x)$ is defined (from B to A) only in Case I.

In Case II, $f^{-1}(x)$ is not defined as pre-image y_1 has more than one images. Also, y_4 has no image in A . Thus, $f^{-1}(x)$ is not defined if $f(x)$ is many-one or into.

With similar reasons, $f^{-1}(x)$ is not defined for Case III and Case IV as well.

Thus, we conclude that $f^{-1}(x)$ is defined if $f(x)$ is bijective.

Also, we observe that domain and range of $f(x)$ are, respectively, range and domain of $f^{-1}(x)$.

INVERSE TRIGONOMETRIC FUNCTIONS

We know that trigonometric functions are periodic and hence, many-one in their actual domain. So, to define an inverse trigonometric function, we have to restrict its actual domain to make function injective. Consider the case of sine function.

The domain of sine function is the set of all real numbers and range is the closed interval $[-1, 1]$. If we restrict its domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ then it becomes one-one and onto with range $[-1, 1]$.

Actually, sine function restricted to any of the intervals $\left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right]$, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ etc. is one-one and its range is $[-1, 1]$. We can, therefore, define the inverse of sine function in each of these intervals. We denote the inverse of sine function by $\sin^{-1}$. Thus, $\sin^{-1}$ is a function whose domain is $[-1, 1]$ and range

could be any of the intervals $\left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right]$, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, and so on. Corresponding to each such interval, we get a branch of the function $\sin^{-1}$. The branch with range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is called

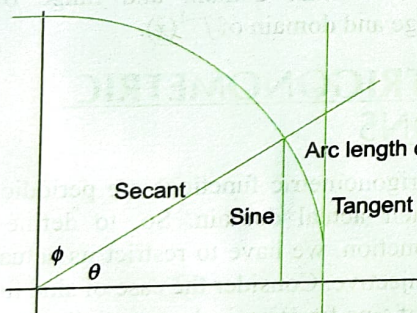
the **principal value** branch whereas other intervals as range give different branches of $\sin^{-1}$. When we refer to the function $\sin^{-1}$, we take it as the function whose domain is $[-1, 1]$ and range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. We write $\sin^{-1} : [-1, 1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. So, range of the inverse trigonometric function is restricted domain of the corresponding trigonometric function.

Similarly, we define other inverse trigonometric functions by restricting their domain. The domain and range of all inverse trigonometric functions are tabulated below:

Function	Domain	Range (Principal value branch)
$f(x) = \sin^{-1}x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$f(x) = \cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$f(x) = \tan^{-1}x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$f(x) = \cot^{-1}x$	R	$(0, \pi)$
$f(x) = \sec^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \{\pi/2\}$
$f(x) = \operatorname{cosec}^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$[-\pi/2, \pi/2] - \{0\}$

Note:

The inverse trigonometric functions are often called “arc functions,” since for the given value of a trigonometric function, they produce the length of arc needed to obtain that value.



In the figure, the arclength of the angle θ is equal to theta (as radius of circle is 1). So, the arclength associated with the length of the sine line, for example, is the same as the angle associated with it.

Hence,

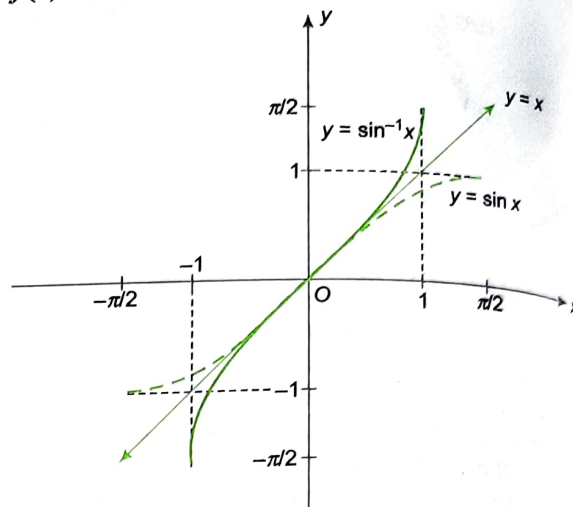
Arc length associated with sine value of $x = \text{Arc of sine} = \arcsine(x) = \theta = \sin^{-1}x$

GRAPH OF INVERSE TRIGONOMETRIC FUNCTION

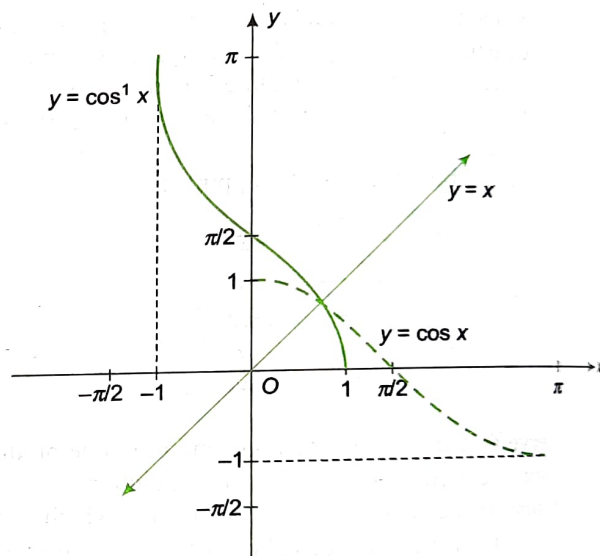
For bijective function $y = f(x)$ if $y_1 = f(x_1)$ then $f^{-1}(x_1) = y_1$. So, if point (x_1, y_1) lies on the graph of $y = f(x)$ then point (y_1, x_1) lies on the graph of $y = f^{-1}(x)$. Now, point (y_1, x_1) is the reflection of the point (x_1, y_1) in the line $y = x$. So, the graph of $y = f^{-1}(x)$ can be

obtained by reflecting the graph $y = f(x)$ in the line $y = f(x)$. Let us now see the graphs of different inverse trigonometric functions.

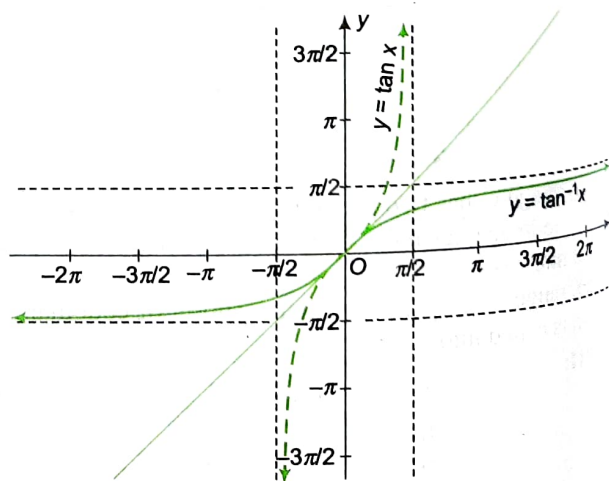
1. $f(x) = \sin^{-1}x$

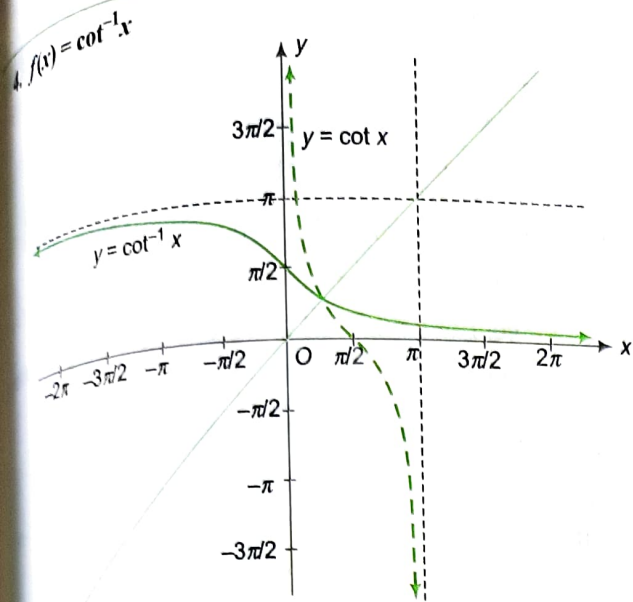


2. $f(x) = \cos^{-1}x$



3. $f(x) = \tan^{-1}x$



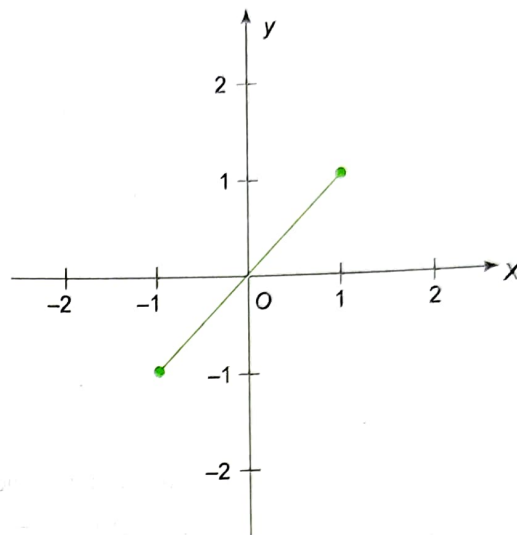


Also, $\sin(\sin^{-1}1) = \sin(\pi/2) = 1$,
 $\sin(\sin^{-1}(1/2)) = \sin(\pi/6) = 1/2$ etc.

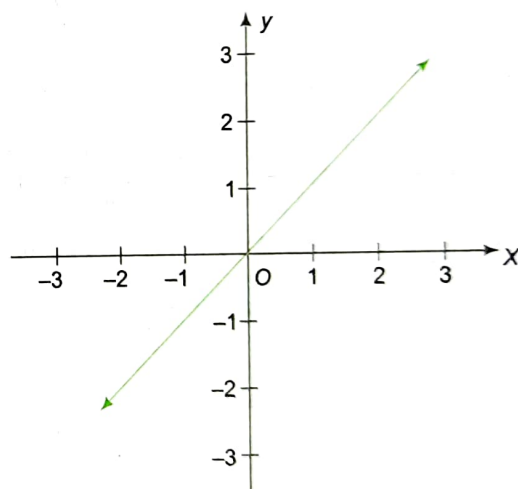
Therefore, $\sin(\sin^{-1}x) = x \forall x \in [-1, 1]$.

Similarly, $\cos(\cos^{-1}x) = x$ for $x \in [-1, 1]$.

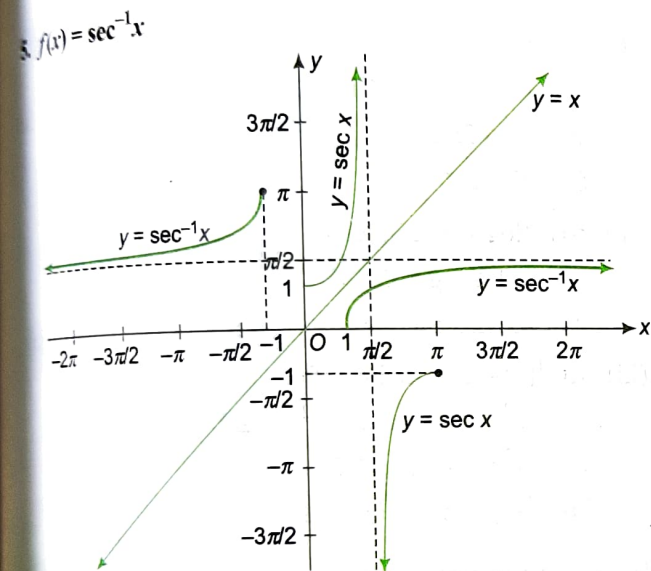
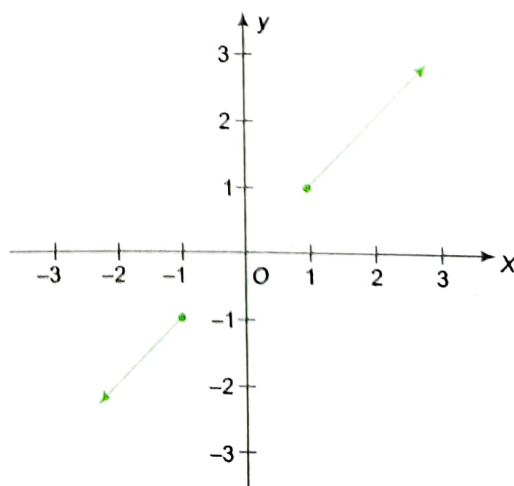
The graphs of $f(x) = \sin(\sin^{-1}x)$ or $f(x) = \cos(\cos^{-1}x)$ are part of the line $y = x$ for $x \in [-1, 1]$.



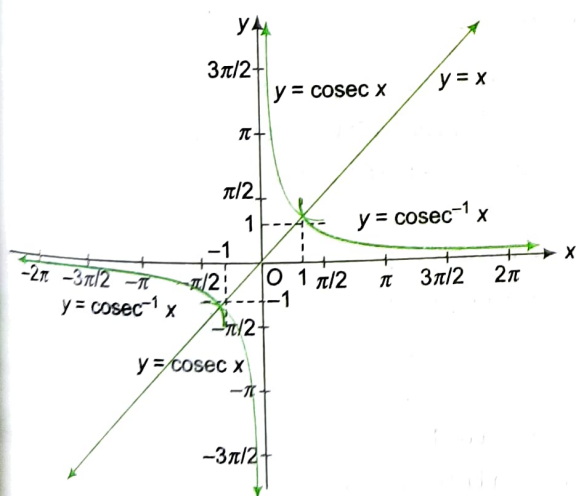
Also, $\tan(\tan^{-1}x) = x$ and $\cot(\cot^{-1}x) = x \forall x \in R$. The graph of the functions is as shown in the following figure.



$\sec(\sec^{-1}x) = x$ and $\operatorname{cosec}(\operatorname{cosec}^{-1}x) = x \forall x \in (-\infty, -1] \cup [1, \infty)$.
 The graph of the functions is as shown in the following figure.



6. $f(x) = \operatorname{cosec}^{-1}x$



FUNCTION $f(f^{-1}(x))$, WHERE $f(x)$ IS TRIGONOMETRIC FUNCTION

Consider the function $f(x) = \sin(\sin^{-1}x)$.

Sine function is defined for all real values of x , but $\sin^{-1}x$ is defined for $x \in [-1, 1]$.

Therefore, domain of $\sin(\sin^{-1}x)$ is $[-1, 1]$.

ILLUSTRATION 7.1

Find the principal value of the following:

(i) $\operatorname{cosec}^{-1}(2)$ (ii) $\tan^{-1}(-\sqrt{3})$ (iii) $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Sol.(i) Let $\operatorname{cosec}^{-1}(2) = y$. Then,

$$\operatorname{cosec} y = 2 = \operatorname{cosec}\left(\frac{\pi}{6}\right)$$

We know that the range of the principal values branch of the function $\operatorname{cosec}^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ and $\operatorname{cosec} \frac{\pi}{6} = 2$.Therefore, the principal value of $\operatorname{cosec}^{-1}(2)$ is $\frac{\pi}{6}$.(ii) Let $\tan^{-1}(-\sqrt{3}) = y$. Then,

$$\tan y = -\sqrt{3} = -\tan \frac{\pi}{3} = \tan\left(-\frac{\pi}{3}\right)$$

We know that the range of the principal values branch of the function $\tan^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$.Therefore, the principal value of $\tan^{-1}(-\sqrt{3})$ is $-\frac{\pi}{3}$.(iii) Let $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = y$. Then,

$$\cos y = -\frac{1}{\sqrt{2}} = -\cos\left(\frac{\pi}{4}\right) = \cos\left(\pi - \frac{\pi}{4}\right) = \cos\left(\frac{3\pi}{4}\right)$$

We know that the range of the principal values branch of the function $\cos^{-1}$ is $[0, \pi]$ and $\cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$.Therefore, the principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ is $\frac{3\pi}{4}$.**ILLUSTRATION 7.2**Solve $\sin^{-1}x > -1$.**Sol.** $\sin^{-1}x > -1$

or $-1 < \sin^{-1}x \leq \frac{\pi}{2}$

or $\sin(-1) < x \leq \sin\left(\frac{\pi}{2}\right)$

$$\Rightarrow x \in (-\sin 1, 1]$$

ILLUSTRATION 7.3Solve $\cos^{-1}x > \cos^{-1}x^2$.**Sol.** $\cos^{-1}x > \cos^{-1}x^2$

or $x < x^2$

or $x^2 - x > 0$

 $(\because \cos^{-1}$ is decreasing)

or $x(x-1) > 0$

$$\Rightarrow x \in [-1, 0)$$

ILLUSTRATION 7.4Solve for x if $(\cot^{-1}x)^2 - 3(\cot^{-1}x) + 2 > 0$.**Sol.** $(\cot^{-1}x - 1)(\cot^{-1}x - 2) > 0$

$$\Rightarrow \cot^{-1}x < 1 \text{ or } \cot^{-1}x > 2$$

$$\Rightarrow x > \cot 1 \text{ or } x < \cot 2 \quad (\because \cot^{-1}x \text{ is a decreasing function})$$

$$\Rightarrow x < \cot 2 \text{ or } x > \cot 1$$

ILLUSTRATION 7.5Find the value of x for which the following expressions are defined.

(i) $\sin^{-1}(3x-2)$ (ii) $\cos^{-1}(\log_e x)$ (iii) $\sec^{-1}(x^2-2)$

Sol.(i) $\sin^{-1}(3x-2)$ is defined if

$$-1 \leq 3x-2 \leq 1$$

$$\therefore 1 \leq 3x \leq 3$$

$$\text{So, } 1/3 \leq x \leq 1$$

(ii) $\cos^{-1}(\log_e x)$ is defined if

$$-1 \leq \log_e x \leq 1$$

$$\therefore e^{-1} \leq x \leq e$$

(iii) $\sec^{-1}(x^2-2)$ is defined if

$$x^2-2 \leq -1 \text{ or } x^2-2 \geq 1$$

$$\therefore x^2 \leq 1 \text{ or } x^2 \geq 3$$

$$\text{So, } -1 \leq x \leq 1 \text{ or } x \leq -\sqrt{3} \text{ or } x \geq \sqrt{3}$$

ILLUSTRATION 7.6Solve $[\cot^{-1}x] + [\cos^{-1}x] = 0$, where $[.]$ denotes the greatest integer function.**Sol.** We have $[\cot^{-1}x] + [\cos^{-1}x] = 0$ Since $\cos^{-1}x, \cot^{-1}x \geq 0$; we must have

$$[\cot^{-1}x] = [\cos^{-1}x] = 0$$

Now, $[\cot^{-1}x] = 0$

$$\Rightarrow 0 < \cot^{-1}x < 1$$

$$\Rightarrow x \in (\cot 1, \infty)$$

Also, $[\cos^{-1}x] = 0$

$$\Rightarrow 0 \leq \cos^{-1}x < 1$$

$$\Rightarrow x \in (\cos 1, 1]$$

From (1) and (2), $x \in (\cot 1, 1]$ **ILLUSTRATION 7.7**

Find the values of

(i) $\sin^{-1}(2^x)$

(ii) $\cos^{-1}\sqrt{x^2-x+1}$

(iii) $\tan^{-1}\frac{x^2}{1+x^2}$

(iv) $\sec^{-1}\left(x + \frac{1}{x}\right)$

Sol. We know that $2^x > 0$
 But for $\sin^{-1}(2^x)$ to be defined, we must have $2^x \leq 1$
 $\therefore 0 < 2^x \leq 1$
 $\Rightarrow 0 < \sin^{-1}(2^x) \leq \pi/2$

$$(ii) \cos^{-1} \sqrt{x^2 - x + 1} = \cos^{-1} \sqrt{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}}$$

$$\text{We have } \sqrt{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}} \geq \frac{\sqrt{3}}{2}$$

$$\therefore \frac{\sqrt{3}}{2} \leq \sqrt{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}} \leq 1$$

$$\therefore 0 \leq \cos^{-1} \sqrt{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}} \leq \frac{\pi}{6}$$

$$(iii) \tan^{-1} \frac{x^2}{1+x^2} = \tan^{-1} \left(1 - \frac{1}{1+x^2}\right)$$

$$\text{Now, } 1 \leq 1+x^2 < \infty$$

$$\therefore 0 < \frac{1}{1+x^2} \leq 1$$

$$\therefore -1 \leq -\frac{1}{1+x^2} < 0$$

$$\therefore 0 \leq 1 - \frac{1}{1+x^2} < 1$$

$$\therefore 0 \leq \tan^{-1} \left(1 - \frac{1}{1+x^2}\right) < \frac{\pi}{4}$$

$$(iv) \text{ Let } x + \frac{1}{x} = y$$

$$\therefore x^2 - yx + 1 = 0$$

Since x is real, $D \geq 0$

$$\therefore y^2 - 4 \geq 0$$

$$\therefore y \leq -2 \text{ or } y \geq 2$$

$$\therefore \sec^{-1} \left(x + \frac{1}{x}\right) \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{2\pi}{3}\right]$$

ILLUSTRATION 7.8

Find the range of $f(x) = |3 \tan^{-1} x - \cos^{-1}(0)| - \cos^{-1}(-1)$.

$$\text{Sol. } f(x) = |3 \tan^{-1} x - \cos^{-1}(0)| - \cos^{-1}(-1) \\ = |3 \tan^{-1} x - (\pi/2)| - \pi$$

$$\text{Now, } -\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}$$

$$\text{or } -\frac{3\pi}{2} < 3 \tan^{-1} x < \frac{3\pi}{2}$$

$$\text{or } -2\pi < 3 \tan^{-1} x - \frac{\pi}{2} < \pi$$

$$\text{or } 0 \leq \left| 3 \tan^{-1} x - \frac{\pi}{2} \right| < 2\pi$$

$$\text{or } -\pi \leq \left| 3 \tan^{-1} x - \frac{\pi}{2} \right| - \pi < \pi$$

ILLUSTRATION 7.9

Find the value of x for which $\sec^{-1} x + \sin^{-1} x = \frac{\pi}{2}$.

Sol. We know that $\sec^{-1} x$ is defined for $x \in (-\infty, -1] \cup [1, \infty)$.
 But $\sin^{-1} x$ is defined for $x \in [-1, 1]$.

Hence, $\sec^{-1} x + \sin^{-1} x = \frac{\pi}{2}$ for $x = \pm 1$.

ILLUSTRATION 7.10

If $\sin^{-1}(x^2 + 2x + 2) + \tan^{-1}(x^2 - 3x - k^2) > \frac{\pi}{2}$, then find the values of k .

$$\text{Sol. } \sin^{-1}(x^2 + 2x + 2) + \tan^{-1}(x^2 - 3x - k^2) > \frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}((x+1)^2 + 1) + \tan^{-1}(x^2 - 3x - k^2) > \frac{\pi}{2}$$

Now, $\sin^{-1}((x+1)^2 + 1)$ is defined if $(x+1)^2 = 0$ or $x = -1$.

Putting $x = -1$, we get

$$\frac{\pi}{2} + \tan^{-1}(4 - k^2) > \frac{\pi}{2}$$

$$\Rightarrow \tan^{-1}(4 - k^2) > 0$$

$$\Rightarrow 4 - k^2 > 0$$

$$\Rightarrow -2 < k < 2$$

ILLUSTRATION 7.11

If $\cos^{-1} \lambda + \cos^{-1} \mu + \cos^{-1} \gamma = 3\pi$, then find the value of $\lambda\mu + \mu\gamma + \gamma\lambda$.

Sol. We know that $0 \leq \cos^{-1} x \leq \pi$.

Hence, from the equation

$$\cos^{-1} \lambda = \pi, \cos^{-1} \mu = \pi, \cos^{-1} \gamma = \pi$$

$$[\because \cos^{-1} \lambda + \cos^{-1} \mu + \cos^{-1} \gamma = 3\pi \text{ is}]$$

possible only when each term attains its maximum.]

$$\Rightarrow \lambda = \mu = \gamma = -1$$

$$\Rightarrow \lambda\mu + \mu\gamma + \gamma\lambda = 3$$

ILLUSTRATION 7.12

If $\sin^{-1} x_1 + \sin^{-1} x_2 + \dots + \sin^{-1} x_n \leq -\frac{n\pi}{2}$, $n \in N$, $n = 2m + 1$,

$m \geq 1$, then find the value of $\frac{x_1^4 + x_3^4 + x_5^4 + \dots + (m+1) \text{ terms}}{x_2^2 + x_4^2 + x_6^2 + \dots + m \text{ terms}}$.

Sol. Since $\sin^{-1} x_i \geq -\frac{\pi}{2}$, $i = 1, 2, 3, \dots, n$;

$$\sin^{-1} x_1 + \sin^{-1} x_2 + \dots + \sin^{-1} x_n \geq -\frac{n\pi}{2}$$

But given that $\sum_{i=1}^n \sin^{-1} x_i \leq -\frac{n\pi}{2}$

$$\therefore \sin^{-1} x_1 = \sin^{-1} x_2 = \dots = \sin^{-1} x_n = -\frac{\pi}{2}$$

$$\therefore x_i = -1; i = 1, 2, 3, \dots, n$$

$$\begin{aligned} \therefore \frac{x_1^1 + x_3^3 + x_5^5 + \dots (m+1) \text{ terms}}{x_2^2 + x_4^4 + x_6^6 + \dots m \text{ terms}} \\ = \frac{(-1) + (-1) + (-1) + \dots (m+1) \text{ times}}{1 + 1 + 1 + \dots m \text{ times}} \\ = \frac{-(m+1)}{m} \end{aligned}$$

ILLUSTRATION 7.13

Find x satisfying $[\tan^{-1} x] + [\cot^{-1} x] = 2$, where $[\cdot]$ represents the greatest integer function.

Sol. $0 < \cot^{-1} x < \pi$ and $-\pi/2 < \tan^{-1} x < \pi/2$

$$\Rightarrow [\cot^{-1} x] \in \{0, 1, 2, 3\} \text{ and } [\tan^{-1} x] \in \{-2, -1, 0, 1\}$$

For $[\tan^{-1} x] + [\cot^{-1} x] = 2$, following cases are possible

Case (i): $[\cot^{-1} x] = [\tan^{-1} x] = 1$

$$\Rightarrow 1 \leq \cot^{-1} x < 2 \text{ and } 1 \leq \tan^{-1} x < \pi/2$$

$$\Rightarrow x \in (\cot 2, \cot 1] \text{ and } x \in [\tan 1, \infty)$$

$$\therefore x \in \phi \text{ as } \cot 1 < \tan 1$$

Case (ii): $[\cot^{-1} x] = 2, [\tan^{-1} x] = 0$

$$\Rightarrow 2 \leq \cot^{-1} x < 3 \text{ and } 0 \leq \tan^{-1} x < 1$$

$$\Rightarrow x \in (\cot 3, \cot 2] \text{ and } x \in [0, \tan 1)$$

$$\therefore x \in \phi \text{ as } \cot 2 < 0$$

So no solution.

Case (iii): $[\cot^{-1} x] = 3, [\tan^{-1} x] = -1$

$$\Rightarrow 3 \leq \cot^{-1} x < \pi \text{ and } -1 \leq \tan^{-1} x < 0$$

$$\Rightarrow x \in (-\infty, \cot 3] \text{ and } x \in [-\tan 1, 0)$$

$$\therefore x \in \phi \text{ as } \cot 3 < -\tan 1$$

Therefore, no such values of x exist.

ILLUSTRATION 7.14

If $\cos(2 \sin^{-1} x) = \frac{1}{9}$, then find the values of x .

Sol. Let $\sin^{-1} x = \theta$. Then,

$$\cos 2\theta = \frac{1}{9}$$

$$\text{or } 1 - 2 \sin^2 \theta = 9$$

$$\text{or } 1 - 2x^2 = \frac{1}{9}$$

$$\text{or } x^2 = \frac{4}{9}$$

$$\text{or } x = \pm \frac{2}{3}$$

ILLUSTRATION 7.15

Find the value of $\sin\left(\frac{1}{2} \cot^{-1}\left(-\frac{3}{4}\right)\right)$.

Sol. Let $\cos^{-1}(-3/4) = \theta$

$$\text{or } \cos \theta = -3/4$$

$$\Rightarrow \theta \in (\pi/2, \pi)$$

$$\Rightarrow \cos \theta = -3/5$$

$$\Rightarrow 1 - 2 \sin^2(\theta/2) = -3/5$$

$$\therefore \sin^2 \theta/2 = 4/5 \text{ or } \sin \theta/2 = 2/\sqrt{5}$$

($\theta \in (\pi/2, \pi)$)

ILLUSTRATION 7.16

Prove that $\cot^{-1}\left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right) = \frac{x}{2}, x \in \left(0, \frac{\pi}{4}\right)$

Sol. Consider $\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}$

$$= \frac{(\sqrt{1+\sin x} + \sqrt{1-\sin x})^2}{(\sqrt{1+\sin x})^2 - (\sqrt{1-\sin x})^2} \quad (\text{by rationalizing})$$

$$= \frac{(1 + \sin x) + (1 - \sin x) + 2\sqrt{(1 + \sin x)(1 - \sin x)}}{1 + \sin x - 1 + \sin x}$$

$$= \frac{2(1 + \sqrt{1 - \sin^2 x})}{2 \sin x} = \frac{1 + \cos x}{\sin x}$$

$$= \frac{2 \cos^2 \frac{x}{2}}{2 \sin \frac{x}{2} \cos \frac{x}{2}} = \cot \frac{x}{2}$$

$$\Rightarrow \cot^{-1}\left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right) = \frac{x}{2}$$

ILLUSTRATION 7.17

Solve $\sin^{-1}(1-x) - 2 \sin^{-1} x = \frac{\pi}{2}$

Sol. Given, $\sin^{-1}(1-x) - 2 \sin^{-1} x = \pi/2$

Let $x = \sin y$

Therefore, the given equation reduces to

$$\sin^{-1}(1 - \sin y) - 2 \sin^{-1} \sin y = \frac{\pi}{2}$$

$$\text{or } \sin^{-1}(1 - \sin y) - 2y = \frac{\pi}{2}$$

$$\text{or } \sin^{-1}(1 - \sin y) = \frac{\pi}{2} + 2y$$

$$\text{or } 1 - \sin y = \sin\left(\frac{\pi}{2} + 2y\right)$$

$$\text{or } 1 - \sin y = \cos 2y$$

$$\text{or } 1 - \cos 2y = \sin y$$

$2 \sin^2 y = \sin y$
 $2 \sin^2 y - \sin y = 0$
 $\sin y(2 \sin y - 1) = 0$
 $\sin y = 0$ or $1/2$
 $x = 0$ or $1/2$
 But, when $x = \frac{1}{2}$, it can be observed that

$$\begin{aligned}
 \text{L.H.S.} &= \sin^{-1}\left(1 - \frac{1}{2}\right) - 2 \sin^{-1} \frac{1}{2} \\
 &= \sin^{-1}\left(\frac{1}{2}\right) - 2 \sin^{-1} \frac{1}{2} \\
 &= -\sin^{-1} \frac{1}{2} \\
 &= -\frac{\pi}{6} \neq \frac{\pi}{2}
 \end{aligned}$$

Therefore, $x = \frac{1}{2}$ is not the solution of the given equation.

Thus, $x = 0$.

ILLUSTRATION 7.18

Find the number of solutions of the equation $\cos(\cos^{-1}x) = \operatorname{cosec}(\operatorname{cosec}^{-1}x)$.

Sol. $\cos(\cos^{-1}x) = x$ for $x \in [-1, 1]$
 $\operatorname{cosec}(\operatorname{cosec}^{-1}x) = x$ for $x \in (-\infty, -1] \cup [1, \infty)$
 $\Rightarrow \cos(\cos^{-1}x) = \operatorname{cosec}(\operatorname{cosec}^{-1}x)$ for $x = \pm 1$ only.
 Hence, there are two roots only.

CONCEPT APPLICATION EXERCISE 7.1

- Find the principal value of
 (a) $\operatorname{cosec}^{-1}(-1)$ (b) $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$
- Solve $\cos^{-1}x < 2$.
- Find the possible values of $\sin^{-1}(1-x) + \cos^{-1}\sqrt{x-2}$.
- Find the real values of x for which the function $f(x) = \cos^{-1}\sqrt{x^2+3x+1} + \cos^{-1}\sqrt{x^2+3x}$ is defined.
- Find the smallest and the largest values of $\tan^{-1}\left(\frac{1-x}{1+x}\right)$, $0 \leq x \leq 1$.
- Find the values of x for which $\sin^{-1}(\cos^{-1}x) < 1$ and $\cos^{-1}(\sin^{-1}x) < 1$.
- Solve $\sin^{-1}x > \tan^{-1}x$.
- Find the range of $f(x) = \cos^{-1}x + \cot^{-1}x$.
- If $(\sin^{-1}x)^2 + (\sin^{-1}y)^2 + (\sin^{-1}z)^2 = \frac{3\pi^2}{4}$, then find the minimum value of $x+y+z$.
- Find the value of $\sin\left(\frac{1}{4}\cos^{-1}\left(\frac{-1}{9}\right)\right)$.
- If $x < 0$, then prove that $\cos^{-1}x = \pi + \tan^{-1}\frac{\sqrt{1-x^2}}{x}$.

12. Prove that $\sin^{-1}\left(\frac{x + \sqrt{1-x^2}}{\sqrt{2}}\right) = \sin^{-1}x + \frac{\pi}{4}$, where $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$.

ANSWERS

- (a) $-\frac{\pi}{2}$ (b) $\frac{2\pi}{3}$
- $x \in (\cos 2, 1]$
- 0
- $x = 0, -3$
- $\left[0, \frac{\pi}{4}\right]$
- $\cos \sin 1 < x < \cos \cos 1$
- $x \in (0, 1]$
- $\left[\frac{\pi}{4}, \frac{7\pi}{4}\right]$
- 3
- $\frac{1}{\sqrt{6}}$
- $\sin^{-1}x + \frac{\pi}{4}$

PRINCIPAL VALUES OF FUNCTION $f^{-1}(f(x))$

PRINCIPAL VALUE OF $\sin^{-1}(\sin x)$

We have $f(x) = \sin^{-1}(\sin x)$.

We know that $(\sin x) \in [-1, 1] \forall x \in \mathbb{R}$. So, $\sin^{-1}(\sin x)$ is defined for all real values of x .

Also, for $(\sin x) \in [-1, 1]$, $\sin^{-1}(\sin x) \in [-\pi/2, \pi/2]$. So, range of function is $[-\pi/2, \pi/2]$.

Therefore, $\sin^{-1}(\sin x) = x$ only if $x \in [-\pi/2, \pi/2]$.

For example,

$$\sin^{-1}\left(\sin \frac{\pi}{4}\right) = \sin^{-1} \frac{1}{\sqrt{2}} = \frac{\pi}{4}$$

$$\sin^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right) = \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

$$\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3} \left(\neq \frac{2\pi}{3}\right)$$

$$\sin^{-1}\left(\sin\left(-\frac{4\pi}{3}\right)\right) = \sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3} \left(\neq -\frac{4\pi}{3}\right)$$

Graph of $y = \sin^{-1}(\sin x)$

$$y = \sin^{-1}(\sin x)$$

$$\therefore \sin y = \sin x$$

$$\Rightarrow y = n\pi + (-1)^n x, n \in \mathbb{Z}$$

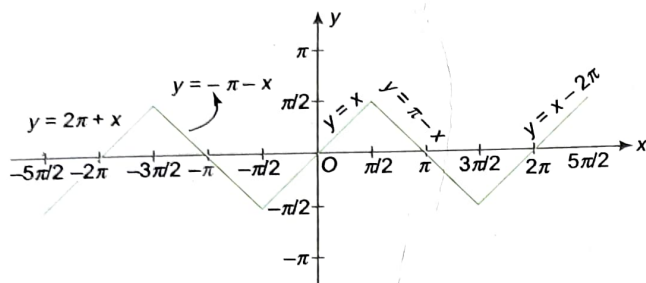
Since $y \in [-\pi/2, \pi/2]$, we have different expressions for $\sin^{-1}(\sin x)$ for different values of x as shown in the following table.

Value of n	Relation	Range of x
...	...	...
...	...	...
$n = -2$	$y = -2\pi + x$	$x \in [3\pi/2, 5\pi/2]$
$n = -1$	$y = -\pi - x$	$x \in [-3\pi/2, -\pi/2]$
$n = 0$	$y = x$	$x \in [-\pi/2, \pi/2]$

7.8 Trigonometry

$n = 1$	$y = \pi - x$	$x \in [\pi/2, 3\pi/2]$
$n = 2$	$y = 2\pi + x$	$x \in [-5\pi/2, -3\pi/2]$
...	...	...
...	...	...

From the preceding discussion, we have following graph of $y = \sin^{-1}(\sin x)$.



Clearly, period of the function is 2π .

ILLUSTRATION 7.19

Find the principal values of the following:

- (i) $\sin^{-1}(\sin 1)$
- (ii) $\sin^{-1}(\sin 2)$
- (iii) $\sin^{-1}(\sin 10)$
- (iv) $\sin^{-1}(\sin 20)$
- (v) $\sin^{-1}(\sin 100)$
- (vi) $\sin^{-1}\left(\sin \frac{29\pi}{5}\right)$

Sol.

- (i) Since $1 \in [-\pi/2, \pi/2]$, $\sin^{-1}(\sin 1) = 1$
- (ii) Since $2 \notin [-\pi/2, \pi/2]$, $\sin^{-1}(\sin 2) \neq 2$
 $\therefore \sin^{-1}(\sin 2) = \sin^{-1}(\sin(\pi - 2))$
 $= \pi - 2$ as $(\pi - 2) \in [-\pi/2, \pi/2]$
- (iii) $\sin^{-1}(\sin 10) = \sin^{-1}(\sin(3\pi - 10))$
 $= 3\pi - 10$ as $(3\pi - 10) \in [-\pi/2, \pi/2]$
- (iv) $\sin^{-1}(\sin(-20)) = \sin^{-1}(\sin(6\pi - 20))$
 $= 6\pi - 20$ as $(6\pi - 20) \in [-\pi/2, \pi/2]$
- (v) $\sin^{-1}(\sin 100) = \sin^{-1}(\sin(100 - 32\pi))$
 $= 100 - 32\pi$ as $(100 - 32\pi) \in [-\pi/2, \pi/2]$
- (vi) $\sin^{-1}\left(\sin \frac{29\pi}{5}\right) = \sin^{-1}\left(\sin\left(6\pi - \frac{\pi}{5}\right)\right)$
 $= \sin^{-1}\left(\sin\left(-\frac{\pi}{5}\right)\right)$
 $= -\frac{\pi}{5}$

ILLUSTRATION 7.20

Solve $\sin^{-1}(\sin 6x) = x$, $x \in [0, \pi]$.

- Sol.** $\sin^{-1}(\sin 6x) = x$
 $\Rightarrow \sin 6x = \sin x$
 $\Rightarrow \sin 6x - \sin x = 0$
 $\Rightarrow 2 \sin \frac{5x}{2} \cos \frac{7x}{2} = 0$

$$\Rightarrow \frac{5x}{2} = n\pi \text{ or } \frac{7x}{2} = (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$$

$$\Rightarrow x = \frac{2n\pi}{5} \text{ or } x = (2n+1)\frac{\pi}{7}$$

$$\Rightarrow x = \frac{\pi}{7}, \frac{3\pi}{7}, \frac{5\pi}{7}, \pi, 0, \frac{2\pi}{5}, \frac{4\pi}{5}$$

But $x = \frac{5\pi}{7}, \pi, \frac{4\pi}{5}$ are not possible as any solution must be

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

ILLUSTRATION 7.21

Solve $\sin^{-1}\left[\sin\left(\frac{2x^2+4}{1+x^2}\right)\right] < \pi - 3$.

Sol. $\sin^{-1}\left[\sin\left(\frac{2x^2+4}{1+x^2}\right)\right] < \pi - 3$

Now, $\frac{2x^2+4}{1+x^2} = \frac{2x^2+2+2}{1+x^2} = 2 + \frac{2}{1+x^2}$

So, $2 + \frac{2}{1+x^2} \in (2, 4]$

Therefore, Eq. (i) reduces to

$$\pi - \frac{2x^2+4}{1+x^2} < \pi - 3$$

or $\frac{2x^2+4}{1+x^2} > 3$

or $2x^2+4 > 3+3x^2$

or $x^2 - 1 < 0$

or $-1 < x < 1$

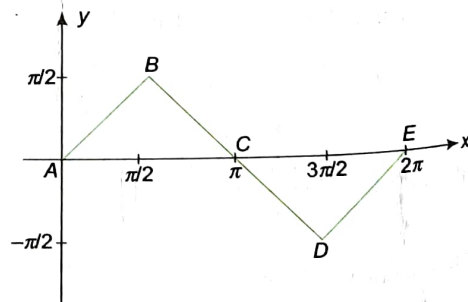
ILLUSTRATION 7.22

Find the area bounded by $y = \sin^{-1}(\sin x)$ and x -axis for $x \in [0, 100\pi]$.

Sol. $y = \sin^{-1}(\sin x)$ is periodic with period 2π .

$\therefore$ Required area = $50 \times$ (area bounded by $y = \sin^{-1}(\sin x)$ and x -axis for $x \in [0, 2\pi]$)

Graph of $y = \sin^{-1}(\sin x)$ for $x \in [0, 2\pi]$ is as shown in the following figure:



$$\begin{aligned}\text{Required area} &= 50 \times (\text{area of } \triangle ABC + \text{area of } \triangle CDE) \\ &= 100 \times (\text{area of } \triangle ABC)\end{aligned}$$

$$\begin{aligned}&= 100 \times \left(\frac{1}{2} \times \pi \times \frac{\pi}{2} \right) \\ &= 25\pi^2\end{aligned}$$

ILLUSTRATION 7.23

Find the values of x for which $f(x) = 2 \sin^{-1} \sqrt{1-x}$ + $\sin^{-1}(2\sqrt{x-x^2})$ is constant.

$$\text{Sol. } f(x) = 2 \sin^{-1} \sqrt{1-(\sqrt{x})^2} + \sin^{-1}(2\sqrt{(\sqrt{x})^2\{1-(\sqrt{x})^2\}})$$

$$\sqrt{x} = \cos \theta, \text{ where } \theta = \cos^{-1} \sqrt{x} \in \left[0, \frac{\pi}{2}\right].$$

$$f(x) = 2 \sin^{-1} \sqrt{1-\cos^2 \theta} + \sin^{-1} 2\sqrt{(\cos^2 \theta)(1-\cos^2 \theta)}$$

$$= 2 \sin^{-1} (\sin \theta) + \sin^{-1} (2 \sin \theta \cos \theta)$$

$$= 2 \sin^{-1} (\sin \theta) + \sin^{-1} (\sin 2\theta)$$

$$= 2\theta + \sin^{-1} (2\theta)$$

$$= 2\theta + (\pi - 2\theta) \text{ if } 2\theta \in \left[\frac{\pi}{2}, \pi\right]$$

$$= \pi, \theta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$$

$$\text{So, } \sqrt{x} = \cos \theta \in \left[0, \frac{1}{\sqrt{2}}\right]$$

$$\text{or } x \in \left[0, \frac{1}{2}\right]$$

PRINCIPAL VALUE OF $\cos^{-1}(\cos x)$

We have $f(x) = \cos^{-1}(\cos x)$.

We know that $(\cos x) \in [-1, 1] \forall x \in R$. So, $\cos^{-1}(\cos x)$ is defined for all real values of x .

Also, for $(\cos x) \in [-1, 1]$, $\cos^{-1}(\cos x) \in [0, \pi]$. So, range of function is $[0, \pi]$.

Therefore, $\cos^{-1}(\cos x) = x$ only if $x \in [0, \pi]$.

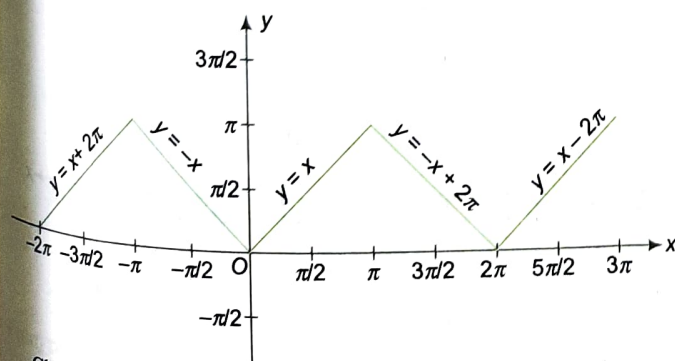
Graph of $y = \cos^{-1}(\cos x)$

$$y = \cos^{-1}(\cos x)$$

$$\therefore \cos y = \cos x$$

$$\Rightarrow y = 2n\pi + x, n \in Z.$$

So, to draw the graph of $y = \cos^{-1}(\cos x)$, we draw all the lines $y = 2n\pi + x, n \in Z$ for $y \in [0, \pi]$ as shown in the following figure.



Clearly, period of the function is 2π .

ILLUSTRATION 7.24

Find the principal values of the following:

(i) $\cos^{-1}(\cos 3)$ (ii) $\cos^{-1}(\cos 4)$

(iii) $\cos^{-1}(\cos 15)$ (iv) $\cos^{-1}(\cos 30)$

(v) $\cos^{-1}(\cos 50)$ (vi) $\cos^{-1}\left(\cos \frac{48\pi}{7}\right)$

Sol.

(i) Since $3 \in [0, \pi]$, $\cos^{-1}(\cos 3) = 3$

(ii) Since $4 \notin [0, \pi]$, $\cos^{-1}(\cos 4) \neq 4$

$$\therefore \cos^{-1}(\cos 4) = 2\pi - 4$$

(iii) $\cos^{-1}(\cos 15) = 15 - 4\pi$

(iv) $\cos^{-1}(\cos 30) = 10\pi - 30$

(v) $\cos^{-1}(\cos 50) = 16\pi - 50$

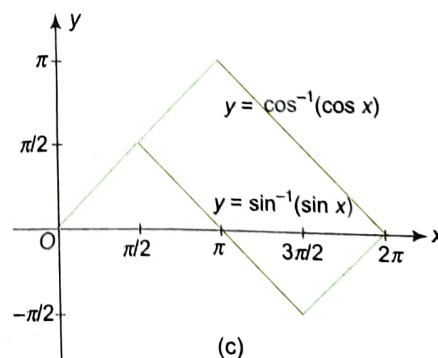
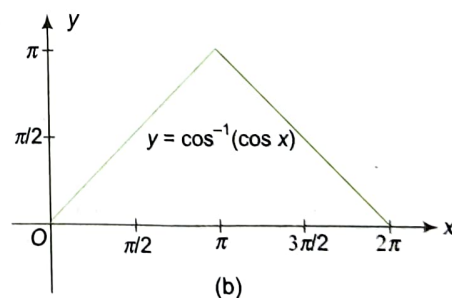
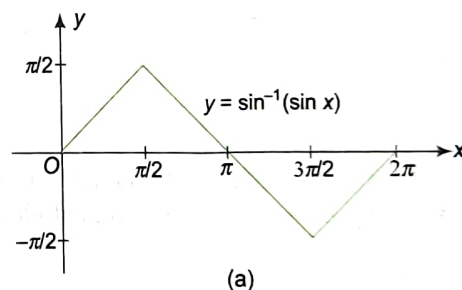
$$\begin{aligned}\text{(vi) } \cos^{-1}\left(\cos \frac{48\pi}{7}\right) &= \cos^{-1}\left(\cos \left(6\pi + \frac{6\pi}{7}\right)\right) \\ &= \cos^{-1}\left(\cos \frac{6\pi}{7}\right) = \frac{6\pi}{7}\end{aligned}$$

ILLUSTRATION 7.25

Solve $\cos^{-1}(\cos x) > \sin^{-1}(\sin x)$, $x \in [0, 2\pi]$.

Sol. We have $\cos^{-1}(\cos x) > \sin^{-1}(\sin x)$

To solve this inequality, we draw the graph of $y = \cos^{-1}(\cos x)$ and $y = \sin^{-1}(\sin x)$ for $x \in [0, 2\pi]$.



From the figure, we can see that graph of $y = \cos^{-1}(\cos x)$ lies above the graph of $y = \sin^{-1}(\sin x)$ for $x \in (\pi/2, 2\pi)$.

Thus, solution of $\cos^{-1}(\cos x) > \sin^{-1}(\sin x)$ is $x \in (\pi/2, 2\pi)$.

PRINCIPAL VALUE OF $\tan^{-1}(\tan x)$

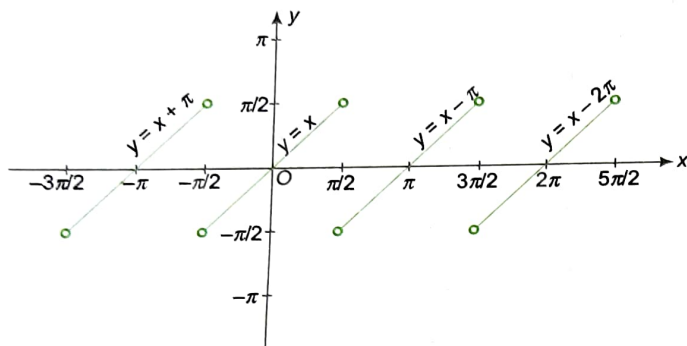
Clearly, domain of the function is $\mathbb{R}$ and range is $(-\pi/2, \pi/2)$.

Also, $y = \tan^{-1}(\tan x)$

$$\therefore \tan y = \tan x$$

$$\Rightarrow y = n\pi + x, n \in \mathbb{Z}.$$

So, to draw the graph of $y = \tan^{-1}(\tan x)$, we draw all the lines $y = n\pi + x, n \in \mathbb{Z}$ for $y \in (-\pi/2, \pi/2)$ as shown in the following figure.



Clearly, period of the function is π .

PRINCIPAL VALUE OF $\cot^{-1}(\cot x)$

Clearly, domain of the function is $\mathbb{R}$ and range is $(0, \pi)$.

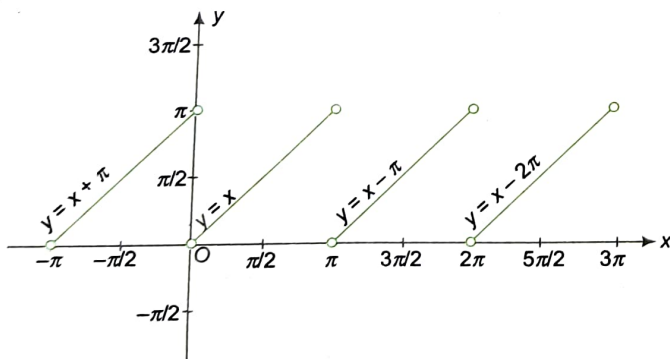
Also, $y = \cot^{-1}(\cot x)$

$$\therefore \cot y = \cot x$$

$$\text{or } \tan y = \tan x$$

$$\Rightarrow y = n\pi + x, n \in \mathbb{Z}.$$

So, to draw the graph of $y = \cot^{-1}(\cot x)$, we draw all the lines $y = n\pi + x, n \in \mathbb{Z}$ for $y \in (0, \pi)$ as shown in the following figure.



Clearly, period of the function is π .

PRINCIPAL VALUE OF $\operatorname{cosec}^{-1}(\operatorname{cosec} x)$

Clearly, domain of the function is $\mathbb{R} - \{n\pi, n \in \mathbb{Z}\}$ and range is $[-\pi/2, \pi/2] - \{0\}$.

Also, $y = \operatorname{cosec}^{-1}(\operatorname{cosec} x)$

$$\therefore \operatorname{cosec} y = \operatorname{cosec} x$$

$$\text{or } \sin y = \sin x$$

Hence, graph of $y = \operatorname{cosec}^{-1}(\operatorname{cosec} x)$ is same as that of $y = \sin^{-1}(\sin x)$, but excluding points $x = n\pi, n \in \mathbb{Z}$.

PRINCIPAL VALUE OF $\sec^{-1}(\sec x)$

Clearly, domain of the function is $\mathbb{R} - \{(2n+1)\frac{\pi}{2}, n \in \mathbb{Z}\}$ and range is $[0, \pi] - \{\pi/2\}$.

Also, $y = \sec^{-1}(\sec x)$

$$\therefore \sec y = \sec x$$

$$\text{or } \cos y = \cos x$$

Hence, graph of $y = \sec^{-1}(\sec x)$ is same as that of $y = \cos^{-1}(\cos x)$, but excluding points $x = (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$.

ILLUSTRATION 7.26

Find the principal values of the following:

(i) $\tan^{-1}\left(\tan \frac{2\pi}{3}\right)$

(ii) $\tan^{-1}(\tan(-6))$

Sol.

(i) $\tan^{-1}\left(\tan \frac{2\pi}{3}\right) \neq \frac{2\pi}{3}$, as $\frac{2\pi}{3}$ does not lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

$$\begin{aligned} \text{Now, } \tan^{-1}\left(\tan \frac{2\pi}{3}\right) &= \tan^{-1}\left(\tan\left(\pi - \frac{\pi}{3}\right)\right) \\ &= \tan^{-1}\left(-\tan \frac{\pi}{3}\right) \\ &= \tan^{-1}\left(\tan\left(-\frac{\pi}{3}\right)\right) \\ &= -\frac{\pi}{3} \end{aligned}$$

(ii) $\tan^{-1}(\tan(-6)) = \tan^{-1}(\tan(2\pi - 6))$
 $= 2\pi - 6$ as $(2\pi - 6) \in (-\pi/2, \pi/2)$

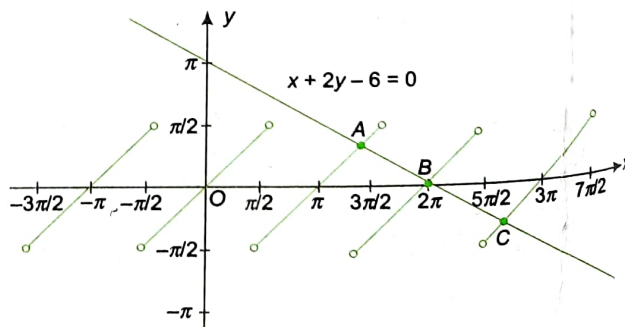
ILLUSTRATION 7.27

Find the number of solutions of $2\tan^{-1}(\tan x) = 6 - x$.

Sol. We have $2\tan^{-1}(\tan x) = 6 - x$

$$\text{or } \tan^{-1}(\tan x) = (6 - x)/2$$

To find the number of solutions, we draw the graphs of $y = \tan^{-1}(\tan x)$ and $y = \frac{6-x}{2}$ or $x + 2y - 6 = 0$ as shown in the following figure.



From the figure number of solutions is three.

CONCEPT APPLICATION EXERCISE 7.2

1. Find the following values:

(a) $\tan^{-1} \tan \frac{13\pi}{5}$

(b) $\sec^{-1} \sec \frac{13\pi}{3}$

(c) $\sin^{-1} \sin \frac{33\pi}{5}$

(d) $\sin^{-1}(\sin 8)$

(e) $\tan^{-1}(\tan 10)$

(f) $\sec^{-1}(\sec 9)$

(g) $\cot^{-1}(\cot 6)$

(h) $\operatorname{cosec}^{-1}(\operatorname{cosec} 7)$

 2. If $f(x) = \sin^{-1}(\sin(\log_2 x))$, then find the value of $f(300)$.

3. Find the maximum value of

$$f(x) = (\sin^{-1}(\sin x))^2 - \sin^{-1}(\sin x)$$

 4. Solve $\sin^{-1}(\sin 5) > x^2 - 4x$.

 5. Consider function $f(x) = \sin^{-1}(\sin x) + \cos^{-1}(\cos x)$, $x \in [0, 2\pi]$.

 (a) Draw the graph of $y = f(x)$.

 (b) Find the range of $f(x)$.

 (c) Find the area bounded by $y = f(x)$ and x -axis.

 6. Find the values of $x \in [0, 2\pi]$ for which functions $f(x) = \tan^{-1}(\tan x)$ and $g(x) = \cos^{-1}(\cos x)$ are identical.

ANSWERS

1. (a) $-\frac{2\pi}{5}$ (b) $\frac{\pi}{3}$ (c) $\frac{2\pi}{5}$ (d) $3\pi - 8$ (e) $10 - 3\pi$

(f) $9 - 2\pi$ (g) $6 - \pi$ (h) $7 - 2\pi$

2. $3\pi - \log_2 300$ 3. $\frac{\pi}{4}(\pi + 2)$

4. $2 - \sqrt{9 - 2\pi} < x < 2 + \sqrt{9 - 2\pi}$

5. Range is $[0, \pi]$, Area = π^2 sq. units 6. $[0, \pi/2) \cup \{2\pi\}$

RELATING DIFFERENT INVERSE TRIGONOMETRIC FUNCTIONS

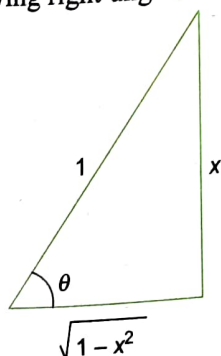
We can write any inverse trigonometric function in the form of the other.

 Consider, $f(x) = \sin^{-1} x$, $x > 0$.

Let $\sin^{-1} x = \theta$,

$$\therefore \sin \theta = \frac{x}{1}$$

So, we have the following right-angled triangle.


 From the figure, $\cos \theta = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$

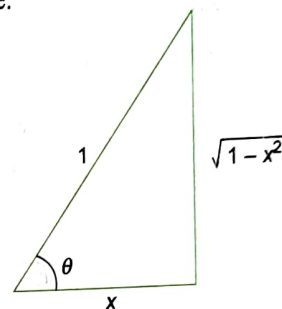
$$\therefore \theta = \sin^{-1} x = \cos^{-1} \sqrt{1-x^2}$$

Also, we have $\theta = \tan^{-1} \frac{x}{\sqrt{1-x^2}}$

$$= \cot^{-1} \frac{\sqrt{1-x^2}}{x}$$

$$= \sec^{-1} \frac{1}{\sqrt{1-x^2}}$$

$$= \operatorname{cosec}^{-1} \frac{1}{x}$$

 Now, for $\theta = \cos^{-1} x$, $x > 0$ or $\cos \theta = x$, we have the following right-angled triangle.


From the figure, $\theta = \sin^{-1} \sqrt{1-x^2}$

$$= \tan^{-1} \frac{\sqrt{1-x^2}}{x}$$

$$= \cot^{-1} \frac{x}{\sqrt{1-x^2}}$$

$$= \sec^{-1} \frac{1}{x}$$

$$= \operatorname{cosec}^{-1} \frac{1}{\sqrt{1-x^2}}$$

ILLUSTRATION 7.28

 Write $\tan^{-1} x$, $x > 0$ in the form of other inverse trigonometric functions.

Sol. For $\theta = \tan^{-1} x$, $x > 0$ or $\tan \theta = x$, we have following right-angled triangle.

From the figure,

$$\theta = \sin^{-1} \frac{x}{\sqrt{1+x^2}}$$

$$= \cos^{-1} \frac{1}{\sqrt{1+x^2}}$$

$$= \cot^{-1} \frac{1}{x}$$

$$= \sec^{-1} \sqrt{1+x^2}$$

$$= \operatorname{cosec}^{-1} \frac{\sqrt{1+x^2}}{x}$$

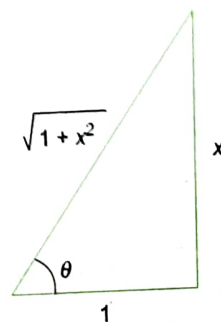


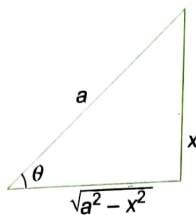
ILLUSTRATION 7.29

Find $\tan^{-1} \frac{x}{\sqrt{a^2 - x^2}}$ in terms of $\sin^{-1}$, where $x \in (0, a)$.

Sol. We have $\tan^{-1} \frac{x}{\sqrt{a^2 - x^2}} = \theta$

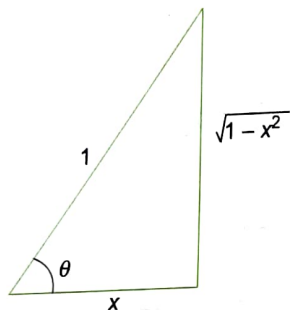
From the triangle,

$$\tan^{-1} \frac{x}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$$

**ILLUSTRATION 7.30**

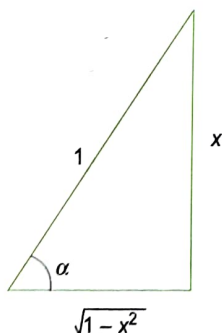
Prove that $\sin(\cot^{-1}(\tan(\cos^{-1} x))) = x$, $x > 0$.

Sol. For $\theta = \cos^{-1} x$, we have the following triangle.



$$\begin{aligned} \therefore E &= \sin(\cot^{-1}(\tan(\cos^{-1} x))) \\ &= \sin\left(\cot^{-1}\left(\tan\left(\tan^{-1} \frac{\sqrt{1-x^2}}{x}\right)\right)\right) \\ &= \sin\left(\cot^{-1} \frac{\sqrt{1-x^2}}{x}\right) \end{aligned}$$

Now, for $\alpha = \cot^{-1} \frac{\sqrt{1-x^2}}{x}$, we have following triangle.



$$\therefore E = \sin\left(\sin^{-1} \frac{x}{1}\right) = x$$

ILLUSTRATION 7.31

If $x < 0$, then prove that $\cos^{-1} x = \pi - \sin^{-1} \sqrt{1-x^2}$.

Sol. Let $x = \cos \theta$ or $\cos^{-1} x = \theta$

Since $x < 0$, $\theta \in [\pi/2, \pi]$

$$\text{Now, } \sin^{-1} \sqrt{1-x^2} = \sin^{-1} \sqrt{1-\cos^2 \theta}$$

$$= \sin^{-1}(\sin \theta) \neq \theta$$

$$= \sin^{-1}(\sin(\pi - \theta)) = \pi - \theta$$

$$\Rightarrow \cos^{-1} x = \pi - \sin^{-1} \sqrt{1-x^2}$$

($\because \theta \in [\pi/2, \pi]$)

ILLUSTRATION 7.32

Prove that $\cos^{-1} \left\{ \sqrt{\frac{1+x}{2}} \right\} = \frac{\cos^{-1} x}{2}$, $-1 < x < 1$.

Sol. Let $x = \cos \theta$, where $\theta \in [0, \pi]$. Then,

$$\begin{aligned} \cos^{-1} \left\{ \sqrt{\frac{1+x}{2}} \right\} &= \cos^{-1} \left\{ \sqrt{\frac{1+\cos \theta}{2}} \right\} \\ &= \cos^{-1} \left\{ \sqrt{\frac{2\cos^2 \frac{\theta}{2}}{2}} \right\} \\ &= \cos^{-1} \left(\cos \frac{\theta}{2} \right) \quad \left(\because \cos \frac{\theta}{2} > 0 \right) \\ &= \frac{\theta}{2} = \frac{\cos^{-1} x}{2} \end{aligned}$$

ILLUSTRATION 7.33

Prove that $\tan^{-1} \left\{ \frac{x}{a + \sqrt{a^2 - x^2}} \right\} = \frac{1}{2} \sin^{-1} \frac{x}{a}$, $-a < x < a$

Sol. Let $x = a \sin \theta$, $-a < x < a$. Then,

$$-a < a \sin \theta < a$$

$$\text{or } -1 < \sin \theta < 1 \Rightarrow \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$$

$$\begin{aligned} \Rightarrow \tan^{-1} \left\{ \frac{x}{a + \sqrt{a^2 - x^2}} \right\} &= \tan^{-1} \left\{ \frac{a \sin \theta}{a + \sqrt{a^2 - a^2 \sin^2 \theta}} \right\} \\ &= \tan^{-1} \left\{ \frac{\sin \theta}{1 + \cos \theta} \right\} \\ &= \tan^{-1} \left\{ \frac{2 \sin(\theta/2) \cos(\theta/2)}{2 \cos^2(\theta/2)} \right\} \\ &= \tan^{-1} \left\{ \tan \frac{\theta}{2} \right\} \\ &= \frac{\theta}{2} = \frac{1}{2} \sin^{-1} \frac{x}{a} \end{aligned}$$

SIMPLIFYING EXPRESSION USING TRIGONOMETRIC SUBSTITUTION

Using some standard trigonometric substitutions, many typical expressions can be simplified.

ILLUSTRATION 7.34

Prove that
 $\sin^{-1} \left\{ \frac{\sqrt{1+x} + \sqrt{1-x}}{2} \right\} = \frac{\pi}{4} + \frac{\cos^{-1} x}{2}, 0 < x < 1.$

Sol. Let $x = \cos \theta$.

Since $0 < x < 1, \theta \in \left(0, \frac{\pi}{2}\right)$

$$\begin{aligned} \Rightarrow \sin^{-1} \left\{ \frac{\sqrt{1+x} + \sqrt{1-x}}{2} \right\} \\ &= \sin^{-1} \left\{ \frac{\sqrt{1+\cos \theta} + \sqrt{1-\cos \theta}}{2} \right\} \\ &= \sin^{-1} \left\{ \frac{\sqrt{2\cos^2 \frac{\theta}{2}} + \sqrt{2\sin^2 \frac{\theta}{2}}}{2} \right\} \\ &= \sin^{-1} \left\{ \frac{\cos \frac{\theta}{2} + \sin \frac{\theta}{2}}{\sqrt{2}} \right\} \\ &= \sin^{-1} \left\{ \sin \left(\frac{\pi}{4} + \frac{\theta}{2} \right) \right\} \\ &= \frac{\pi}{4} + \frac{\theta}{2} \quad \left[\because \theta \in \left(0, \frac{\pi}{2}\right) \Rightarrow \left(\frac{\pi}{4} + \frac{\theta}{2}\right) \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \right] \\ &= \frac{\pi}{4} + \frac{\cos^{-1} x}{2} \end{aligned}$$

ILLUSTRATION 7.35

Prove that $\cos^{-1} \left(\frac{1-x^{2n}}{1+x^{2n}} \right) = 2 \tan^{-1} x^n, 0 < x < \infty.$

Sol. Since $0 < x < \infty; 0 < x^n < \infty$

Let $x^n = \tan \theta \Rightarrow \theta \in (0, \pi/2)$

$$\begin{aligned} \cos^{-1} \left(\frac{1-x^{2n}}{1+x^{2n}} \right) &= \cos^{-1} \left(\frac{1-\tan^2 \theta}{1+\tan^2 \theta} \right) \\ &= \cos^{-1} (\cos 2\theta) \\ &= 2\theta \quad [\because \theta \in (0, \pi)] \\ &= 2 \tan^{-1} x^n \end{aligned}$$

ILLUSTRATION 7.36

If $x \in [-1, 0]$, then find the value of $\cos^{-1}(2x^2 - 1) - 2 \sin^{-1} x$.

Sol. Let $x = \cos \theta$, for $x \in [-1, 0], \theta \in (\pi/2, \pi]$

$$\begin{aligned} \Rightarrow \cos^{-1}(2x^2 - 1) - 2 \sin^{-1} x \\ &= \cos^{-1}(2 \cos^2 \theta - 1) - 2 \sin^{-1}(\cos \theta) \\ &= \cos^{-1}(\cos 2\theta) - 2 \sin^{-1}(\sin(\pi/2 - \theta)) \end{aligned}$$

$$\begin{aligned} &= \cos^{-1}(\cos 2\theta) - 2 \left(\frac{\pi}{2} - \theta \right) \\ &= 2\pi - 2\theta + 2\theta - \pi \\ &= \pi \end{aligned}$$

ILLUSTRATION 7.37

If $\frac{1}{\sqrt{2}} < x < 1$, then prove that

$$\cos^{-1} x + \cos^{-1} \left(\frac{x + \sqrt{1-x^2}}{\sqrt{2}} \right) = \frac{\pi}{4}.$$

Sol. $\cos^{-1} x + \cos^{-1} \left(\frac{x + \sqrt{1-x^2}}{\sqrt{2}} \right)$

Let $\cos^{-1} x = \theta$ or $x = \cos \theta$

For $\frac{1}{\sqrt{2}} < x < 1, 0 < \theta < \frac{\pi}{4}$

$$\begin{aligned} \Rightarrow \cos^{-1} x + \cos^{-1} \left(\frac{x + \sqrt{1-x^2}}{\sqrt{2}} \right) \\ &= \theta + \cos^{-1} \left(\frac{\cos \theta + \sqrt{1-\cos^2 \theta}}{\sqrt{2}} \right) \\ &= \theta + \cos^{-1} \left(\frac{\cos \theta + \sin \theta}{\sqrt{2}} \right) \\ &= \theta + \cos^{-1} \left(\cos \left(\theta - \frac{\pi}{4} \right) \right) \end{aligned}$$

Now, $\left(\theta - \frac{\pi}{4} \right) \in \left(-\frac{\pi}{4}, 0 \right)$, which is not the principal values of $\cos^{-1}$ function.

But $\left(\frac{\pi}{4} - \theta \right) \in \left(0, \frac{\pi}{4} \right)$

$$\begin{aligned} \Rightarrow \theta + \cos^{-1} \left(\cos \left(\theta - \frac{\pi}{4} \right) \right) &= \theta + \cos^{-1} \left(\cos \left(\frac{\pi}{4} - \theta \right) \right) \\ &= \theta + \left(\frac{\pi}{4} - \theta \right) \\ &= \frac{\pi}{4} \end{aligned}$$

RELATING $f^{-1}(x)$ WITH $f^{-1}(-x)$

We have the following results:

(i) $\sin^{-1}(-x) = -\sin^{-1}(x)$, for $x \in [-1, 1]$

Proof:

Clearly, $-x \in [-1, 1]$ for all $x \in [-1, 1]$

Let $\sin^{-1}(-x) = \theta$

$\therefore -x = \sin \theta$

$\Rightarrow x = -\sin \theta$

$\Rightarrow x = \sin(-\theta)$

...(1)

$$\Rightarrow -\theta = \sin^{-1} x \quad [\because x \in [-1, 1] \text{ and } -\theta \in [-\pi/2, \pi/2] \text{ for all } \theta \in [-\pi/2, \pi/2]]$$

$$\Rightarrow \theta = -\sin^{-1} x \quad \dots(2)$$

From (1) and (2), we get

$$\sin^{-1}(-x) = -\sin^{-1} x$$

$$(ii) \tan^{-1}(-x) = -\tan^{-1} x, \text{ for } x \in \mathbb{R}$$

$$(iii) \operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x, \text{ for } x \in (-\infty, -1] \cup [1, \infty)$$

Results (ii) and (iii) can be proved in the same way as Result (i) is proved.

$$(iv) \cos^{-1}(-x) = \pi - \cos^{-1} x, \text{ for } x \in [-1, 1]$$

Proof:

Clearly, $-x \in [-1, 1]$ for all $x \in [-1, 1]$

$$\text{Let } \cos^{-1}(-x) = \theta \quad \dots(1)$$

$$\therefore -x = \cos \theta$$

$$\Rightarrow x = -\cos \theta$$

$$\Rightarrow x = \cos(\pi - \theta)$$

$$\Rightarrow \cos^{-1} x = \pi - \theta \quad [\because x \in [-1, 1] \text{ and } \pi - \theta \in [0, \pi] \text{ for all } \theta \in [0, \pi]]$$

$$\Rightarrow \theta = \pi - \cos^{-1} x \quad \dots(2)$$

From (1) and (2), we get

$$\cos^{-1}(-x) = \pi - \cos^{-1} x$$

$$(v) \sec^{-1}(-x) = \pi - \sec^{-1} x, \text{ for } x \in (-\infty, -1] \cup [1, \infty)$$

$$(vi) \cot^{-1}(-x) = \pi - \cot^{-1} x, \text{ for } x \in \mathbb{R}$$

Results (v) and (vi) can be proved in the same way as Result (iv) is proved.

RELATING $f^{-1}(x)$ WITH $f^{-1}\left(\frac{1}{x}\right)$

We have following results:

$$(i) \sin^{-1} \frac{1}{x} = \operatorname{cosec}^{-1} x, \text{ for } x \in (-\infty, -1] \cup [1, \infty)$$

Proof:

$$\text{Let } \operatorname{cosec}^{-1} x = \theta \quad \dots(1)$$

Here, $\theta \in [-\pi/2, \pi/2] - \{0\}$.

and $x \in (-\infty, -1] \cup [1, \infty)$

$$\therefore x = \operatorname{cosec} \theta$$

$$\Rightarrow \frac{1}{x} = \sin \theta$$

$$\Rightarrow \sin^{-1} \frac{1}{x} = \sin^{-1}(\sin \theta)$$

$$\Rightarrow \theta = \sin^{-1} \frac{1}{x} \text{ (as } \theta \in [-\pi/2, \pi/2] - \{0\}) \quad \dots(2)$$

From (1) and (2), we get

$$\sin^{-1} \frac{1}{x} = \operatorname{cosec}^{-1} x$$

$$(ii) \cos^{-1} \frac{1}{x} = \sec^{-1} x, \text{ for } x \in (-\infty, -1] \cup [1, \infty)$$

Proof:

$$\text{Let } \sec^{-1} x = \theta \quad \dots(1)$$

Here, $\theta \in [0, \pi] - \{\pi/2\}$ and $x \in (-\infty, -1] \cup [1, \infty)$.

$$\text{Now, } \sec^{-1} x = \theta$$

$$\therefore x = \sec \theta$$

$$\Rightarrow \frac{1}{x} = \cos \theta$$

$$\Rightarrow \cos^{-1} \frac{1}{x} = \cos^{-1}(\cos \theta)$$

$$\Rightarrow \theta = \cos^{-1} \frac{1}{x} \text{ (as } \theta \in [0, \pi] - \{\pi/2\})$$

From (1) and (2), we get

$$\cos^{-1} \frac{1}{x} = \sec^{-1} x$$

$$(iii) \tan^{-1} \frac{1}{x} = \begin{cases} \cot^{-1} x, & \text{for } x > 0 \\ -\pi + \cot^{-1} x, & \text{for } x < 0 \end{cases}$$

Proof:

Let $\cot^{-1} x = \theta$, where $\theta \in (0, \pi)$ and $x \in \mathbb{R}$.

$$\therefore x = \cot \theta$$

$$\Rightarrow \frac{1}{x} = \tan \theta$$

$$\Rightarrow \tan^{-1} \frac{1}{x} = \tan^{-1}(\tan \theta)$$

Now, $\tan^{-1}(\tan \theta) = \theta$ only if $\theta \in (0, \pi/2)$

For $\theta \in (\pi/2, \pi)$, $\tan^{-1}(\tan \theta)$

$$= \tan^{-1}(\tan(-\pi + \theta))$$

$$= -\pi + \theta \quad [\text{as } (-\pi + \theta) \in (-\pi/2, 0)]$$

$$\text{Thus, } \tan^{-1} \frac{1}{x} = \begin{cases} \theta, & 0 < \theta < \pi/2 \\ -\pi + \theta, & \pi/2 < \theta < \pi \end{cases}$$

$$= \begin{cases} \cot^{-1} x, & 0 < \cot^{-1} x < \pi/2 \\ -\pi + \cot^{-1} x, & \pi/2 < \cot^{-1} x < \pi \end{cases}$$

$$\therefore \tan^{-1} \frac{1}{x} = \begin{cases} \cot^{-1} x, & x > 0 \\ -\pi + \cot^{-1} x, & x < 0 \end{cases}$$

ILLUSTRATION 7.38

Find the value of $\sin^{-1}(\sin 5) + \cos^{-1}(\cos 10) + \tan^{-1}\{\tan(-6)\} + \cot^{-1}\{\cot(-10)\}$.

$$\begin{aligned} \text{Sol. } & \sin^{-1}(\sin 5) + \cos^{-1}(\cos 10) + \tan^{-1}\{\tan(-6)\} \\ & \quad + \cot^{-1}\{\cot(-10)\} \\ &= \sin^{-1}(\sin 5) + \cos^{-1}(\cos 10) - \tan^{-1}\{\tan(6)\} \\ & \quad + \pi - \cot^{-1}\{\cot(10)\} \\ &= (5 - 2\pi) + (4\pi - 10) - (6 - 2\pi) + \pi - (10 - 3\pi) \\ &= 8\pi - 21 \end{aligned}$$

ILLUSTRATION 7.39

Find the minimum value of the function

$$f(x) = \frac{\pi^2}{16 \cot^{-1}(-x)} - \cot^{-1} x.$$

$$\begin{aligned}
 \text{Sol. } f(x) &= \frac{\pi^2}{16 \cot^{-1}(-x)} - (\pi - \cot^{-1}(-x)) \\
 &= \cot^{-1}(-x) + \frac{\pi^2}{16 \cot^{-1}(-x)} - \pi \\
 &= \left(\sqrt{\cot^{-1}(-x)} - \frac{\pi}{4\sqrt{\cot^{-1}(-x)}} \right)^2 + \frac{\pi}{2} - \pi \geq -\frac{\pi}{2}
 \end{aligned}$$

$$f_{\min} = -\frac{\pi}{2}$$

ILLUSTRATION 7.40

Find the range of $y = (\cot^{-1} x)(\cot^{-1}(-x))$.

$$\begin{aligned}
 \text{Sol. } y &= (\cot^{-1} x)(\cot^{-1}(-x)) \\
 &= \cot^{-1}(x)(\pi - \cot^{-1}(x)) \\
 \text{Now, } \cot^{-1}(x) \text{ and } (\pi - \cot^{-1}(x)) &> 0
 \end{aligned}$$

Using A.M. $\geq$ G.M., we get

$$\frac{\cot^{-1} x + (\pi - \cot^{-1} x)}{2} \geq \sqrt{(\cot^{-1} x)(\pi - \cot^{-1} x)}$$

$$0 < \cot^{-1}(x)(\pi - \cot^{-1}(x))$$

$$\leq \left(\frac{\cot^{-1} x + (\pi - \cot^{-1} x)}{2} \right)^2 = \frac{\pi^2}{4}$$

$$0 < y \leq \frac{\pi^2}{4}$$

ILLUSTRATION 7.41

Prove that $2 \tan^{-1}(\operatorname{cosec} \tan^{-1} x - \tan \cot^{-1} x) = \tan^{-1} x$ ($x \neq 0$).

Sol. Case I: $x > 0$

$$\begin{aligned}
 &2 \tan^{-1}(\operatorname{cosec} \tan^{-1} x - \tan \cot^{-1} x) \\
 &= 2 \tan^{-1} \left(\operatorname{cosec} \left(\operatorname{cosec}^{-1} \frac{\sqrt{1+x^2}}{x} \right) - \tan \left(\tan^{-1} \frac{1}{x} \right) \right) \\
 &= 2 \tan^{-1} \left(\frac{\sqrt{1+x^2}}{x} - \frac{1}{x} \right) \\
 &= 2 \tan^{-1} \left(\frac{\sqrt{1+x^2} - 1}{x} \right) \\
 &= 2 \tan^{-1} \left(\frac{\sec \theta - 1}{\tan \theta} \right) \quad (\text{Putting } x = \tan \theta) \\
 &= 2 \tan^{-1} \left(\frac{1 - \cos \theta}{\sin \theta} \right) \\
 &= 2 \tan^{-1} \left(\frac{2 \sin^2 \theta / 2}{2 \sin \theta / 2 \cos \theta / 2} \right) \\
 &= 2 \tan^{-1} \left(\tan \frac{\theta}{2} \right) = 2 \times \frac{\theta}{2} \\
 &= \theta = \tan^{-1} x
 \end{aligned}$$

Case II: $x < 0$

Let $y = -x$.

$\therefore y > 0$.

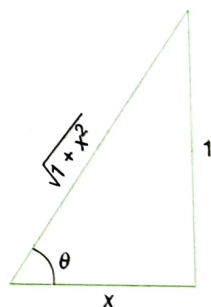
$$\begin{aligned}
 &\therefore 2 \tan^{-1}(\operatorname{cosec} \tan^{-1} x - \tan \cot^{-1} x) \\
 &= 2 \tan^{-1}(\operatorname{cosec} \tan^{-1}(-y) - \tan \cot^{-1}(-y)) \\
 &= 2 \tan^{-1}(\operatorname{cosec}(-\tan^{-1} y) - \tan(\pi - \cot^{-1} y)) \\
 &= 2 \tan^{-1}(-\operatorname{cosec}(\tan^{-1} y) + \tan(\cot^{-1} y)) \\
 &= 2 \tan^{-1} \left(-\operatorname{cosec} \left(\operatorname{cosec}^{-1} \frac{\sqrt{1+y^2}}{y} \right) + \tan \left(\tan^{-1} \frac{1}{y} \right) \right) \\
 &= 2 \tan^{-1} \left(-\frac{\sqrt{1+y^2}}{y} + \frac{1}{y} \right) \\
 &= 2 \tan^{-1} \left(\frac{\sqrt{1+y^2} - 1}{y} \right) \\
 &= -\tan^{-1} y \\
 &= \tan^{-1}(-y) \\
 &= \tan^{-1} x
 \end{aligned}$$

ILLUSTRATION 7.42

Prove that $\cos(\tan^{-1}(\sin(\cot^{-1} x))) = \frac{\sqrt{x^2+1}}{\sqrt{x^2+2}}$.

Sol. $E = \cos(\tan^{-1}(\sin(\cot^{-1} x)))$

Let $\cot^{-1} x = \theta$.



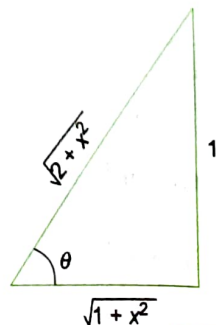
$$\therefore E = \cos \left[\tan^{-1} \left(\sin \left(\sin^{-1} \frac{1}{\sqrt{1+x^2}} \right) \right) \right] \text{ if } x > 0$$

$$\text{or } E = \cos \left[\tan^{-1} \left(\sin \left(\pi - \sin^{-1} \frac{1}{\sqrt{1+x^2}} \right) \right) \right] \text{ if } x < 0$$

In each case,

$$E = \cos \left[\tan^{-1} \frac{1}{\sqrt{1+x^2}} \right]$$

$$\begin{aligned}
 \therefore E &= \cos \left[\cos^{-1} \frac{\sqrt{1+x^2}}{\sqrt{2+x^2}} \right] \\
 &= \frac{\sqrt{x^2+1}}{\sqrt{x^2+2}}
 \end{aligned}$$



CONCEPT APPLICATION EXERCISE 7.3

- Express $\sin^{-1} \frac{\sqrt{x}}{\sqrt{x+a}}$ as a function of $\tan^{-1}$.
- If $\tan(\cos^{-1} x) = \sin\left(\cot^{-1} \frac{1}{2}\right)$, then find the value of x .
- Prove that $\operatorname{cosec}(\tan^{-1}(\cos(\cot^{-1}(\sec(\sin^{-1} a)))) = \sqrt{3-a^2}$, where $a \in [0, 1]$.
- Prove that $\sin \cot^{-1} \tan \cos^{-1} x = \sin \operatorname{cosec}^{-1} \cot \tan^{-1} x = x$, where $x \in (0, 1]$.
- Prove that $\tan^{-1}\left(\frac{\sqrt{1+a^2x^2}-1}{ax}\right) = \frac{\tan^{-1} ax}{2}$, where $x \neq 0$.
- Prove that $\sin\left[2 \tan^{-1}\left\{\sqrt{\frac{1-x}{1+x}}\right\}\right] = \sqrt{1-x^2}$.
- Prove that $\tan^{-1} \frac{1}{\sqrt{x^2-1}} = \frac{\pi}{2} - \sec^{-1} x$, $x > 1$.
- Prove that $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \frac{\pi}{4} - \frac{1}{2} \cos^{-1} x$.
- If $x \leq 0$, then prove that $\cos^{-1}\left(\frac{1+x}{\sqrt{2(1+x^2)}}\right) = \frac{\pi}{4} - \tan^{-1} x$.
- Find the value of $\tan^{-1}\left(-\tan \frac{13\pi}{8}\right) + \cot^{-1}\left(-\cot\left(\frac{19\pi}{8}\right)\right)$.
- Find the value of $\tan\left\{\left(\cos^{-1}\left(-\frac{2}{7}\right) - \frac{\pi}{2}\right)\right\}$.
- If $\tan^{-1}\left(\frac{1}{y}\right) = -\pi + \cot^{-1} y$, where $y = x^2 - 3x + 2$, then find the value of x .

ANSWERS

- $\tan^{-1}\left(\sqrt{\frac{x}{a}}\right)$
- $x = \frac{\sqrt{5}}{3}$
- π
- $\frac{2}{3\sqrt{5}}$
- $x \in (1, 2)$

COMPLEMENTARY ANGLES

Results

- $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, for all $x \in [-1, 1]$
- $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$, for all $x \in \mathbb{R}$
- $\sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2}$ for all $x \in (-\infty, -1] \cup [1, \infty)$

Proof:

- Let $\sin^{-1} x = \theta$
where $\theta \in [-\pi/2, \pi/2]$

$$\Rightarrow -\frac{\pi}{2} \leq -\theta \leq \frac{\pi}{2}$$

$$\Rightarrow 0 \leq \frac{\pi}{2} - \theta \leq \pi$$

$$\Rightarrow \frac{\pi}{2} - \theta \in [0, \pi]$$

$$\text{Now, } \sin^{-1} x = \theta$$

$$\text{or } x = \sin \theta$$

$$\text{or } x = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\text{or } \cos^{-1} x = \frac{\pi}{2} - \theta$$

$\because x \in [-1, 1]$ and $(\pi/2 - \theta) \in [0, \pi]$

$$\text{or } \theta + \cos^{-1} x = \frac{\pi}{2}$$

From Eqs. (i) and (ii), we get $\sin^{-1} x + \cos^{-1} x = \pi/2$. Similarly, we get the other results.

ILLUSTRATION 7.43

If $\sin^{-1} x = \pi/5$, for some $x \in (-1, 1)$, then find the value of $\cos^{-1} x$.

Sol.

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \quad \text{or} \quad \cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x = \frac{\pi}{2} - \frac{\pi}{5} = \frac{3\pi}{10}$$

ILLUSTRATION 7.44

If $\sin\left(\sin^{-1} \frac{1}{5} + \cos^{-1} x\right) = 1$, then find the value of x .

$$\text{Sol. } \sin\left(\sin^{-1} \frac{1}{5} + \cos^{-1} x\right) = 1$$

$$\text{or } \sin^{-1} \frac{1}{5} + \cos^{-1} x = \frac{\pi}{2}$$

$$\text{or } \sin^{-1} \frac{1}{5} = \frac{\pi}{2} - \cos^{-1} x$$

$$\text{or } \sin^{-1} \frac{1}{5} = \sin^{-1} x$$

$$\text{or } x = \frac{1}{5}$$

ILLUSTRATION 7.45

Solve $\sin^{-1} x \leq \cos^{-1} x$.

$$\text{Sol. } \cos^{-1} x \geq \sin^{-1} x$$

$$\text{or } \frac{\pi}{2} - \sin^{-1} x \geq \sin^{-1} x$$

$$\frac{\pi}{2} \geq 2 \sin^{-1} x$$

$$-\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{4}$$

$$-1 \leq x \leq \sin\left(\frac{\pi}{4}\right)$$

$$\Rightarrow x \in \left[-1, \frac{1}{\sqrt{2}}\right]$$

ILLUSTRATION 7.46

Find the range of $f(x) = \sin^{-1} x + \tan^{-1} x + \cos^{-1} x$.

Sol. Clearly, the domain of the function is $[-1, 1]$.

Also, $\tan^{-1} x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ for $x \in [-1, 1]$.

Now, $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \quad \forall x \in [-1, 1]$.

Thus, $f(x) = \tan^{-1} x + \frac{\pi}{2}$, where $x \in [-1, 1]$.

Hence, the range is $\left[-\frac{\pi}{4} + \frac{\pi}{2}, \frac{\pi}{4} + \frac{\pi}{2}\right] = \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$.

ILLUSTRATION 7.47

Find the minimum value of $(\sec^{-1} x)^2 + (\operatorname{cosec}^{-1} x)^2$.

Sol. Let $I = (\sec^{-1} x)^2 + (\operatorname{cosec}^{-1} x)^2$

$$= (\sec^{-1} x + \operatorname{cosec}^{-1} x)^2 - 2 \sec^{-1} x \operatorname{cosec}^{-1} x$$

$$= \frac{\pi^2}{4} - 2 \sec^{-1} x \left(\frac{\pi}{2} - \sec^{-1} x \right)$$

$$= \frac{\pi^2}{4} + 2(\sec^{-1} x)^2 - \pi \sec^{-1} x$$

$$= \frac{\pi^2}{4} + 2 \left[(\sec^{-1} x)^2 - 2 \frac{\pi}{4} \sec^{-1} x + \left(\frac{\pi}{4} \right)^2 \right] - \frac{\pi^2}{8}$$

$$= 2 \left(\sec^{-1} x - \frac{\pi}{4} \right)^2 + \frac{\pi^2}{8} \Rightarrow I \geq \frac{\pi^2}{8}$$

ILLUSTRATION 7.48

Find the range of $f(x) = (\sin^{-1} x)^2 + 2\pi \cos^{-1} x + \pi^2$.

Sol. $f(x) = (\sin^{-1} x)^2 + 2\pi \left(\frac{\pi}{2} - \sin^{-1} x \right) + \pi^2$

$$= (\sin^{-1} x)^2 + \pi^2 - 2\pi \sin^{-1} x + \pi^2$$

$$= (\sin^{-1} x - \pi)^2 + \pi^2$$

$$f_{\min} = \left(\frac{\pi}{2} - \pi \right)^2 + \pi^2 = \frac{\pi^2}{4} + \pi^2 = \frac{5\pi^2}{4}$$

$$f_{\max} = \left(-\frac{\pi}{2} - \pi \right)^2 + \pi^2 = \frac{9\pi^2}{4} + \pi^2 = \frac{13\pi^2}{4}$$

Therefore, range is $\left[\frac{5\pi^2}{4}, \frac{13\pi^2}{4} \right]$.

ILLUSTRATION 7.49

Solve $\sin^{-1} \frac{14}{|x|} + \sin^{-1} \frac{2\sqrt{15}}{|x|} = \frac{\pi}{2}$.

Sol. $\sin^{-1} \frac{14}{|x|} + \sin^{-1} \frac{2\sqrt{15}}{|x|} = \frac{\pi}{2}$

$$\Rightarrow \sin^{-1} \frac{14}{|x|} = \frac{\pi}{2} - \sin^{-1} \frac{2\sqrt{15}}{|x|}$$

$$= \cos^{-1} \frac{2\sqrt{15}}{|x|} = \sin^{-1} \sqrt{1 - \left(\frac{2\sqrt{15}}{|x|} \right)^2}$$

$$\Rightarrow \left(\frac{14}{|x|} \right)^2 = 1 - \left(\frac{2\sqrt{15}}{|x|} \right)^2$$

$$\Rightarrow |x| = 16 \quad \text{or} \quad x = \pm 16, \text{ which satisfies } |x| \geq 2\sqrt{15}.$$

ILLUSTRATION 7.50

If $\alpha = \sin^{-1}(\cos(\sin^{-1} x))$ and $\beta = \cos^{-1}(\sin(\cos^{-1} x))$, then find $\tan \alpha \cdot \tan \beta$.

Sol. $\beta = \cos^{-1} \left(\sin \left(\frac{\pi}{2} - \sin^{-1} x \right) \right)$

$$= \cos^{-1} [\cos(\sin^{-1} x)]$$

Also, $\alpha = \sin^{-1} [\cos(\sin^{-1} x)]$

$$\therefore \alpha + \beta = \frac{\pi}{2}$$

$$\Rightarrow \tan \alpha = \cot \beta \text{ or } \tan \alpha \cdot \tan \beta = 1$$

ILLUSTRATION 7.51

If $\sec^{-1} x = \operatorname{cosec}^{-1} y$, then find the value of

$$\cos^{-1} \frac{1}{x} + \cos^{-1} \frac{1}{y}.$$

Sol. $\sec^{-1} x = \operatorname{cosec}^{-1} y$

or $\cos^{-1} \frac{1}{x} = \sin^{-1} \frac{1}{y}$

or $\cos^{-1} \frac{1}{x} = \frac{\pi}{2} - \cos^{-1} \frac{1}{y}$

or $\cos^{-1} \frac{1}{x} + \cos^{-1} \frac{1}{y} = \frac{\pi}{2}$

ILLUSTRATION 7.52

Prove that $\tan^{-1}x + \tan^{-1}\frac{1}{x} = \begin{cases} \pi/2, & \text{if } x > 0 \\ -\pi/2, & \text{if } x < 0 \end{cases}$

Sol. We know that $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1}x, & x > 0 \\ -\pi + \cot^{-1}x, & x < 0 \end{cases}$

$$\begin{aligned} \Rightarrow \tan^{-1}x + \tan^{-1}\frac{1}{x} &= \begin{cases} \tan^{-1}x + \cot^{-1}x, & x > 0 \\ -\pi + \cot^{-1}x + \tan^{-1}x, & x < 0 \end{cases} \\ &= \begin{cases} \frac{\pi}{2}, & x > 0 \\ -\pi + \frac{\pi}{2}, & x < 0 \end{cases} \\ &= \begin{cases} \frac{\pi}{2}, & x > 0 \\ -\frac{\pi}{2}, & x < 0 \end{cases} \end{aligned}$$

ILLUSTRATION 7.53

For which values of x , function $f(x) = \sin^{-1}x + \sin^{-1}\frac{1}{x} + \cos^{-1}x + \cos^{-1}\frac{1}{x}$ is defined? Also, find the range.

Sol. $f(x) = \sin^{-1}x + \sin^{-1}\frac{1}{x} + \cos^{-1}x + \cos^{-1}\frac{1}{x}$

is defined only when $x, \frac{1}{x} \in [-1, 1]$

which is possible only when $x = \pm 1$.

Also, $(\sin^{-1}x + \cos^{-1}x) + \left(\sin^{-1}\frac{1}{x} + \cos^{-1}\frac{1}{x}\right) = \frac{\pi}{2} + \frac{\pi}{2} = \pi$.

So, range is $\{\pi\}$.

ILLUSTRATION 7.54

Find the value of $\sum_{r=1}^{10} \sum_{s=1}^{10} \tan^{-1}\left(\frac{r}{s}\right)$.

Sol. $S = \sum_{r=1}^{10} \sum_{s=1}^{10} \tan^{-1}\left(\frac{r}{s}\right)$

$$\therefore S = \sum_{r=1}^{10} \sum_{s=1}^{10} \tan^{-1}\left(\frac{s}{r}\right) \quad (\text{As } r \text{ and } s \text{ are independent})$$

On adding, we get

$$2S = \sum_{r=1}^{10} \sum_{s=1}^{10} \left(\tan^{-1}\left(\frac{r}{s}\right) + \tan^{-1}\left(\frac{s}{r}\right) \right)$$

$$\Rightarrow 2S = \sum_{r=1}^{10} \sum_{s=1}^{10} \frac{\pi}{2} = \frac{\pi}{2} \sum_{r=1}^{10} 10 = \frac{100\pi}{2}$$

$$\therefore S = 25\pi$$

ILLUSTRATION 7.55

If $x_i \in [0, 1] \forall i = 1, 2, 3, \dots, 28$ then find the maximum value of

$$\sqrt{\sin^{-1}x_1} \sqrt{\cos^{-1}x_2} + \sqrt{\sin^{-1}x_2} \sqrt{\cos^{-1}x_3} + \sqrt{\sin^{-1}x_3} \sqrt{\cos^{-1}x_4} + \dots + \sqrt{\sin^{-1}x_{28}} \sqrt{\cos^{-1}x_1}$$

Sol. We have

$$E = \sqrt{\sin^{-1}x_1} \sqrt{\cos^{-1}x_2} + \sqrt{\sin^{-1}x_2} \sqrt{\cos^{-1}x_3} + \sqrt{\sin^{-1}x_3} \sqrt{\cos^{-1}x_4} + \dots + \sqrt{\sin^{-1}x_{28}} \sqrt{\cos^{-1}x_1}$$

$$x_i \in [0, 1] \forall i = 1, 2, 3, \dots, 28$$

$$\therefore \sin^{-1}x_i > 0$$

Now using A.M. $\geq$ G.M., we have

$$\frac{a^2 + b^2}{2} \geq ab, \text{ where } a, b > 0$$

$$\therefore \sqrt{\sin^{-1}x_1} \sqrt{\cos^{-1}x_2} \leq \left(\frac{\sin^{-1}x_1 + \cos^{-1}x_2}{2} \right)$$

$$\sqrt{\sin^{-1}x_2} \sqrt{\cos^{-1}x_3} \leq \left(\frac{\sin^{-1}x_2 + \cos^{-1}x_3}{2} \right)$$

$$\vdots$$

$$\sqrt{\sin^{-1}x_{28}} \sqrt{\cos^{-1}x_1} \leq \left(\frac{\sin^{-1}x_{28} + \cos^{-1}x_1}{2} \right)$$

On adding all, we get

$$E \leq \sum_{i=1}^{28} \frac{\sin^{-1}x_i + \cos^{-1}x_i}{2}$$

$$\therefore E \leq \frac{28\left(\frac{\pi}{2}\right)}{2}$$

$$\therefore E_{\max} = 7\pi$$

CONCEPT APPLICATION EXERCISE 7.4

1. If $\sin^{-1}x + \sin^{-1}y = \frac{2\pi}{3}$, then find the value of $\cos^{-1}x + \cos^{-1}y$.
2. Solve $\cot^{-1}x + \tan^{-1}3 = \frac{\pi}{2}$.
3. Solve $2\cos^{-1}x + \sin^{-1}x = \frac{2\pi}{3}$.
4. Prove that $\sin^{-1}\cos(\sin^{-1}x) + \cos^{-1}\sin(\cos^{-1}x) = \frac{\pi}{2}, |x| \leq 1$.
5. Solve for x and y :
 $\sin^{-1}x + \sin^{-1}y = \frac{2\pi}{3}$ and $\cos^{-1}x - \cos^{-1}y = \frac{\pi}{3}$.
6. Solve $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \frac{5\pi^2}{8}$.
7. Solve $\sec^{-1}x > \operatorname{cosec}^{-1}x$.
8. Solve $\tan^{-1}x > \cot^{-1}x$.

9. If α is the only real root of the equation $x^3 + bx^2 + cx + 1 = 0$ ($b < c$), then find the value of $\tan^{-1} \alpha + \tan^{-1}(\alpha^{-1})$.
10. If $\alpha \in \left(-\frac{\pi}{2}, 0\right)$, then find the value of $\tan^{-1}(\cot \alpha) - \cot^{-1}(\tan \alpha)$.
11. Find the maximum value of $(\sec^{-1} x)(\operatorname{cosec}^{-1} x)$, $x \geq 1$.
12. If equation $\sin^{-1}(4 \sin^2 \theta + \sin \theta) + \cos^{-1}(6 \sin \theta - 1) = \frac{\pi}{2}$ has 10 solutions for $\theta \in [0, n\pi]$, then find the minimum value of n .

ANSWERS

1. $\frac{\pi}{3}$ 2. $x = 3$ 3. $\frac{\sqrt{3}}{2}$
5. $x = \frac{1}{2}, y = 1$ 6. $x = -1$ 7. $x \in (-\infty, -1] \cup (\sqrt{2}, \infty)$
8. $x > 1$ 9. $-\frac{\pi}{2}$ 10. $-\pi$
11. $\frac{\pi^2}{16}$ 12. 9

SUM AND DIFFERENCE OF ANGLES IN TERMS OF $\tan^{-1}$

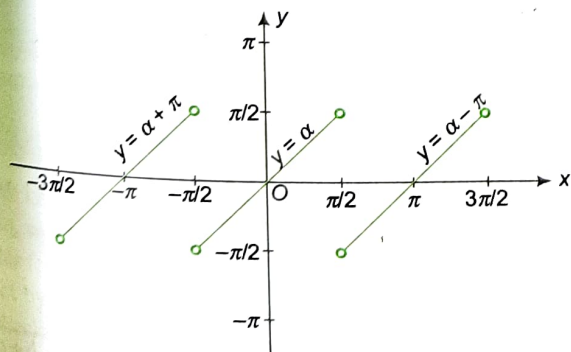
We have the following formulas for sum of angles when angles are in terms of $\tan^{-1}$:

(i) $\tan^{-1} x + \tan^{-1} y$

$$= \begin{cases} \tan^{-1} \left(\frac{x+y}{1-xy} \right), & \text{if } xy < 1 \\ \pi + \tan^{-1} \left(\frac{x+y}{1-xy} \right), & \text{if } x > 0, y > 0 \text{ and } xy > 1 \\ -\pi + \tan^{-1} \left(\frac{x+y}{1-xy} \right), & \text{if } x < 0, y < 0 \text{ and } xy > 1 \end{cases}$$

Let $\tan^{-1} x = A$ and $\tan^{-1} y = B$;

where $A, B \in (-\pi/2, \pi/2)$.



$$\text{Now, } \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{x+y}{1-xy}$$

$$\Rightarrow \tan^{-1} \left(\frac{x+y}{1-xy} \right) = \tan^{-1} \tan(A+B) = \tan^{-1} \tan \alpha$$

Here, $(A+B) = \alpha \in (-\pi, \pi)$

Now, consider graph of $y = \tan^{-1} \tan \alpha$.

From the graph,

$$\tan^{-1} \left(\frac{x+y}{1-xy} \right) = \tan^{-1}(\tan \alpha)$$

$$= \begin{cases} \alpha + \pi, & -\pi < \alpha < -\pi/2 \\ \alpha, & -\pi/2 \leq \alpha \leq \pi/2 \quad \dots (1) \\ \alpha - \pi, & \pi/2 < \alpha < \pi \end{cases}$$

Case I:

When $-\pi < \alpha < -\pi/2$

$$\therefore -\pi < \tan^{-1} x + \tan^{-1} y < -\pi/2$$

$$\Rightarrow x < 0, y < 0$$

$$\text{Also, } \tan^{-1} x < -\frac{\pi}{2} - \tan^{-1} y$$

$$\Rightarrow \tan^{-1} x < -\left(\frac{\pi}{2} - \tan^{-1}(-y)\right)$$

$$\Rightarrow \tan^{-1} x < -\tan^{-1} \left(-\frac{1}{y} \right)$$

$$\Rightarrow x < -\left(-\frac{1}{y} \right)$$

$$\Rightarrow xy > 1 \quad (\text{as } y < 0)$$

Case II:

When $\pi/2 < \tan^{-1} x + \tan^{-1} y < \pi$

$$\Rightarrow x, y > 0$$

$$\text{Also, } \tan^{-1} x > \pi/2 - \tan^{-1} y$$

$$\Rightarrow \tan^{-1} x > \tan^{-1} \frac{1}{y}$$

$$\Rightarrow x > \frac{1}{y}$$

$$\Rightarrow xy > 1$$

Case III:

When $-\pi/2 < \tan^{-1} x + \tan^{-1} y < \pi/2$

If $x, y > 0$ then $0 < \tan^{-1} x + \tan^{-1} y < \pi/2$

$$\Rightarrow \tan^{-1} x + \tan^{-1} y < \pi/2$$

$$\Rightarrow \tan^{-1} x < \frac{\pi}{2} - \tan^{-1} y$$

$$\Rightarrow \tan^{-1} x < \tan^{-1} \frac{1}{y}$$

$$\Rightarrow x < \frac{1}{y}$$

$$\Rightarrow xy < 1$$

If $x, y < 0$ then $-\pi/2 < \tan^{-1} x + \tan^{-1} y < 0$

$$\Rightarrow \tan^{-1} x + \tan^{-1} y > -\pi/2$$

$$\Rightarrow \tan^{-1} x > -\frac{\pi}{2} - \tan^{-1} y$$

$$\Rightarrow \tan^{-1} x > -\left(\frac{\pi}{2} - \tan^{-1}(-y)\right)$$

$$\Rightarrow \tan^{-1} x > -\tan^{-1} \left(-\frac{1}{y} \right)$$

$$\Rightarrow x > \frac{1}{y}$$

$$\Rightarrow xy < 1 \quad (\text{as } y < 0)$$

If x and y are of opposite sign, then $xy < 1$.

Thus, from (1), we have

$$\tan^{-1} x + \tan^{-1} y$$

$$= \begin{cases} \tan^{-1} \left(\frac{x+y}{1-xy} \right), & \text{if } xy < 1 \\ \pi + \tan^{-1} \left(\frac{x+y}{1-xy} \right), & \text{if } x > 0, y > 0 \text{ and } xy > 1 \dots (2) \\ -\pi + \tan^{-1} \left(\frac{x+y}{1-xy} \right), & \text{if } x < 0, y < 0 \text{ and } xy > 1 \end{cases}$$

Case IV:

If $xy = 1$ then

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} x + \tan^{-1} \frac{1}{x} = \frac{\pi}{2}$$

(ii) In above set of formulas replacing y by $(-y)$, we get

$$\tan^{-1} x - \tan^{-1} y$$

$$= \begin{cases} \tan^{-1} \left(\frac{x-y}{1+xy} \right), & \text{if } xy > -1 \\ \pi + \tan^{-1} \left(\frac{x-y}{1+xy} \right), & \text{if } x > 0, y < 0 \text{ and } xy < -1 \\ -\pi + \tan^{-1} \left(\frac{x-y}{1+xy} \right), & \text{if } x < 0, y > 0 \text{ and } xy < -1 \end{cases}$$

(iii) In set of formulas of (i), replacing y by x we get

$$2 \tan^{-1} x = \begin{cases} \tan^{-1} \frac{2x}{1-x^2}, & -1 < x < 1 \\ \pi + \tan^{-1} \frac{2x}{1-x^2}, & x > 1 \\ -\pi + \tan^{-1} \frac{2x}{1-x^2}, & x < -1 \end{cases}$$

ILLUSTRATION 7.56

If two angles of a triangle are $\tan^{-1}(2)$ and $\tan^{-1}(3)$, then find the third angle.

Sol. Given two angles are $\tan^{-1}(2)$ and $\tan^{-1}(3)$. Now $(2)(3) > 1$

$$\Rightarrow \tan^{-1}(2) \text{ and } \tan^{-1}(3) = \pi + \tan^{-1} \left(\frac{2+3}{1-2 \times 3} \right)$$

$$= \pi + \tan^{-1}(-1) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

Hence, the third angle is $\pi - \frac{3\pi}{4} = \frac{\pi}{4}$.

ILLUSTRATION 7.57

Prove that $\cos^{-1} \frac{4}{5} + \cos^{-1} \frac{12}{13} = \cos^{-1} \frac{33}{65}$.

Sol. $\cos^{-1} \frac{4}{5} + \cos^{-1} \frac{12}{13}$

$$\begin{aligned} &= \tan^{-1} \frac{3}{4} + \tan^{-1} \frac{5}{12} = \tan^{-1} \frac{\frac{3}{4} + \frac{5}{12}}{1 - \left(\frac{3}{4} \right) \left(\frac{5}{12} \right)} \quad \left(\because \frac{3}{4} \times \frac{5}{12} < 1 \right) \\ &= \tan^{-1} \left(\frac{36+20}{48-15} \right) = \tan^{-1} \frac{56}{33} \\ &= \cos^{-1} \frac{33}{\sqrt{33^2 + 56^2}} = \cos^{-1} \frac{33}{65} \end{aligned}$$

ILLUSTRATION 7.58

Find the value of $\tan^{-1} \left(\frac{1}{2} \tan 2A \right) + \tan^{-1}(\cot A)$

+ $\tan^{-1}(\cot^3 A)$, for $0 < A < \frac{\pi}{4}$.

Sol. For $0 < A < (\pi/4)$, $\cot A > 1$

$$\Rightarrow (\cot A)(\cot^3 A) > 1$$

Then, $\tan^{-1} \left(\frac{1}{2} \tan 2A \right) + \tan^{-1}(\cot A) + \tan^{-1}(\cot^3 A)$

$$= \tan^{-1} \left(\frac{\tan A}{1 - \tan^2 A} \right) + \pi + \tan^{-1} \left(\frac{\cot A + \cot^3 A}{1 - \cot^4 A} \right)$$

$$= \tan^{-1} \left(\frac{\tan A}{1 - \tan^2 A} \right) + \pi + \tan^{-1} \left(\frac{\cot A}{1 - \cot^2 A} \right)$$

$$= \tan^{-1} \left(\frac{\tan A}{1 - \tan^2 A} \right) + \pi + \tan^{-1} \left(\frac{\tan A}{\tan^2 A - 1} \right) = \pi$$

ILLUSTRATION 7.59

Let a, b and c be positive real numbers. Then prove that

$$\begin{aligned} &\tan^{-1} \sqrt{\frac{a(a+b+c)}{bc}} + \tan^{-1} \sqrt{\frac{b(a+b+c)}{ca}} \\ &\quad + \tan^{-1} \sqrt{\frac{c(a+b+c)}{ab}} = \pi. \end{aligned}$$

Sol. Let

$$\begin{aligned} S &= \tan^{-1} \sqrt{\frac{a(a+b+c)}{bc}} + \tan^{-1} \sqrt{\frac{b(a+b+c)}{ca}} \\ &\quad + \tan^{-1} \sqrt{\frac{c(a+b+c)}{ab}} \end{aligned}$$

$$\text{Now, } \sqrt{\frac{a(a+b+c)}{bc}} \sqrt{\frac{b(a+b+c)}{ca}} = \frac{a+b+c}{c} = 1 + \frac{b}{c} + \frac{a}{c} > 1$$

$$\begin{aligned}
 S &= \pi + \tan^{-1} \frac{\sqrt{\frac{a(a+b+c)}{bc}} + \sqrt{\frac{b(a+b+c)}{ca}}}{1 - \sqrt{\frac{a(a+b+c)}{bc}} \sqrt{\frac{b(a+b+c)}{ca}}} \\
 &\quad + \tan^{-1} \sqrt{\frac{c(a+b+c)}{ab}} \\
 &= \pi + \tan^{-1} \frac{\sqrt{\frac{a+b+c}{c}} \left(\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} \right)}{1 - \frac{a+b+c}{c}} + \tan^{-1} \sqrt{\frac{c(a+b+c)}{ab}} \\
 &= \pi + \tan^{-1} \left(-\sqrt{\frac{c(a+b+c)}{ab}} \right) + \tan^{-1} \sqrt{\frac{c(a+b+c)}{ab}} \\
 &= \pi
 \end{aligned}$$

ILLUSTRATION 7.60

Simplify

$$\left[\frac{3 \sin 2\alpha}{5 + 3 \cos 2\alpha} \right] + \tan^{-1} \left[\frac{\tan \alpha}{4} \right], \text{ where } -\frac{\pi}{2} < \alpha < \frac{\pi}{2}.$$

Sol.

$$\begin{aligned}
 &\tan^{-1} \left[\frac{3 \sin 2\alpha}{5 + 3 \cos 2\alpha} \right] + \tan^{-1} \left[\frac{\tan \alpha}{4} \right] \\
 &= \tan^{-1} \left(\frac{6 \tan \alpha}{8 + 2 \tan^2 \alpha} \right) + \tan^{-1} \left(\frac{\tan \alpha}{4} \right) \\
 &= \tan^{-1} \left(\frac{\frac{3 \tan \alpha}{4 + \tan^2 \alpha} + \frac{\tan \alpha}{4}}{1 - \frac{3 \tan^2 \alpha}{16 + 4 \tan^2 \alpha}} \right) \quad \left[\because \frac{3 \tan^2 \alpha}{16 + 4 \tan^2 \alpha} < 1 \right] \\
 &= \tan^{-1} \left(\frac{12 \tan \alpha + 4 \tan \alpha + \tan^3 \alpha}{16 + \tan^2 \alpha} \right) \\
 &= \tan^{-1} (\tan \alpha) = \alpha
 \end{aligned}$$

ILLUSTRATION 7.61

Solve the equation $\tan^{-1} 2x + \tan^{-1} 3x = \pi/4$.

Sol. While solving such equations we use the following formulas:

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy} \quad \dots(i)$$

We know that this formula is valid only when $xy < 1$. So, we may get some extra solutions by solving equations with this formula. The extra solutions can be removed from the solution set by putting the values obtained in the original equation and checking whether it satisfies the equation or not.

$$\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$$

$$\text{or } \tan^{-1} \left(\frac{2x+3x}{1-6x^2} \right) = \frac{\pi}{4}$$

$$\text{or } \frac{5x}{1-6x^2} = 1$$

$$\text{or } 6x^2 + 5x - 1 = 0$$

$$\text{or } (6x-1)(x+1) = 0$$

$$\text{or } x = 1/6, -1$$

But for $x = -1$, $\tan^{-1} 2x + \tan^{-1} 3x < 0$.

So, it does not satisfy Eq. (i).

$$\text{Hence, } x = \frac{1}{6}.$$

ILLUSTRATION 7.62

Solve $\tan^{-1} x + \sin^{-1} x = \tan^{-1} 2x$.

$$\text{Sol. } \tan^{-1} x + \sin^{-1} x = \tan^{-1} 2x$$

$$\Rightarrow \sin^{-1} x = \tan^{-1} 2x - \tan^{-1} x$$

$$\Rightarrow \tan^{-1} \frac{x}{\sqrt{1-x^2}} = \tan^{-1} \left[\frac{2x-x}{1+2x^2} \right]$$

$$\Rightarrow \frac{x}{\sqrt{1-x^2}} = \frac{x}{1+2x^2}$$

$$\Rightarrow 2x^3 + x = x\sqrt{1-x^2}$$

$$\Rightarrow 2x^3 - x\sqrt{1-x^2} + x = 0$$

$$\Rightarrow x(2x^2 - \sqrt{1-x^2} + 1) = 0$$

$$\Rightarrow x = 0 \text{ or } 2x^2 + 1 = \sqrt{1-x^2}$$

$$\Rightarrow x = 0 \text{ or } 4x^4 + 4x^2 + 1 = 1 - x^2$$

$$\Rightarrow x = 0 \text{ or } 4x^4 + 5x^2 = 0$$

$$\Rightarrow x = 0 \text{ is the only solution}$$

ILLUSTRATION 7.63

Solve $\cot^{-1} \left(\frac{3x^2+1}{x} \right) = \cot^{-1} \left(\frac{1-3x^2}{x} \right) - \tan^{-1} 6x$.

$$\text{Sol. } \cot^{-1} \left(\frac{3x^2+1}{x} \right) = \cot^{-1} \left(\frac{1-3x^2}{x} \right) - \tan^{-1} 6x \quad \dots(i)$$

$$\Rightarrow \tan^{-1} 6x = \tan^{-1} \left(\frac{3x^2+1}{x} \right) - \tan^{-1} \left(\frac{1-3x^2}{x} \right)$$

$$\Rightarrow 6x = \frac{\frac{3x^2+1}{x} - \frac{1-3x^2}{x}}{1 + \left(\frac{3x^2+1}{x} \right) \left(\frac{1-3x^2}{x} \right)}$$

$$\Rightarrow 6x = \frac{6x^2}{x} \times \frac{x^2}{x^2+1-9x^4}$$

$$\Rightarrow x^2 + 1 - 9x^4 = x^2$$

$$\Rightarrow 1 - 9x^4 = 0$$

$$\Rightarrow x = \pm \frac{1}{\sqrt{3}}$$

Both values satisfy given equation (i).

($\because x \neq 0$)

ILLUSTRATION 7.64

If $x > y > z > 0$, then find the value of

$$\cot^{-1} \frac{xy+1}{x-y} + \cot^{-1} \frac{yz+1}{y-z} + \cot^{-1} \frac{zx+1}{z-x}.$$

Sol. $\cot^{-1} \frac{xy+1}{x-y} + \cot^{-1} \frac{yz+1}{y-z} + \cot^{-1} \frac{zx+1}{z-x}$

$$\tan^{-1} \frac{x-y}{1+xy} + \tan^{-1} \frac{y-z}{1+yz} + \pi + \tan^{-1} \frac{z-x}{1+zx}$$

$$= \left[\because \tan^{-1} \left(\frac{1}{x} \right) = \begin{cases} \cot^{-1} x, & \text{for } x > 0 \\ -\pi + \cot^{-1} x, & \text{for } x < 0 \end{cases} \right]$$

$$= \tan^{-1} x - \tan^{-1} y + \tan^{-1} y - \tan^{-1} z + \pi + \tan^{-1} z - \tan^{-1} x$$

$$= \pi$$

ILLUSTRATION 7.65

Solve $\tan^{-1} x + \cot^{-1}(-x) = 2 \tan^{-1} 6x$.

Sol. If $x > 0$,

$$\tan^{-1} x + \cot^{-1}(-x) = 2 \tan^{-1} 6x$$

$$\Rightarrow \tan^{-1} x + \pi - \cot^{-1} x = 2 \tan^{-1} 6x$$

$$\Rightarrow \frac{\pi}{2} + 2 \tan^{-1} x = 2 \tan^{-1} 6x$$

$$\Rightarrow \tan^{-1} 6x - \tan^{-1} x = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1} \frac{6x-x}{1+6x^2} = \frac{\pi}{4}$$

$$\Rightarrow \frac{5x}{1+6x^2} = 1$$

$$\Rightarrow 6x^2 - 5x + 1 = 0$$

$$\Rightarrow x = \frac{1}{2}, \frac{1}{3}$$

If $x < 0$,

$$\tan^{-1} x + \cot^{-1} x = 2 \tan^{-1} 6x$$

$$\Rightarrow \tan^{-1} 6x = \frac{\pi}{4}; \text{ This is not possible as } x < 0.$$

ILLUSTRATION 7.66

If $a_1, a_2, a_3, \dots, a_n$ is an A.P. with common difference d , then prove that

$$\tan \left[\tan^{-1} \left(\frac{d}{1+a_1 a_2} \right) + \tan^{-1} \left(\frac{d}{1+a_2 a_3} \right) + \dots \right. \\ \left. + \tan^{-1} \left(\frac{d}{1+a_{n-1} a_n} \right) \right] = \frac{(n-1)d}{1+a_1 a_n}$$

Sol. We have

$$\tan^{-1} \left(\frac{d}{1+a_1 a_2} \right) + \tan^{-1} \left(\frac{d}{1+a_2 a_3} \right) + \dots + \tan^{-1} \left(\frac{d}{1+a_{n-1} a_n} \right)$$

$$= \tan^{-1} \left(\frac{a_2 - a_1}{1+a_1 a_2} \right) + \tan^{-1} \left(\frac{a_3 - a_2}{1+a_2 a_3} \right) + \dots + \tan^{-1} \left(\frac{a_n - a_{n-1}}{1+a_{n-1} a_n} \right)$$

$$= (\tan^{-1} a_2 - \tan^{-1} a_1) + (\tan^{-1} a_3 - \tan^{-1} a_2) + \dots$$

$$+ (\tan^{-1} a_n - \tan^{-1} a_{n-1})$$

$$= \tan^{-1} a_n - \tan^{-1} a_1 = \tan^{-1} \left(\frac{a_n - a_1}{1 + a_n a_1} \right) = \tan^{-1} \left(\frac{(n-1)d}{1 + a_1 a_n} \right)$$

$$\Rightarrow \tan \left[\tan^{-1} \left(\frac{d}{1+a_1 a_2} \right) + \tan^{-1} \left(\frac{d}{1+a_2 a_3} \right) + \dots \right. \\ \left. + \tan^{-1} \left(\frac{d}{1+a_{n-1} a_n} \right) \right] = \frac{(n-1)d}{1+a_1 a_n}$$

ILLUSTRATION 7.67

Find the value of $\sum_{r=0}^{\infty} \tan^{-1} \left(\frac{1}{1+r+r^2} \right)$.

Sol. $\tan^{-1} \left(\frac{1}{1+r+r^2} \right) = \tan^{-1} \left(\frac{r+1-r}{1+r(r+1)} \right)$

$$= \tan^{-1}(r+1) - \tan^{-1}(r)$$

$$\Rightarrow \sum_{r=0}^n [\tan^{-1}(r+1) - \tan^{-1}(r)] = \tan^{-1}(n+1) - \tan^{-1}(0)$$

$$= \tan^{-1}(n+1)$$

$$\Rightarrow \sum_{r=0}^{\infty} \tan^{-1} \left(\frac{1}{1+r+r^2} \right) = \tan^{-1}(\infty) = \frac{\pi}{2}$$

ILLUSTRATION 7.68

Find the sum $\sum_{r=1}^{\infty} \tan^{-1} \left(\frac{2(2r-1)}{4+r^2(r^2-2r+1)} \right)$.

Sol. $S_n = \sum_{r=1}^n \tan^{-1} \left(\frac{2(2r-1)}{4+r^2(r^2-2r+1)} \right)$

$$= \sum_{r=1}^n \tan^{-1} \left(\frac{2(2r-1)}{4+r^2(r-1)^2} \right)$$

$$= \sum_{r=1}^n \tan^{-1} \left(\frac{\frac{2r-1}{2}}{1+\frac{r^2}{2} \cdot \frac{(r-1)^2}{2}} \right)$$

$$= \sum_{r=1}^n \tan^{-1} \left(\frac{\frac{r^2}{2} - \frac{(r-1)^2}{2}}{1+\frac{r^2}{2} \cdot \frac{(r-1)^2}{2}} \right)$$

$$= \sum_{r=1}^n \left(\tan^{-1} \frac{r^2}{2} - \tan^{-1} \frac{(r-1)^2}{2} \right)$$

$$= \tan^{-1} \frac{n^2}{2}$$

$$\therefore S_{\infty} = \lim_{n \rightarrow \infty} \tan^{-1} \frac{n^2}{2} = \frac{\pi}{2}$$

ILLUSTRATION 7.69

Find the value of $4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{70} + \tan^{-1} \frac{1}{99}$.

$$\begin{aligned} \text{Sol. } & 4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{70} + \tan^{-1} \frac{1}{99} \\ &= 2 \tan^{-1} \left[\frac{\frac{2}{5}}{1 - \frac{1}{25}} \right] - \tan^{-1} \frac{1}{70} + \tan^{-1} \frac{1}{99} \\ &= 2 \tan^{-1} \left(\frac{5}{12} \right) + \tan^{-1} \left[\frac{\frac{1}{99} - \frac{1}{70}}{1 + \frac{1}{99} \times \frac{1}{70}} \right] \\ &= \tan^{-1} \left[\frac{\frac{5}{6}}{1 - \frac{25}{144}} \right] + \tan^{-1} \left(\frac{-29}{6931} \right) \\ &= \tan^{-1} \left(\frac{120}{119} \right) - \tan^{-1} \left(\frac{1}{239} \right) \\ &= \tan^{-1} \left[\frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119} \times \frac{1}{239}} \right] = \tan^{-1} (1) = \frac{\pi}{4} \end{aligned}$$

ILLUSTRATION 7.70

If $(x-1)(x^2+1) > 0$, then find the value of

$$\sin \left(\frac{1}{2} \tan^{-1} \frac{2x}{1-x^2} - \tan^{-1} x \right).$$

Sol. $(x-1)(x^2+1) > 0$

$$\Rightarrow x > 1$$

$$\begin{aligned} \therefore \sin \left[\frac{1}{2} \tan^{-1} \left(\frac{2x}{1-x^2} \right) - \tan^{-1} x \right] \\ = \sin \left[\frac{1}{2} (-\pi + 2 \tan^{-1} x) - \tan^{-1} x \right] = \sin \left(-\frac{\pi}{2} \right) = -1 \end{aligned}$$

ILLUSTRATION 7.71

Prove that

$$3 \tan^{-1} x = \begin{cases} \tan^{-1} \left(\frac{3x-x^3}{1-3x^2} \right), & \text{if } -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}} \\ \pi + \tan^{-1} \left(\frac{3x-x^3}{1-3x^2} \right), & \text{if } x > \frac{1}{\sqrt{3}} \\ -\pi + \tan^{-1} \left(\frac{3x-x^3}{1-3x^2} \right), & \text{if } x < -\frac{1}{\sqrt{3}} \end{cases}$$

Sol. Let $x = \tan \theta$, where $\theta \in (-\pi/2, \pi/2)$.

$$\tan^{-1} \frac{3x-x^3}{1-3x^2} = \tan^{-1} \frac{3 \tan \theta - \tan^3 \theta}{1-3 \tan^2 \theta}$$

$$= \tan^{-1} (\tan 3\theta), \text{ where } 3\theta \in (-3\pi/2, 3\pi/2)$$

$$\therefore \tan^{-1} \frac{3x-x^3}{1-3x^2} = \begin{cases} 3\theta, & \text{if } -\frac{\pi}{2} < 3\theta < \frac{\pi}{2} \\ 3\theta - \pi, & \text{if } \frac{\pi}{2} < 3\theta < \frac{3\pi}{2} \\ 3\theta + \pi, & \text{if } -\frac{3\pi}{2} < 3\theta < -\frac{\pi}{2} \end{cases}$$

$$\text{Now, if } -\frac{\pi}{2} < 3\theta < \frac{\pi}{2}$$

$$-\frac{\pi}{2} < 3 \tan^{-1} x < \frac{\pi}{2}$$

$$\Rightarrow -\frac{\pi}{6} < \tan^{-1} x < \frac{\pi}{6}$$

$$\Rightarrow -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$$

Similarly, from $\frac{\pi}{2} < 3\theta < \frac{3\pi}{2}$, we get $x > \frac{1}{\sqrt{3}}$

And from $-\frac{3\pi}{2} < 3\theta < -\frac{\pi}{2}$, we get $x < -\frac{1}{\sqrt{3}}$

$$\text{Thus, } \tan^{-1} \frac{3x-x^3}{1-3x^2} = \begin{cases} 3 \tan^{-1} x, & \text{if } -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}} \\ 3 \tan^{-1} x - \pi, & \text{if } \frac{1}{\sqrt{3}} < x < \infty \\ 3 \tan^{-1} x + \pi, & \text{if } -\infty < x < -\frac{1}{\sqrt{3}} \end{cases}$$

CONCEPT APPLICATION EXERCISE 7.5

- Find the value of $\sin^{-1} \left(\frac{3}{5} \right) + \tan^{-1} \left(\frac{1}{7} \right)$.
- If $x > y > 0$, then find the value of $\tan^{-1} \frac{x}{y} + \tan^{-1} \left[\frac{x+y}{x-y} \right]$.
- Prove that $\tan^{-1} \frac{1}{\sqrt{2}} + \sin^{-1} \frac{1}{\sqrt{5}} - \cos^{-1} \frac{1}{\sqrt{10}} = -\pi + \cot^{-1} \left(\frac{1+\sqrt{2}}{1-\sqrt{2}} \right)$.
- Solve $\tan^{-1} \frac{x-1}{x+2} + \tan^{-1} \frac{x+1}{x+2} = \frac{\pi}{4}$.
- Find the number of solutions of the equation $\tan^{-1} \left(\frac{x}{1-x^2} \right) + \tan^{-1} \left(\frac{1}{x^3} \right) = \frac{3\pi}{4}$.
- Prove that $2 \sin^{-1} \frac{3}{5} = \tan^{-1} \frac{24}{7}$.
- Prove that $\tan^{-1} \left(\frac{3a^2x-x^3}{a^3-3ax^2} \right) = 3 \tan^{-1} \frac{x}{a}$; $a > 0$, $-\frac{a}{\sqrt{3}} \leq x \leq \frac{a}{\sqrt{3}}$.
- Solve the equation $2 \tan^{-1} (\cos x) = \tan^{-1} (2 \operatorname{cosec} x)$.

9. Solve $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2} \tan^{-1}x$, ($x > 0$).
10. If $x + y + z = xyz$, and $x, y, z > 0$, then find the value of $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z$.
11. If α and β ($\alpha > \beta$) are the roots of $x^2 + kx - 1 = 0$, then find the value of $\tan^{-1}\alpha - \tan^{-1}\beta$.
12. Find the sum $\cot^{-1}2 + \cot^{-1}8 + \cot^{-1}18 + \dots \infty$.
13. Prove that $\sum_{r=1}^n \tan^{-1}\left(\frac{2^{r-1}}{1+2^{2r-1}}\right) = \tan^{-1}(2^n) - \frac{\pi}{4}$.

ANSWERS

1. $\frac{\pi}{4}$ 2. $\frac{3\pi}{4}$ 4. $x = \sqrt{\frac{5}{2}}$ 5. no solution
8. $x = \frac{\pi}{4}$ 9. $x = \frac{1}{\sqrt{3}}$ 10. π
11. π 12. $\frac{\pi}{4}$

SUM AND DIFFERENCE OF ANGLES IN TERMS OF $\sin^{-1}$ AND $\cos^{-1}$

(i) $\sin^{-1}x + \sin^{-1}y$

$$= \begin{cases} \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2}) & x \geq 0, y \geq 0, \text{ and } x^2 + y^2 \leq 1 \\ \pi - \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2}), & x \geq 0, y \geq 0, \text{ and } x^2 + y^2 > 1 \end{cases}$$

Proof:

Let $\sin^{-1}x = A$ and $\sin^{-1}y = B$, where $x \geq 0$ and $y \geq 0$

$$\Rightarrow A, B \in [0, \pi/2]$$

$$\Rightarrow A + B \in [0, \pi]$$

Now, $\sin(A+B) = \sin A \cos B + \sin B \cos A$

$$= x\sqrt{1-y^2} + y\sqrt{1-x^2}$$

$$\Rightarrow \sin^{-1}(\sin(A+B)) = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$$

$$\Rightarrow \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2}) = \begin{cases} A+B, & 0 \leq A+B \leq (\pi/2) \\ \pi - (A+B), & (\pi/2) < A+B \leq \pi \end{cases} \quad (i)$$

Now, $A+B \leq (\pi/2)$

$$\Rightarrow A \leq (\pi/2) - B$$

$$\Rightarrow \sin A \leq \cos B$$

$$\Rightarrow x \leq \sqrt{1-y^2}$$

$$\Rightarrow x^2 + y^2 \leq 1$$

And $A+B > \frac{\pi}{2}$

$$\Rightarrow x^2 + y^2 > 1$$

Hence from Eq. (i), we get

$$\sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2}) = \begin{cases} \sin^{-1}x + \sin^{-1}y, & x^2 + y^2 \leq 1 \\ \pi - (\sin^{-1}x + \sin^{-1}y), & x^2 + y^2 > 1 \end{cases}$$

$$\Rightarrow \sin^{-1}x + \sin^{-1}y$$

$$= \begin{cases} \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2}), & x \geq 0, y \geq 0 \text{ and } x^2 + y^2 \leq 1 \\ \pi - \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2}), & x \geq 0, y \geq 0 \text{ and } x^2 + y^2 > 1 \end{cases}$$

Note:

For $x < 0$ and $y < 0$, these identities can be used with the help of property $\sin^{-1}(-x) = -\sin^{-1}x$, i.e., change x and y to $-x$ and $-y$, respectively. Also for $\sin^{-1}x - \sin^{-1}y$, replace y by $-y$ in above results.

(ii) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2})$, $x \geq 0, y \geq 0$

Proof:

Let $\cos^{-1}x = A$ and $\cos^{-1}y = B$, where $x \geq 0$ and $y \geq 0$

$$\Rightarrow A, B \in [0, \pi/2]$$

$$\Rightarrow A + B \in [0, \pi]$$

Now, $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$= xy - \sqrt{1-x^2}\sqrt{1-y^2}$$

$$\Rightarrow \cos^{-1}(\cos(A+B)) = \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2})$$

$$\Rightarrow A+B = \cos^{-1}x + \cos^{-1}y = \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2})$$

(iii) $\cos^{-1}x - \cos^{-1}y$

$$= \begin{cases} \cos^{-1}(xy + \sqrt{1-x^2}\sqrt{1-y^2}), & x \geq 0, y \geq 0 \text{ and } x \leq y \\ -\cos^{-1}(xy + \sqrt{1-x^2}\sqrt{1-y^2}), & x \geq 0, y \geq 0 \text{ and } x > y \end{cases}$$

Proof:

Let $\cos^{-1}x = A$ and $\cos^{-1}y = B$, where $x \geq 0$ and $y \geq 0$

$$\Rightarrow A, B \in [0, \pi/2]$$

If $x \leq y$, then $\cos^{-1}x \geq \cos^{-1}y$ [$\because \cos^{-1}$ is a decreasing function]

$$\Rightarrow A \geq B$$

$$\Rightarrow A - B \in [0, \pi/2]$$

Now, $\cos(A-B) = \cos A \cos B + \sin A \sin B$

$$= xy + \sqrt{1-x^2}\sqrt{1-y^2}$$

$$\Rightarrow \cos^{-1}(\cos(A-B)) = \cos^{-1}(xy + \sqrt{1-x^2}\sqrt{1-y^2})$$

$$\Rightarrow A-B = \cos^{-1}x - \cos^{-1}y = \cos^{-1}(xy + \sqrt{1-x^2}\sqrt{1-y^2})$$

If $x > y$, then $\cos^{-1}x < \cos^{-1}y$

$$\Rightarrow A < B \Rightarrow A - B \in [-\pi/2, 0]$$

$$\Rightarrow \cos^{-1}x - \cos^{-1}y = -\cos^{-1}(xy + \sqrt{1-x^2}\sqrt{1-y^2})$$

Note:

For $x < 0$ and $y < 0$, these identities can be used with the help of property $\cos^{-1}(-x) = \pi - \cos^{-1}x$, i.e., change x and y to $-x$ and $-y$, respectively.

ILLUSTRATION 7.72

Prove that $\cot^{-1} \frac{3}{4} + \sin^{-1} \frac{5}{13} = \sin^{-1} \frac{63}{65}$.

$$\begin{aligned} \text{Sol. } \cot^{-1} \frac{3}{4} + \sin^{-1} \frac{5}{13} &= \sin^{-1} \frac{4}{5} + \sin^{-1} \frac{5}{13} \\ &= \sin^{-1} \left(\frac{4}{5} \sqrt{1 - \left(\frac{5}{13}\right)^2} + \frac{5}{13} \sqrt{1 - \left(\frac{4}{5}\right)^2} \right) \\ &= \sin^{-1} \left(\frac{4}{5} \cdot \frac{12}{13} + \frac{5}{13} \cdot \frac{3}{5} \right) = \sin^{-1} \frac{63}{65} \end{aligned}$$

ILLUSTRATION 7.73

Solve $\sin^{-1} x + \sin^{-1} 2x = \frac{\pi}{3}$.

$$\text{Sol. } \sin^{-1} x + \sin^{-1} 2x = \frac{\pi}{3}$$

$$\begin{aligned} \text{or } \sin^{-1} 2x &= \sin^{-1} \frac{\sqrt{3}}{2} - \sin^{-1} x \\ &= \sin^{-1} \left[\frac{\sqrt{3}}{2} \sqrt{1-x^2} - x \sqrt{1-\frac{3}{4}} \right] \end{aligned}$$

$$\text{or } 2x = \frac{\sqrt{3}}{2} \sqrt{1-x^2} - \frac{x}{2}$$

$$\left(\frac{5x}{2} \right)^2 = \frac{3}{4} (1-x^2)$$

$$\text{or } 28x^2 = 3$$

$$\text{or } x = \sqrt{\frac{3}{28}} = \frac{1}{2} \sqrt{\frac{3}{7}}$$

$$\left(\because x = -\frac{1}{2} \sqrt{\frac{3}{7}} \text{ makes L.H.S of Eq. (i) negative} \right)$$

ILLUSTRATION 7.74

Solve $\sin^{-1} x + \sin^{-1} (1-x) = \cos^{-1} x$.

$$\text{Sol. } \text{We have, } \sin^{-1} x + \sin^{-1} (1-x) = \cos^{-1} x$$

$$\Rightarrow \sin^{-1} \left[x \sqrt{1-(1-x)^2} + \sqrt{1-x^2} (1-x) \right] = \sin^{-1} \sqrt{1-x^2}$$

$$\Rightarrow x \sqrt{1-(1-x)^2} + \sqrt{1-x^2} (1-x) = \sqrt{1-x^2}$$

$$\Rightarrow x \sqrt{1-(1-x)^2} = x \sqrt{1-x^2}$$

$$\Rightarrow x = 0 \text{ or } 2x - x^2 = 1 - x^2$$

$$\Rightarrow x = 0 \text{ or } x = \frac{1}{2}$$

ILLUSTRATION 7.75

Solve $\cos^{-1} \left(\frac{1}{2} x^2 + \sqrt{1-x^2} \sqrt{1-\frac{x^2}{4}} \right) = \cos^{-1} \frac{x}{2} - \cos^{-1} x$.

$$\begin{aligned} \text{Sol. } \cos^{-1} \left(\frac{1}{2} x^2 + \sqrt{1-x^2} \sqrt{1-\frac{x^2}{4}} \right) \\ = \cos^{-1} \left(x \cdot \frac{x}{2} + \sqrt{1-x^2} \sqrt{1-\left(\frac{x}{2}\right)^2} \right) \end{aligned}$$

$$\text{For } \cos^{-1} \left(\frac{1}{2} x^2 + \sqrt{1-x^2} \sqrt{1-\frac{x^2}{4}} \right) = \cos^{-1} \frac{x}{2} - \cos^{-1} x,$$

L.H.S. > 0 , hence R.H.S. > 0

$$\Rightarrow \cos^{-1} \frac{x}{2} - \cos^{-1} x > 0 \text{ or } \cos^{-1} \frac{x}{2} > \cos^{-1} x$$

Since $\cos^{-1} x$ is a decreasing function, we get

$$\frac{x}{2} \leq x \Rightarrow \frac{x}{2} \geq 0 \Rightarrow x \geq 0 \Rightarrow x \in [0, 1]$$

ILLUSTRATION 7.76

If $x \in \left(0, \frac{\pi}{2} \right)$, then show that

$$\begin{aligned} \cos^{-1} \left(\frac{7}{2} (1 + \cos 2x) + \sqrt{(\sin^2 x - 48 \cos^2 x) \sin x} \right) \\ = x - \cos^{-1} (7 \cos x) \end{aligned}$$

$$\begin{aligned} \text{Sol. } y &= \cos^{-1} \left(\frac{7}{2} (1 + \cos 2x) + \sqrt{(\sin^2 x - 48 \cos^2 x) \sin x} \right) \\ &= \cos^{-1} ((7 \cos x)(\cos x) + \sqrt{1-49 \cos^2 x} \sqrt{1-\cos^2 x}) \\ &= \cos^{-1} (\cos x) - \cos^{-1} (7 \cos x) \quad [\because \cos x < 7 \cos x \text{ for } x \in \left(0, \frac{\pi}{2} \right)] \\ &= x - \cos^{-1} (7 \cos x) \end{aligned}$$

ILLUSTRATION 7.77

Which of the following angles is greater?

$$\theta_1 = \sin^{-1} \frac{4}{5} + \sin^{-1} \frac{1}{3} \text{ and } \theta_2 = \cos^{-1} \frac{4}{5} + \cos^{-1} \frac{1}{3}$$

$$\begin{aligned} \text{Sol. } \theta_1 + \theta_2 &= \sin^{-1} \frac{4}{5} + \sin^{-1} \frac{1}{3} + \cos^{-1} \frac{4}{5} + \cos^{-1} \frac{1}{3} \\ &= \left(\sin^{-1} \frac{4}{5} + \cos^{-1} \frac{4}{5} \right) + \left(\sin^{-1} \frac{1}{3} + \cos^{-1} \frac{1}{3} \right) \\ &= \frac{\pi}{2} + \frac{\pi}{2} = \pi \\ \theta_2 &= \cos^{-1} \left[\left(\frac{4}{5} \right) \left(\frac{1}{3} \right) - \sqrt{1-\left(\frac{4}{5}\right)^2} \sqrt{1-\left(\frac{1}{3}\right)^2} \right] \\ &= \cos^{-1} \left[\frac{4}{5} \cdot \frac{1}{3} - \frac{3}{5} \cdot \frac{\sqrt{8}}{3} \right] \end{aligned}$$

$$\text{Now, } \frac{4-6\sqrt{2}}{15} < 0$$

Therefore, θ_2 is obtuse and hence θ_1 is acute.

$$\therefore \theta_1 < \theta_2$$

ILLUSTRATION 7.78

Find the value

$$\lim_{n \rightarrow \infty} \sum_{k=2}^n \cos^{-1} \left(\frac{1 + \sqrt{(k-1)k(k+1)(k+2)}}{k(k+1)} \right)$$

Sol.
$$S_{n-1} = \sum_{k=2}^n \cos^{-1} \left(\frac{1 + \sqrt{(k-1)k(k+1)(k+2)}}{k(k+1)} \right)$$

$$= \cos^{-1} \left(\frac{1}{k} \cdot \frac{1}{(k+1)} + \frac{\sqrt{(k-1)k(k+1)(k+2)}}{k(k+1)} \right)$$

$$= \cos^{-1} \left(\frac{1}{k} \cdot \frac{1}{(k+1)} + \sqrt{1 - \frac{1}{k^2}} \sqrt{1 - \frac{1}{(k+1)^2}} \right)$$

$$= \cos^{-1} \frac{1}{k+1} - \cos^{-1} \frac{1}{k} = t_k$$

$$\left(\because \frac{1}{k+1} < \frac{1}{k}, \text{ so } \cos^{-1} \frac{1}{k+1} > \cos^{-1} \frac{1}{k} \right)$$

Substituting $k = 2, 3, 4, \dots$, we get

$$t_2 = \cos^{-1} \left(\frac{1}{3} \right) - \cos^{-1} \left(\frac{1}{2} \right)$$

$$t_3 = \cos^{-1} \left(\frac{1}{4} \right) - \cos^{-1} \left(\frac{1}{3} \right)$$

$$\vdots$$

$$t_n = \cos^{-1} \left(\frac{1}{n+1} \right) - \cos^{-1} \left(\frac{1}{n} \right)$$

$$\therefore S_{n-1} = \cos^{-1} \left(\frac{1}{n+1} \right) - \cos^{-1} \left(\frac{1}{2} \right)$$

$$\therefore S = \lim_{n \rightarrow \infty} \left[\cos^{-1} \left(\frac{1}{n+1} \right) - \cos^{-1} \left(\frac{1}{2} \right) \right]$$

$$= \cos^{-1}(0) - \cos^{-1} \left(\frac{1}{2} \right)$$

$$= \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$$

MULTIPLE ANGLES IN TERMS OF $\sin^{-1}x$ AND $\cos^{-1}x$ **Results:**

$$(i) \quad 2 \sin^{-1}x = \begin{cases} \sin^{-1}(2x\sqrt{1-x^2}), & -\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}} \\ \pi - \sin^{-1}(2x\sqrt{1-x^2}), & x > \frac{1}{\sqrt{2}} \\ -\pi - \sin^{-1}(2x\sqrt{1-x^2}), & x < -\frac{1}{\sqrt{2}} \end{cases}$$

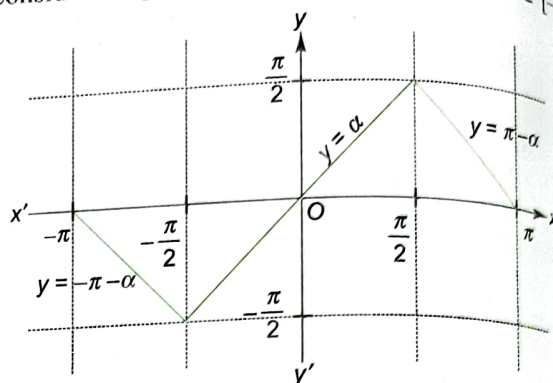
$$(ii) \quad 3 \sin^{-1}x = \begin{cases} \sin^{-1}(3x - 4x^3), & -\frac{1}{2} \leq x \leq \frac{1}{2} \\ \pi - \sin^{-1}(3x - 4x^3), & x > \frac{1}{2} \\ -\pi - \sin^{-1}(3x - 4x^3), & x < -\frac{1}{2} \end{cases}$$

Proof:Let $x = \sin \theta$, $\theta \in [-\pi/2, \pi/2]$. Then $\theta = \sin^{-1}x$.

$$\text{Now, } \sin^{-1}(2x\sqrt{1-x^2}) = \sin^{-1}(2 \sin \theta \cos \theta)$$

$$= \sin^{-1}(\sin 2\theta)$$

$$= \sin^{-1}(\sin \alpha)$$

Now, consider the graph of $y = \sin^{-1}(\sin \alpha)$, where $\alpha \in [-\pi, \pi]$.

From the graph, we have

$$\sin^{-1}(2x\sqrt{1-x^2}) = \sin^{-1}(\sin \alpha)$$

$$= \begin{cases} -\alpha - \pi, & -\pi < \alpha < (-\pi/2) \\ \alpha, & (-\pi/2) \leq \alpha \leq (\pi/2) \\ -\alpha + \pi, & (\pi/2) < \alpha < \pi \end{cases}$$

$$= \begin{cases} -2 \sin^{-1}x - \pi, & -\pi \leq 2 \sin^{-1}x < (-\pi/2) \\ 2 \sin^{-1}x, & (-\pi/2) \leq 2 \sin^{-1}x \leq (\pi/2) \\ -2 \sin^{-1}x - \pi, & (\pi/2) < 2 \sin^{-1}x < \pi \end{cases}$$

$$= \begin{cases} -2 \sin^{-1}x - \pi, & (-\pi/2) \leq \sin^{-1}x < (-\pi/4) \\ 2 \sin^{-1}x, & (-\pi/4) \leq \sin^{-1}x \leq (\pi/4) \\ -2 \sin^{-1}x + \pi, & (\pi/4) < \sin^{-1}x < (\pi/2) \end{cases}$$

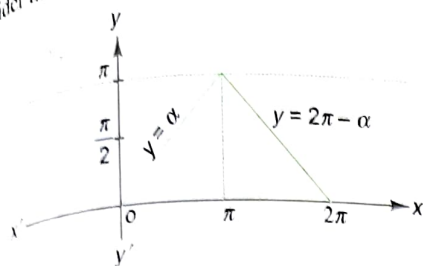
$$= \begin{cases} -2 \sin^{-1}x - \pi, & x < -\frac{1}{\sqrt{2}} \\ 2 \sin^{-1}x, & -\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}} \\ -2 \sin^{-1}x + \pi, & x > \frac{1}{\sqrt{2}} \end{cases}$$

$$\Rightarrow 2 \sin^{-1}x = \begin{cases} \sin^{-1}(2x\sqrt{1-x^2}), & -\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}} \\ \pi - \sin^{-1}(2x\sqrt{1-x^2}), & x > \frac{1}{\sqrt{2}} \\ -\pi - \sin^{-1}(2x\sqrt{1-x^2}), & x < -\frac{1}{\sqrt{2}} \end{cases}$$

$$(iii) \quad 2 \cos^{-1}x = \begin{cases} 2\pi - \cos^{-1}(2x^2 - 1), & \text{if } -1 \leq x < 0 \\ \cos^{-1}(2x^2 - 1), & \text{if } 0 \leq x \leq 1 \end{cases}$$

Proof:Let $\cos^{-1}x = \theta$, where $\theta \in [0, \pi]$
or $\cos \theta = x$

$\cos^{-1}(2x^2 - 1) = \cos^{-1}(2 \cos^2 \theta - 1)$
 $= \cos^{-1}(\cos 2\theta)$
 $= \cos^{-1}(\cos \alpha)$, where $\alpha \in [0, 2\pi]$
 Now consider the graph of $y = \cos^{-1}(\cos \alpha)$ for $\alpha \in [0, 2\pi]$.



From the graph, we have

$$\begin{aligned}
 \cos^{-1}(2x^2 - 1) &= \begin{cases} \alpha, & \text{if } 0 \leq \alpha < \pi \\ 2\pi - \alpha, & \text{if } \pi \leq \alpha \leq 2\pi \end{cases} \\
 &= \begin{cases} 2 \cos^{-1} x, & \text{if } 0 \leq 2 \cos^{-1} x \leq \pi \\ 2\pi - 2 \cos^{-1} x, & \text{if } \pi < 2 \cos^{-1} x \leq 2\pi \end{cases} \\
 &= \begin{cases} 2 \cos^{-1} x, & \text{if } 0 \leq \cos^{-1} x \leq (\pi/2) \\ 2\pi - 2 \cos^{-1} x, & \text{if } (\pi/2) < \cos^{-1} x \leq \pi \end{cases} \\
 &= \begin{cases} 2 \cos^{-1} x, & \text{if } 0 \leq x \leq 1 \\ 2\pi - 2 \cos^{-1} x, & \text{if } -1 \leq x < 0 \end{cases}
 \end{aligned}$$

$$2 \cos^{-1} x = \begin{cases} 2\pi - \cos^{-1}(2x^2 - 1), & \text{if } -1 \leq x < 0 \\ \cos^{-1}(2x^2 - 1), & \text{if } 0 \leq x \leq 1 \end{cases}$$

ILLUSTRATION 7.79

If $f(x) = \sin^{-1} x$ then prove that

$$\lim_{x \rightarrow \frac{1}{2}^+} f(3x - 4x^3) = \pi - 3 \lim_{x \rightarrow \frac{1}{2}^+} \sin^{-1} x.$$

$$\sin^{-1}(3x - 4x^3) = \pi - 3 \sin^{-1} x \text{ if } \frac{1}{2} < x < 1$$

$$\begin{aligned}
 \lim_{x \rightarrow \frac{1}{2}^+} f(3x - 4x^3) &= \lim_{x \rightarrow \frac{1}{2}^+} (\pi - 3 \sin^{-1} x) \\
 &= \pi - 3 \lim_{x \rightarrow \frac{1}{2}^+} \sin^{-1} x
 \end{aligned}$$

ILLUSTRATION 7.80

Solve $\sin^{-1} x - \cos^{-1} x = \sin^{-1}(3x - 2)$.

$$\sin^{-1} x - \cos^{-1} x = \sin^{-1}(3x - 2)$$

This equation is valid if $-1 \leq x \leq 1$ and $-1 \leq 3x - 2 \leq 1$

$$x \in \left[\frac{1}{3}, 1 \right]$$

Given equation can be rewritten as:

$$\frac{\pi}{2} - \cos^{-1} x - \cos^{-1} x = \frac{\pi}{2} - \cos^{-1}(3x - 2)$$

$$\begin{aligned}
 \Rightarrow 2 \cos^{-1} x &= \cos^{-1}(3x - 2) \\
 \Rightarrow \cos^{-1}(2x^2 - 1) &= \cos^{-1}(3x - 2) \\
 \Rightarrow 2x^2 - 1 &= 3x - 2 \\
 \Rightarrow 2x^2 - 3x + 1 &= 0 \\
 \Rightarrow x &= 1 \text{ or } \frac{1}{2}
 \end{aligned}$$

ILLUSTRATION 7.81

If $A = 2 \tan^{-1}(2\sqrt{2} - 1)$ and $B = 3 \sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{3}{5}\right)$, then which is greater?

Sol. We have $A = 2 \tan^{-1}(2\sqrt{2} - 1) = 2 \tan^{-1}(1.828)$

$$\Rightarrow A > 2 \tan^{-1} \sqrt{3}$$

$$\Rightarrow A > \frac{2\pi}{3}$$

$$\text{Now, } \sin^{-1}\left(\frac{1}{3}\right) < \sin^{-1}\left(\frac{1}{2}\right)$$

$$\Rightarrow \sin^{-1}\left(\frac{1}{3}\right) < \frac{\pi}{6}$$

$$\Rightarrow 3 \sin^{-1} \frac{1}{3} < \frac{\pi}{2}$$

$$\text{Further, } \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}(0.6) < \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow \sin^{-1}\left(\frac{3}{5}\right) < \frac{\pi}{3}$$

$$\Rightarrow B = 3 \sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{3}{5}\right) < \frac{\pi}{2} + \frac{\pi}{3}$$

$$\Rightarrow B < \frac{5\pi}{6}$$

From this, we really cannot relate A and B .

$$\text{Now, } 3 \sin^{-1}\left(\frac{1}{3}\right) = \sin^{-1}\left[3 \cdot \frac{1}{3} - 4\left(\frac{1}{3}\right)^3\right]$$

$$= \sin^{-1}\left(\frac{23}{27}\right)$$

$$= \sin^{-1}(0.852)$$

$$\Rightarrow 3 \sin^{-1}\left(\frac{1}{3}\right) < \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

$$\text{Hence, } B = 3 \sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{3}{5}\right) < \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$$

$$\therefore A > B.$$

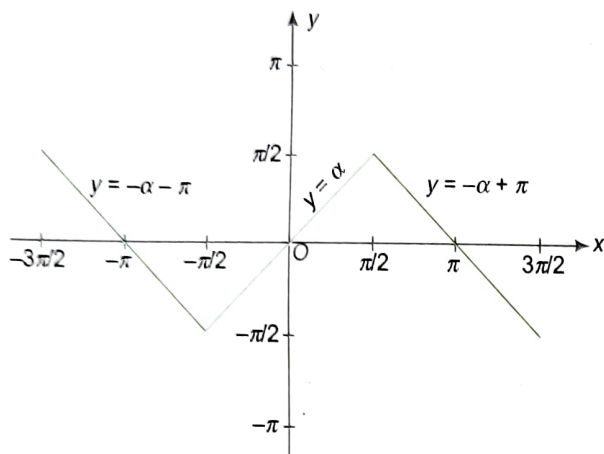
$$(iv) \quad 2 \tan^{-1} x = \begin{cases} \sin^{-1}\left(\frac{2x}{1+x^2}\right), & \text{if } -1 \leq x \leq 1 \\ \pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right), & \text{if } x > 1 \\ -\pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right), & \text{if } x < -1 \end{cases}$$

Proof:

Let $\tan^{-1} x = \theta$, where $\theta \in (-\pi/2, \pi/2)$.

Then $\tan \theta = x$.

$$\begin{aligned}\text{Now, } \sin^{-1}\left(\frac{2x}{1+x^2}\right) &= \sin^{-1}\left(\frac{2 \tan \theta}{1 + \tan^2 \theta}\right) \\ &= \sin^{-1}(\sin 2\theta) \\ &= \sin^{-1}(\sin \alpha), \text{ where } \alpha \in (-\pi, \pi)\end{aligned}$$



From the graph,

$$\begin{aligned}\sin^{-1}\left(\frac{2x}{1+x^2}\right) &= \sin^{-1}(\sin \alpha) \\ &= \begin{cases} -\alpha - \pi, & -\pi < \alpha < -\pi/2 \\ \alpha, & -\pi/2 \leq \alpha \leq \pi/2 \\ -\alpha + \pi, & \pi/2 < \alpha < \pi \end{cases} \\ &= \begin{cases} -2 \tan^{-1} x - \pi, & -\pi < 2 \tan^{-1} x < -\pi/2 \\ 2 \tan^{-1} x, & -\pi/2 \leq 2 \tan^{-1} x \leq \pi/2 \\ -2 \tan^{-1} x + \pi, & \pi/2 < 2 \tan^{-1} x < \pi \end{cases} \\ &= \begin{cases} -2 \tan^{-1} x - \pi, & -\pi/2 < \tan^{-1} x < -\pi/4 \\ 2 \tan^{-1} x, & -\pi/4 \leq \tan^{-1} x \leq \pi/4 \\ -2 \tan^{-1} x + \pi, & \pi/4 < \tan^{-1} x < \pi/2 \end{cases} \\ &= \begin{cases} -2 \tan^{-1} x - \pi, & x < -1 \\ 2 \tan^{-1} x, & -1 \leq x \leq 1 \\ -2 \tan^{-1} x + \pi, & x > 1 \end{cases}\end{aligned}$$

$$(v) \quad 2 \tan^{-1} x = \begin{cases} \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), & \text{if } x \geq 0 \\ -\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), & \text{if } x < 0 \end{cases}$$

Proof:

Let $\tan^{-1} x = \theta$, where $\theta \in (-\pi/2, \pi/2)$

Then $\tan \theta = x$.

$$\begin{aligned}\text{Now, } \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) &= \cos^{-1}\left(\frac{1-\tan^2 \theta}{1+\tan^2 \theta}\right) \\ &= \cos^{-1}(\cos 2\theta), \text{ where } 2\theta \in (-\pi, \pi)\end{aligned}$$

$$= \begin{cases} -2\theta, & -\pi < 2\theta < 0 \\ 2\theta, & 0 \leq 2\theta < \pi \end{cases}$$

$$= \begin{cases} -2 \tan^{-1} x, & -\pi < 2 \tan^{-1} x < 0 \\ 2 \tan^{-1} x, & 0 \leq 2 \tan^{-1} x < \pi \end{cases}$$

$$= \begin{cases} -2 \tan^{-1} x, & -\pi/2 < \tan^{-1} x < 0 \\ 2 \tan^{-1} x, & 0 \leq \tan^{-1} x < \pi/2 \end{cases}$$

$$= \begin{cases} -2 \tan^{-1} x, & x < 0 \\ 2 \tan^{-1} x, & x \geq 0 \end{cases}$$

ILLUSTRATION 7.82

If $\sin^{-1} \frac{2x}{1+x^2} = \tan^{-1} \frac{2x}{1-x^2}$, then find the value of x .

Sol. $\sin^{-1} \frac{2x}{1+x^2} = 2 \tan^{-1} x, \quad -1 \leq x \leq 1$

$$\tan^{-1} \frac{2x}{1-x^2} = 2 \tan^{-1} x, \quad -1 < x < 1$$

Therefore, equation is satisfied by $-1 < x < 1$.

ILLUSTRATION 7.83

If $\sin^{-1}\left(\frac{4x}{x^2+4}\right) + 2 \tan^{-1}\left(-\frac{x}{2}\right)$ is independent of x , find the values of x .

Sol.

$$E = \sin^{-1}\left(\frac{4x}{x^2+4}\right) + 2 \tan^{-1}\left(-\frac{x}{2}\right)$$

$$= \sin^{-1}\left(\frac{2 \times \frac{x}{2}}{\left(\frac{x}{2}\right)^2 + 1}\right) - 2 \tan^{-1} \frac{x}{2}$$

$$= 2 \tan^{-1} \frac{x}{2} - 2 \tan^{-1} \frac{x}{2}$$

$$= 0$$

(For E to be independent of x)

$$\Rightarrow \left|\frac{x}{2}\right| \leq 1$$

$$\text{or } |x| \leq 2 \quad \text{or } -2 \leq x \leq 2$$

ILLUSTRATION 7.84

If $\cos^{-1} \frac{6x}{1+9x^2} = -\frac{\pi}{2} + 2 \tan^{-1} 3x$, then find the values of x .

Sol. $\cos^{-1} \frac{6x}{1+9x^2} = -\frac{\pi}{2} + 2 \tan^{-1} 3x$

$$\frac{\pi}{2} - \sin^{-1} \frac{6x}{1+9x^2} = -\frac{\pi}{2} + 2 \tan^{-1} 3x$$

$$\sin^{-1} \frac{6x}{1+9x^2} = \pi - 2 \tan^{-1} 3x$$

$$\sin^{-1} \frac{2 \times 3x}{1+(3x)^2} = \pi - 2 \tan^{-1} 3x$$

is true when $3x > 1$

$$x > \frac{1}{3}$$

$$x \in \left(\frac{1}{3}, \infty\right)$$

ILLUSTRATION 7.85

Find the value of $2 \cos^{-1} \frac{3}{\sqrt{13}} + \cot^{-1} \frac{16}{63} + \frac{1}{2} \cos^{-1} \frac{7}{25}$.

$$\text{Sol. } E = 2 \cos^{-1} \frac{3}{\sqrt{13}} + \cot^{-1} \frac{16}{63} + \frac{1}{2} \cos^{-1} \frac{7}{25}$$

$$= 2 \tan^{-1} \frac{2}{3} + \tan^{-1} \frac{63}{16} + \frac{1}{2} \cos^{-1} \frac{7}{25}$$

$$\text{Now, } 2 \tan^{-1} \frac{2}{3} = \tan^{-1} \frac{2 \left(\frac{2}{3}\right)}{1 - \frac{4}{9}}$$

$$= \tan^{-1} \frac{12}{5}$$

$$\text{Let } \frac{1}{2} \cos^{-1} \frac{7}{25} = \tan^{-1} x$$

$$\cos^{-1} \frac{7}{25} = 2 \tan^{-1} x$$

$$\tan^{-1} \frac{24}{7} = \tan^{-1} \frac{2x}{1-x^2}$$

$$\frac{24}{7} = \frac{2x}{1-x^2}$$

$$12x^2 + 7x - 12 = 0$$

$$(4x-3)(3x+4) = 0$$

$$x = 3/4$$

$$E = \tan^{-1} \frac{12}{5} + \tan^{-1} \frac{63}{16} + \tan^{-1} \frac{3}{4}$$

$$= \pi + \tan^{-1} \frac{\frac{12}{5} + \frac{3}{4}}{1 - \left(\frac{12}{5}\right)\left(\frac{3}{4}\right)} + \tan^{-1} \frac{63}{16}$$

$$= \pi + \tan^{-1} \left(-\frac{63}{16}\right) + \tan^{-1} \frac{63}{16}$$

$$= \pi$$

CONCEPT APPLICATION EXERCISE 7.6

1. If $\cos^{-1} \frac{x}{2} + \cos^{-1} \frac{y}{3} = \frac{\pi}{6}$, then prove that

$$\frac{x^2}{4} - \frac{xy}{2\sqrt{3}} + \frac{y^2}{9} = \frac{1}{4}$$

2. Find the set of values of x for which the equation

$$\cos^{-1} x + \cos^{-1} \left(\frac{x}{2} + \frac{1}{2} \sqrt{3-3x^2} \right) = \frac{\pi}{3}$$
 holds good.

3. Solve the following equation:

$$\sec^{-1} \frac{x}{a} - \sec^{-1} \frac{x}{b} = \sec^{-1} b - \sec^{-1} a, \quad a \geq 1, b \geq 1, a \neq b.$$

4. If $a^2 + b^2 = c^2, c \neq 0$, then find the non-zero solution of the equation:

$$\sin^{-1} \frac{ax}{c} + \sin^{-1} \frac{bx}{c} = \sin^{-1} x$$

5. If $\cos(\theta - \alpha) = a$ and $\sin(\theta - \beta) = b$ ($0 < \theta - \alpha, \theta - \beta < \pi/2$), then prove that $\cos^2(\alpha - \beta) + 2ab \sin(\alpha - \beta) = a^2 + b^2$.

6. Find the values of x which satisfy equation

$$2 \tan^{-1} 2x = \sin^{-1} \frac{4x}{1+4x^2}$$

7. If $x \in (0, 1)$, then find the value of

$$\tan^{-1} \left(\frac{1-x^2}{2x} \right) + \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right)$$

8. If $x \in [-1, 0)$ then find the value of $\cos^{-1}(2x^2 - 1) - 2 \sin^{-1} x$.

9. Find the value of $\sin(2 \sin^{-1}(0.8))$.

ANSWERS

2. $x \in \left[\frac{1}{2}, 1\right]$

3. $x = ab$

4. $x = \pm 1$

6. $-\frac{1}{2} \leq x \leq \frac{1}{2}$

7. $\frac{\pi}{2}$

8. π

9. 0.96

Solved Examples

EXAMPLE 7.1

Solve $2 \cos^{-1} x = \sin^{-1} (2x \sqrt{1-x^2})$.

Sol. Let $x = \cos y$, where $0 \leq y \leq \pi, |x| \leq 1$

$$2 \cos^{-1} x = \sin^{-1} (2x \sqrt{1-x^2}) \quad \dots(i)$$

$$\begin{aligned} \Rightarrow 2 \cos^{-1}(\cos y) &= \sin^{-1}(2 \cos y \sqrt{1-\cos^2 y}) \\ &= \sin^{-1}(2 \cos y \sin y) \\ &= \sin^{-1}(\sin 2y) \end{aligned}$$

$$\Rightarrow \sin^{-1}(\sin 2y) = 2y \quad \text{for } -\pi/4 \leq y \leq \pi/4$$

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and $2 \cos^{-1}(\cos y) = 2y$ for $0 \leq y \leq \pi$

Thus, Eq. (i) holds only when

$$y \in [0, \pi/4]$$

$$\Rightarrow x \in [1/\sqrt{2}, 1]$$

EXAMPLE 7.2

Find the domain for $f(x) = \sin^{-1}\left(\frac{1+x^2}{2x}\right)$.

Sol. $f(x) = \sin^{-1}\left(\frac{1+x^2}{2x}\right)$ is defined for $-1 \leq \frac{1+x^2}{2x} \leq 1$

or $\left|\frac{1+x^2}{2x}\right| \leq 1$

$$\Rightarrow |1+x^2| \leq |2x|, \text{ for all } x$$

$$\Rightarrow 1+x^2 \leq |2x|, \text{ for all } x \text{ (as } 1+x^2 > 0\text{)}$$

$$\Rightarrow x^2 - 2|x| + 1 \leq 0$$

$$\Rightarrow |x|^2 - 2|x| + 1 \leq 0 \text{ (as } x^2 = |x|^2\text{)}$$

$$\Rightarrow (|x| - 1)^2 \leq 0$$

But $(|x| - 1)^2$ is always either positive or zero. Thus,

$$(|x| - 1)^2 = 0$$

or $|x| = 1$ or $x = \pm 1$

Hence, domain for $f(x)$ is $\{-1, 1\}$.

EXAMPLE 7.3

Find the range of $f(x) = \cot^{-1}(2x - x^2)$.

Sol. Let $\theta = \cot^{-1}(2x - x^2)$, where $\theta \in (0, \pi)$

$$\Rightarrow \cot \theta = 2x - x^2, \text{ where } \theta \in (0, \pi)$$

$$= 1 - (1 - 2x + x^2), \text{ where } \theta \in (0, \pi)$$

$$= 1 - (1 - x)^2, \text{ where } \theta \in (0, \pi)$$

$$\Rightarrow \cot \theta \leq 1, \text{ where } \theta \in (0, \pi)$$

$$\Rightarrow \frac{\pi}{4} \leq \theta < \pi$$

$$\Rightarrow \text{Range of } f(x) \text{ is } \left[\frac{\pi}{4}, \pi\right)$$

EXAMPLE 7.4

Find the set of values of parameter a so that the equation $(\sin^{-1} x)^3 + (\cos^{-1} x)^3 = a\pi^3$ has a solution.

Sol. $(\sin^{-1} x)^3 + (\cos^{-1} x)^3 = a\pi^3$

$$\Rightarrow (\sin^{-1} x + \cos^{-1} x) ((\sin^{-1} x + \cos^{-1} x)^2 - 3 \sin^{-1} x \cos^{-1} x) = a\pi^3$$

$$\Rightarrow \frac{\pi^2}{4} - 3 \sin^{-1} x \cos^{-1} x = 2a\pi^2$$

$$\Rightarrow \sin^{-1} x \left(\frac{\pi}{2} - \sin^{-1} x\right) = \frac{\pi^2}{12} (1 - 8a)$$

$$\Rightarrow (\sin^{-1} x)^2 - \frac{\pi}{2} \sin^{-1} x = -\frac{\pi^2}{12} (1 - 8a)$$

$$\Rightarrow \left(\sin^{-1} x - \frac{\pi}{4}\right)^2 = \frac{\pi^2}{12} (8a - 1) + \frac{\pi^2}{16} = \frac{\pi^2}{48} (32a - 1)$$

$$\text{Now, } \sin^{-1} x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\Rightarrow -\frac{3\pi}{4} \leq \sin^{-1} x - \frac{\pi}{4} \leq \frac{\pi}{4}$$

$$\Rightarrow 0 \leq \left(\sin^{-1} x - \frac{\pi}{4}\right)^2 \leq \frac{9\pi^2}{16}$$

$$\Rightarrow 0 \leq \frac{\pi^2}{48} (32a - 1) \leq \frac{9\pi^2}{16}$$

$$\Rightarrow 0 \leq 32a - 1 \leq 27$$

$$\Rightarrow \frac{1}{32} \leq a \leq \frac{7}{8}$$

Thus, the required set of values of a is $\left[\frac{1}{32}, \frac{7}{8}\right]$.

EXAMPLE 7.5

Solve the equation $\sqrt{\sin^{-1} |\cos x| + |\cos^{-1} |\sin x||}$

$$= \sin^{-1} |\cos x| - \cos^{-1} |\sin x|, \frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$$

Sol. Given $\sqrt{\sin^{-1} |\cos x| + |\cos^{-1} |\sin x||}$
 $= \sin^{-1} (|\cos x|) - \cos^{-1} |\sin x|$

When $x \in \left[0, \frac{\pi}{2}\right]$, $0 \leq \frac{\pi}{2} - x \leq \frac{\pi}{2}$, we have

$$\begin{aligned} & \sqrt{\sin^{-1} \sin\left(\frac{\pi}{2} - x\right) + \cos^{-1} \cos\left(\frac{\pi}{2} - x\right)} \\ &= \sin^{-1} \left(\sin\left(\frac{\pi}{2} - x\right)\right) - \cos^{-1} \left(\cos\left(\frac{\pi}{2} - x\right)\right) \end{aligned}$$

$$\text{or } \sqrt{2\left(\frac{\pi}{2} - x\right)} = \frac{\pi}{2} - x - \frac{\pi}{2} + x$$

$$\text{or } x = \frac{\pi}{2}$$

When $x \in \left[-\frac{\pi}{2}, 0\right]$, $\frac{\pi}{2} \leq \frac{\pi}{2} - x \leq \pi$, we have

$$\begin{aligned} & \sqrt{\sin^{-1} \left(\sin\left(\frac{\pi}{2} - x\right)\right) + \cos^{-1} \left(-\cos\left(\frac{\pi}{2} - x\right)\right)} \\ &= \sin^{-1} \left(\sin\left(\frac{\pi}{2} - x\right)\right) - \cos^{-1} \left(-\cos\left(\frac{\pi}{2} - x\right)\right) \end{aligned}$$

$$\begin{aligned}
 & \Rightarrow \sqrt{\pi - \left(\frac{\pi}{2} - x\right) + \pi - \cos^{-1}\left(\cos\left(\frac{\pi}{2} - x\right)\right)} \\
 & = \pi - \left(\frac{\pi}{2} - x\right) - \pi + \cos^{-1}\left(\cos\left(\frac{\pi}{2} - x\right)\right) \\
 & \Rightarrow \sqrt{\frac{\pi}{2} + x + \pi - \left(\frac{\pi}{2} - x\right)} = \frac{\pi}{2} + x - \pi + \frac{\pi}{2} - x \\
 & \Rightarrow \sqrt{\pi + 2x} = 0 \\
 & \Rightarrow x = -\frac{\pi}{2}
 \end{aligned}$$

EXAMPLE 7.6

If $p > q > 0$ and $pr < -1 < qr$, then find the value of

$$\tan^{-1} \frac{p-q}{1+pq} + \tan^{-1} \frac{q-r}{1+qr} + \tan^{-1} \frac{r-p}{1+rp}$$

Sol. Since $p, q > 0$, we have $pq > 0$.

$$\tan^{-1} \frac{p-q}{1+pq} = \tan^{-1} p - \tan^{-1} q \quad \dots(i)$$

Since $qr > -1$, we have

$$\tan^{-1} \frac{q-r}{1+qr} = \tan^{-1} q - \tan^{-1} r \quad \dots(ii)$$

Since $pr < -1$ and $r < 0$, we have

$$\tan^{-1} \frac{r-p}{1+rp} = \pi + \tan^{-1} r - \tan^{-1} p \quad \dots(iii)$$

On adding Eqs. (i), (ii), and (iii), we get

$$\tan^{-1} \frac{p-q}{1+pq} + \tan^{-1} \frac{q-r}{1+qr} + \tan^{-1} \frac{r-p}{1+rp} = \pi$$

EXAMPLE 7.7

If $0 < a_1 < a_2 < \dots < a_n$ and $x, y > 0$ then prove that

$$\begin{aligned}
 & \tan^{-1} \left(\frac{a_1 x - y}{x + a_1 y} \right) + \tan^{-1} \left(\frac{a_2 - a_1}{1 + a_2 a_1} \right) + \tan^{-1} \left(\frac{a_3 - a_2}{1 + a_3 a_2} \right) + \dots \\
 & + \tan^{-1} \left(\frac{a_n - a_{n-1}}{1 + a_n a_{n-1}} \right) + \tan^{-1} \left(\frac{1}{a_n} \right) = \tan^{-1} \frac{x}{y}
 \end{aligned}$$

Sol. Here,

$$\tan^{-1} \left(\frac{a_1 x - y}{x + a_1 y} \right) = \tan^{-1} \left(\frac{a_1 - \frac{y}{x}}{1 + a_1 \frac{y}{x}} \right) = \tan^{-1} a_1 - \tan^{-1} \frac{y}{x}$$

$$\tan^{-1} \left(\frac{a_2 - a_1}{1 + a_2 a_1} \right) = \tan^{-1} a_2 - \tan^{-1} a_1$$

$$\tan^{-1} \left(\frac{a_3 - a_2}{1 + a_3 a_2} \right) = \tan^{-1} a_3 - \tan^{-1} a_2$$

$\vdots$

$$\tan^{-1} \left(\frac{a_n - a_{n-1}}{1 + a_n a_{n-1}} \right) = \tan^{-1} a_n - \tan^{-1} a_{n-1}$$

$$\tan^{-1} \left(\frac{1}{a_n} \right) = \cot^{-1} a_n$$

Adding, we get

$$\begin{aligned}
 \text{L.H.S.} &= \tan^{-1} a_n + \cot^{-1} a_n - \tan^{-1} \frac{y}{x} = \frac{\pi}{2} - \tan^{-1} \frac{y}{x} \\
 &= \cot^{-1} \frac{y}{x} = \tan^{-1} \frac{x}{y} = \text{R.H.S.}
 \end{aligned}$$

EXAMPLE 7.8

Find the number of positive integral solutions of the equation

$$\tan^{-1} x + \cos^{-1} \frac{y}{\sqrt{1-y^2}} = \sin^{-1} \frac{3}{\sqrt{10}}$$

Sol. Here, $\tan^{-1} x + \cos^{-1} \frac{y}{\sqrt{1-y^2}} = \sin^{-1} \frac{3}{\sqrt{10}}$

$$\Rightarrow \tan^{-1} x + \tan^{-1} \left(\frac{1}{y} \right) = \tan^{-1} (3)$$

$$\Rightarrow \tan^{-1} \left(\frac{1}{y} \right) = \tan^{-1} 3 - \tan^{-1} (x)$$

$$\Rightarrow \tan^{-1} \left(\frac{1}{y} \right) = \tan^{-1} \left(\frac{3-x}{1+3x} \right)$$

$$\therefore y = \frac{1+3x}{3-x}$$

As x, y are positive integers, $x = 1, 2$ and corresponding $y = 2, 7$. Therefore, the solutions are $(x, y) = (1, 2)$ and $(2, 7)$, i.e., there are two solutions.

EXAMPLE 7.9

If $\tan^{-1} y = 4 \tan^{-1} x$ $\left(|x| < \tan \frac{\pi}{8} \right)$, find y as an algebraic function of x , and, hence, prove that $\tan \pi/8$ is a root of the equation $x^4 - 6x^2 + 1 = 0$.

Sol. We have $\tan^{-1} y = 4 \tan^{-1} x$

$$\Rightarrow \tan^{-1} y = 2 \tan^{-1} \frac{2x}{1-x^2} \quad (\text{as } |x| < 1)$$

$$= \tan^{-1} \frac{\frac{4x}{1-x^2}}{1 - \frac{4x^2}{(1-x^2)^2}}$$

$$= \tan^{-1} \frac{4x(1-x^2)}{x^4 - 6x^2 + 1}$$

$$\left(\text{as } \left| \frac{2x}{1-x^2} \right| < 1 \right)$$

or $y = \frac{4x(1-x^2)}{x^4 - 6x^2 + 1}$

If $x = \tan \frac{\pi}{8}$

$$\Rightarrow \tan^{-1} y = 4 \tan^{-1} x = \frac{\pi}{2}$$

$$\Rightarrow y \rightarrow \infty$$

$$\Rightarrow x^4 - 6x^2 + 1 = 0$$

Hence, $\tan \frac{\pi}{8}$ is a root of $x^4 - 6x^2 + 1 = 0$.

EXAMPLE 7.10

Find the sum $\operatorname{cosec}^{-1} \sqrt{10} + \operatorname{cosec}^{-1} \sqrt{50} + \operatorname{cosec}^{-1} \sqrt{170} + \dots$
 $+ \operatorname{cosec}^{-1} \sqrt{(n^2 + 1)(n^2 + 2n + 2)}$.

Sol. Let $\theta = \operatorname{cosec}^{-1} \sqrt{(n^2 + 1)(n^2 + 2n + 2)}$

$$\begin{aligned} \text{or } \operatorname{cosec}^2 \theta &= (n^2 + 1)(n^2 + 2n + 2) \\ &= (n^2 + 1)^2 + 2n(n^2 + 1) + n^2 + 1 \\ &= (n^2 + n + 1)^2 + 1 \end{aligned}$$

$$\Rightarrow \cot^2 \theta = (n^2 + n + 1)^2$$

$$\Rightarrow \tan \theta = \frac{1}{n^2 + n + 1} = \frac{(n+1) - n}{1 + (n+1)n}$$

$$\Rightarrow \theta = \tan^{-1} \left[\frac{(n+1) - n}{1 + (n+1)n} \right] = \tan^{-1}(n+1) - \tan^{-1} n$$

Thus, sum of n terms of the given series

$$\begin{aligned} &= (\tan^{-1} 2 - \tan^{-1} 1) + (\tan^{-1} 3 - \tan^{-1} 2) \\ &\quad + (\tan^{-1} 4 - \tan^{-1} 3) + \dots + (\tan^{-1}(n+1) - \tan^{-1} n) \\ &= \tan^{-1}(n+1) - \pi/4 \end{aligned}$$

EXAMPLE 7.11

Let $f(x) = \sin x + \cos x + \tan x + \sin^{-1} x + \cos^{-1} x + \tan^{-1} x$.
 Then find the maximum and minimum values of $f(x)$.

Sol. $f(x) = \sin x + \cos x + \tan x + \sin^{-1} x + \cos^{-1} x + \tan^{-1} x$
 $f(x) = \sin x + \cos x + \tan x + \frac{\pi}{2} + \tan^{-1} x$

Clearly, domain of f is $[-1, 1]$.

$$f'(x) = \cos x - \sin x + \sec^2 x + 0 + \frac{1}{1+x^2}$$

$$\text{Now, } \sec^2 x \geq 1 \text{ and } \frac{1}{1+x^2} \in \left[\frac{1}{2}, 1 \right]$$

Hence, $f'(x) > 0$

$\Rightarrow f$ is increasing

$\Rightarrow$ Range is $[f(-1), f(1)]$

$$\begin{aligned} \text{Now, } f(-1) &= -\sin 1 + \cos 1 - \tan 1 + \frac{\pi}{2} - \frac{\pi}{4} \\ &= \frac{\pi}{4} + \cos 1 - \sin 1 - \tan 1 \end{aligned}$$

$$\begin{aligned} \text{And } f(1) &= \sin 1 + \cos 1 + \tan 1 + \frac{\pi}{2} + \frac{\pi}{4} \\ &= \frac{3\pi}{4} + \cos 1 + \sin 1 + \tan 1 \end{aligned}$$

Thus, least value is $\frac{\pi}{4} + \cos 1 - \sin 1 - \tan 1$ and greatest value
 is $\frac{3\pi}{4} + \cos 1 + \sin 1 + \tan 1$.

Exercises

Single Correct Answer Type

- $\cos^{-1}(\cos(2 \cot^{-1}(\sqrt{2}-1)))$ is equal to
 (1) $\sqrt{2}-1$ (2) $\frac{\pi}{4}$
 (3) $\frac{3\pi}{4}$ (4) none of these
- The value of $\sin^{-1}\left(\cot\left(\sin^{-1}\sqrt{\frac{2-\sqrt{3}}{4}} + \cos^{-1}\frac{\sqrt{12}}{4} + \sec^{-1}\sqrt{2}\right)\right)$ is
 (1) 0 (2) $\frac{\pi}{2}$
 (3) $\frac{\pi}{3}$ (4) none of these
- If $\cot^{-1}\frac{n}{\pi} > \frac{\pi}{6}$, $n \in N$, then the maximum value of n is
 (1) 6 (2) 7
 (3) 5 (4) none of these
- If $\operatorname{cosec}^{-1}(\operatorname{cosec} x)$ and $\operatorname{cosec}(\operatorname{cosec}^{-1} x)$ are equal functions, then the maximum range of value of x is
 (1) $\left[-\frac{\pi}{2}, -1\right] \cup \left[1, \frac{\pi}{2}\right]$ (2) $\left[-\frac{\pi}{2}, 0\right) \cup \left[0, \frac{\pi}{2}\right]$
 (3) $(-\infty, -1] \cup [1, \infty)$ (4) $[-1, 0) \cup [0, 1]$
- $\sec^2(\tan^{-1} 2) + \operatorname{cosec}^2(\cot^{-1} 3)$ is equal to
 (1) 5 (2) 13 (3) 15 (4) 6
- The maximum value of $f(x) = \tan^{-1}\left(\frac{(\sqrt{12}-2)x^2}{x^4+2x^2+3}\right)$ is
 (1) 18° (2) 36° (3) 22.5° (4) 15°
- For the equation $\cos^{-1} x + \cos^{-1} 2x + \pi = 0$, the number of real solution is
 (1) 1 (2) 2 (3) 0 (4) ∞
- The number of real solutions of the equation $\tan^{-1}\sqrt{x^2-3x+2} + \cos^{-1}\sqrt{4x-x^2-3} = \pi$ is
 (1) one (2) two (3) zero (4) infinite
- If $\sin^{-1}(x-1) + \cos^{-1}(x-3) + \tan^{-1}\left(\frac{x}{2-x^2}\right) = \cos^{-1} k + \pi$, then the value of k is
 (1) 1 (2) $-\frac{1}{\sqrt{2}}$
 (3) $\frac{1}{\sqrt{2}}$ (4) none of these
- The number of real solutions of the equation $\sqrt{1+\cos 2x} = \sqrt{2} \sin^{-1}(\sin x)$, $-\pi \leq x \leq \pi$, is
 (1) 0 (2) 1 (3) 2 (4) infinite
- The equation $3 \cos^{-1} x - \pi x - \frac{\pi}{2} = 0$ has
 (1) one negative solution (2) one positive solution
 (3) no solution (4) more than one solution
- Range of $f(x) = \sin^{-1} x + \tan^{-1} x + \sec^{-1} x$ is
 (1) $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ (2) $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$
 (3) $\left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$ (4) none of these
- The value of $\lim_{|x| \rightarrow \infty} \cos(\tan^{-1}(\sin(\tan^{-1} x)))$ is equal to
 (1) -1 (2) $\sqrt{2}$
 (3) $-\frac{1}{\sqrt{2}}$ (4) $\frac{1}{\sqrt{2}}$
- Range of $\tan^{-1}\left(\frac{2x}{1+x^2}\right)$ is
 (1) $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ (2) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
 (3) $\left(-\frac{\pi}{2}, \frac{\pi}{4}\right]$ (4) $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$
- Equation $[\cot^{-1} x] + 2[\tan^{-1} x] = 0$, where $[\cdot]$ denotes the greatest integer function, is satisfied by
 (1) $(0, \cot 1)$ (2) $(0, \tan 1)$
 (3) $(\tan 1, \infty)$ (4) $(\cot 1, \tan 1)$
- The number of integral values of k for which the equation $\sin^{-1} x + \tan^{-1} x = 2k + 1$ has a solution is
 (1) 1 (2) 2 (3) 3 (4) 4
- The range of values of p for which the equation $\sin \cos^{-1}(\cos(\tan^{-1} x)) = p$ has a solution is
 (1) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ (2) $[0, 1)$
 (3) $\left(\frac{1}{\sqrt{2}}, 1\right)$ (4) $(-1, 1)$
- The sum of the solutions of the equation $2 \sin^{-1} \sqrt{x^2+x+1} + \cos^{-1} \sqrt{x^2+x} = \frac{3\pi}{2}$ is
 (1) 0 (2) -1 (3) 1 (4) 2
- Complete solution set of $\tan^2(\sin^{-1} x) > 1$ is
 (1) $\left(-1, -\frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, 1\right)$
 (2) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \sim \{0\}$
 (3) $(-1, 1) \sim \{0\}$
 (4) None of these

20. The trigonometric equation $\sin^{-1}x = 2 \sin^{-1}a$ has a solution for

(1) all real values (2) $|a| < \frac{1}{2}$
 (3) $|a| \leq \frac{1}{\sqrt{2}}$ (4) $\frac{1}{2} < |a| < \frac{1}{\sqrt{2}}$

21. The number of solution of equation $\sin^{-1}x + n \sin^{-1}(1-x) = \frac{m\pi}{2}$, where $n > 0$, $m \leq 0$, is

(1) 3 (2) 1
 (3) 2 (4) None of these

22. If $\left| \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right) \right| < \frac{\pi}{3}$, then

(1) $x \in \left[-\frac{1}{3}, \frac{1}{\sqrt{3}} \right]$ (2) $x \in \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$
 (3) $x \in \left(0, \frac{1}{\sqrt{3}} \right)$ (4) none of these

23. The value of $\sin^{-1}(\sin 12) + \cos^{-1}(\cos 12)$ is equal to

(1) zero (2) $24 - 2\pi$
 (3) $4\pi - 24$ (4) none of these

24. The value of the expression $\sin^{-1} \left(\sin \frac{22\pi}{7} \right)$

$\cos^{-1} \left(\cos \frac{5\pi}{3} \right) + \tan^{-1} \left(\tan \frac{5\pi}{7} \right) + \sin^{-1}(\cos 2)$ is

(1) $\frac{17\pi}{42} - 2$ (2) -2
 (3) $\frac{-\pi}{21} - 2$ (4) none of these

25. The value of $\sin^{-1}(\cos(\cos^{-1}(\cos x) + \sin^{-1}(\sin x)))$, where $x \in \left(\frac{\pi}{2}, \pi \right)$, is equal to

(1) $\frac{\pi}{2}$ (2) $-\pi$
 (3) π (4) $-\frac{\pi}{2}$

26. If $\alpha \in \left(-\frac{3\pi}{2}, -\pi \right)$, then the value of $\tan^{-1}(\cot \alpha)$

$-\cot^{-1}(\tan \alpha) + \sin^{-1}(\sin \alpha) + \cos^{-1}(\cos \alpha)$ is equal to

(1) $2\pi + \alpha$ (2) $\pi + \alpha$
 (3) 0 (4) $\pi - \alpha$

27. $\tan^{-1} \left[\frac{\cos x}{1 + \sin x} \right]$ is equal to

(1) $\frac{\pi}{4} - \frac{x}{2}$, for $x \in \left(-\frac{\pi}{2}, \frac{3\pi}{2} \right)$
 (2) $\frac{\pi}{4} - \frac{x}{2}$, for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$

(3) $\frac{\pi}{4} - \frac{x}{2}$, for $x \in \left(\frac{3\pi}{2}, \frac{5\pi}{2} \right)$

(4) $\frac{\pi}{4} - \frac{x}{2}$, for $x \in \left(-\frac{3\pi}{2}, -\frac{\pi}{2} \right)$

28. If $f(x) = x^{11} + x^9 - x^7 + x^3 + 1$ and $f(\sin^{-1}(\sin 8)) = \alpha$, where α is a constant, then $f(\tan^{-1}(\tan 8))$ is equal to

(1) α (2) $\alpha - 2$ (3) $\alpha + 2$ (4) $2 - \alpha$

29. If $\sin^{-1} : [-1, 1] \rightarrow \left[\frac{\pi}{2}, \frac{3\pi}{2} \right]$ and $\cos^{-1} : [-1, 1] \rightarrow [0, \pi]$

be two bijective functions, respectively inverses of bijective functions $\sin : \left[\frac{\pi}{2}, \frac{3\pi}{2} \right] \rightarrow [-1, 1]$ and $\cos : [0, \pi] \rightarrow [-1, 1]$, then $\sin^{-1}x + \cos^{-1}x$ is

(1) $\frac{\pi}{2}$ (2) π
 (3) $\frac{3\pi}{2}$ (4) not a constant

30. Which of the following is the solution set of the equation

$2\cos^{-1}x = \cot^{-1} \left(\frac{2x^2 - 1}{2x\sqrt{1-x^2}} \right)$?

(1) $(0, 1)$ (2) $(-1, 1) - \{0\}$
 (3) $(-1, 0)$ (4) $[-1, 1]$

31. The value of $\tan(\sin^{-1}(\cos(\sin^{-1}x))) \tan(\cos^{-1}(\sin(\cos^{-1}x)))$, where $x \in (0, 1)$, is equal to

(1) 0 (2) 1
 (3) -1 (4) none of these

32. There exists a positive real number x satisfying

$\cos(\tan^{-1}x) = x$. Then the value of $\cos^{-1} \left(\frac{x^2}{2} \right)$ is

(1) $\frac{\pi}{10}$ (2) $\frac{\pi}{5}$
 (3) $\frac{2\pi}{5}$ (4) $\frac{4\pi}{5}$

33. If $\tan^{-1} \frac{\sqrt{1+x^2}-1}{x} = 4^\circ$, then

(1) $x = \tan 2^\circ$ (2) $x = \tan 4^\circ$
 (3) $x = \tan(1/4)^\circ$ (4) $x = \tan 8^\circ$

34. The value of

$\frac{\alpha^3}{2} \operatorname{cosec}^2 \left(\frac{1}{2} \tan^{-1} \left(\frac{\alpha}{\beta} \right) \right) + \frac{\beta^3}{2} \sec^2 \left(\frac{1}{2} \tan^{-1} \left(\frac{\beta}{\alpha} \right) \right)$

is equal to

(1) $(\alpha - \beta)(\alpha^2 + \beta^2)$ (2) $(\alpha + \beta)(\alpha^2 - \beta^2)$
 (3) $(\alpha + \beta)(\alpha^2 + \beta^2)$ (4) none of these

35. $\tan \left(\frac{\pi}{4} + \frac{1}{2} \cos^{-1}x \right) + \tan \left(\frac{\pi}{4} - \frac{1}{2} \cos^{-1}x \right)$, $x \neq 0$, is equal to

(2) $2x$

(4) none of these

36. If $\sin^{-1}x + \sin^{-1}y = \frac{\pi}{2}$, then $\frac{1+x^4+y^4}{x^2-x^2y^2+y^2}$ is equal to

(2) 2

(4) none of these

37. The value of $2 \tan^{-1}(\operatorname{cosec} \tan^{-1}x - \tan \cot^{-1}x)$ is equal to

(2) $\cot^{-1} \frac{1}{x}$

(4) none of these

38. If $\sin^{-1}a + \sin^{-1}b + \sin^{-1}c = \pi$, then the value of $a\sqrt{1-a^2} + b\sqrt{1-b^2} + c\sqrt{1-c^2}$ will be

(2) abc

(4) $\frac{1}{3}abc$

39. If $a \sin^{-1}x - b \cos^{-1}x = c$, then $a \sin^{-1}x + b \cos^{-1}x$ is equal to

(2) $\frac{\pi ab + c(b-a)}{a+b}$

(4) $\frac{\pi ab + c(a-b)}{a+b}$

40. The solution of the inequality $\log_{1/2} \sin^{-1}x > \log_{1/2} \cos^{-1}x$ is

(1) $x \in \left[0, \frac{1}{\sqrt{2}}\right]$

(2) $x \in \left[\frac{1}{\sqrt{2}}, 1\right]$

(3) $x \in \left(0, \frac{1}{\sqrt{2}}\right)$

(4) None of these

41. For $0 < \theta < 2\pi$, $\sin^{-1}(\sin \theta) > \cos^{-1}(\sin \theta)$ is true when

(1) $\left(\frac{\pi}{4}, \pi\right)$

(2) $\left(\pi, \frac{3\pi}{2}\right)$

(3) $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$

(4) $\left(\frac{3\pi}{4}, 2\pi\right)$

42. If $|\sin^{-1}x| + |\cos^{-1}x| = \frac{\pi}{2}$, then $x \in$

(1) R (2) $[-1, 1]$ (3) $[0, 1]$ (4) ϕ

43. If $(\sin^{-1}x)^2 - (\cos^{-1}x)^2 = a\pi^2$ then find the range of a .

(1) $\left[-\frac{3}{4}, \frac{1}{4}\right]$

(2) $\left[-\frac{3}{4}, \frac{3}{4}\right]$

(3) $[-1, 1]$

(4) $\left[-1, \frac{3}{4}\right]$

44. The number of integer x satisfying

$\sin^{-1}|x-2| + \cos^{-1}(1-|3-x|) = \frac{\pi}{2}$ is

(1) 1

(2) 2

(3) 3

(4) 4

45. The number of solutions of the equation

$$\cos^{-1}\left(\frac{1+x^2}{2x}\right) - \cos^{-1}x = \frac{\pi}{2} + \sin^{-1}x \text{ is}$$

(1) 0

(2) 1

(3) 2

(4) 3

46. $f(x) = \tan^{-1}x + \tan^{-1}\left(\frac{1}{x}\right)$; $g(x) = \sin^{-1}x + \cos^{-1}x$ are identical functions if

(1) $x \in R$

(2) $x > 0$

(3) $x \in [-1, 1]$

(4) $x \in (0, 1]$

47. The value of a for which $ax^2 + \sin^{-1}(x^2 - 2x + 2) + \cos^{-1}(x^2 - 2x + 2) = 0$ has a real solution is

(1) $\frac{\pi}{2}$

(2) $-\frac{\pi}{2}$

(3) $\frac{2}{\pi}$

(4) $-\frac{2}{\pi}$

48. If $\sin^{-1}\left(\frac{5}{x}\right) + \sin^{-1}\left(\frac{12}{x}\right) = \frac{\pi}{2}$, then x is equal to

(1) $\frac{7}{13}$

(2) $\frac{4}{3}$

(3) 13

(4) $\frac{13}{7}$

49. If $\cos^{-1}\sqrt{p} + \cos^{-1}\sqrt{1-p} + \cos^{-1}\sqrt{1-q} = \frac{3\pi}{4}$, then the value of q is

(1) 1

(2) $\frac{1}{\sqrt{2}}$

(3) $\frac{1}{3}$

(4) $\frac{1}{2}$

50. If $\tan^{-1}(\sin^2\theta - 2\sin\theta + 3) + \cot^{-1}(5^{\sec^2\theta} + 1) = \frac{\pi}{2}$, then the value of $\cos^2\theta - \sin\theta$ is equal to

(1) 0

(2) -1

(3) 1

(4) none of these

51. The product of all values of x satisfying the equation

$$\sin^{-1} \cos\left(\frac{2x^2 + 10|x| + 4}{x^2 + 5|x| + 3}\right) = \cot\left(\cot^{-1}\left(\frac{2 - 18|x|}{9|x|}\right)\right) + \frac{\pi}{2}$$

is

(1) 9

(2) -9

(3) -3

(4) -1

52. The exhaustive set of values of a for which $a - \cot^{-1}3x = 2\tan^{-1}3x + \cos^{-1}x\sqrt{3} + \sin^{-1}x\sqrt{3}$ may have solution, is

(1) $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

(3) $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

(3) $\left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]$

(4) $\left[-\frac{3\pi}{6}, \frac{7\pi}{6}\right]$

53. If $u = \cot^{-1}\sqrt{\tan\alpha} - \tan^{-1}\sqrt{\tan\alpha}$, then $\tan\left(\frac{\pi}{4} - \frac{u}{2}\right)$ is equal to

(1) $\sqrt{\tan\alpha}$

(2) $\sqrt{\cot\alpha}$

(3) $\tan\alpha$

(4) $\cot\alpha$

54. The solution set of the equation

$$\sin^{-1} \sqrt{1-x^2} + \cos^{-1} x = \cot^{-1} \frac{\sqrt{1-x^2}}{x} - \sin^{-1} x \text{ is}$$

- (1) $[-1, 1] - \{0\}$ (2) $(0, 1] \cup \{-1\}$
 (3) $[-1, 0] \cup \{1\}$ (4) $[-1, 1]$

55. The value of $\cos^{-1} \frac{\sqrt{2}}{3} - \cos^{-1} \frac{\sqrt{6}+1}{2\sqrt{3}}$ is equal to

- (1) $\frac{\pi}{3}$ (2) $\frac{\pi}{4}$ (3) $\frac{\pi}{2}$ (4) $\frac{\pi}{6}$

56. $\theta = \tan^{-1}(2 \tan^2 \theta) - \tan^{-1}\left(\frac{1}{3} \tan \theta\right)$ then $\tan \theta =$

- (1) -2 (2) -1 (3) $2/3$ (4) 2

57. If $y = \tan^{-1} \frac{1}{2} + \tan^{-1} b$, ($0 < b < 1$) and $0 < y \leq \frac{\pi}{4}$, then the maximum value of b is

- (1) $1/2$ (2) $1/3$ (3) $1/4$ (4) $2/3$

58. If x, y, z are natural numbers such that $\cot^{-1} x + \cot^{-1} y = \cot^{-1} z$ then the number of ordered triplets (x, y, z) that satisfy the equation is

- (1) 0 (2) 1
 (3) 2 (4) Infinite solutions

59. The value of α such that $\sin^{-1} \frac{2}{\sqrt{5}}$, $\sin^{-1} \frac{3}{\sqrt{10}}$, $\sin^{-1} \alpha$ are the angles of a triangle is

- (1) $\frac{-1}{\sqrt{2}}$ (2) $\frac{1}{2}$ (3) $\frac{1}{\sqrt{3}}$ (4) $\frac{1}{\sqrt{2}}$

60. The number of solutions of the equation

$$\tan^{-1}(1+x) + \tan^{-1}(1-x) = \frac{\pi}{2} \text{ is}$$

- (1) 2 (2) 3 (3) 1 (4) 0

61. The sum of roots of the equation

$$\tan^{-1} \frac{1}{1+2x} + \tan^{-1} \frac{1}{1+4x} = \tan^{-1} \frac{2}{x^2} \text{ is}$$

- (1) 2 (2) 3
 (3) 4 (4) none of these

62. If $\cot^{-1} x + \cot^{-1} y + \cot^{-1} z = \frac{\pi}{2}$, $x, y, z > 0$ and $xy < 1$, then $x+y+z$ is also equal to

- (1) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ (2) xyz
 (3) $xy + yz + zx$ (4) none of these

63. If $x^2 + y^2 + z^2 = r^2$, then

$$\tan^{-1}\left(\frac{xy}{zr}\right) + \tan^{-1}\left(\frac{yz}{xr}\right) + \tan^{-1}\left(\frac{xz}{yr}\right) \text{ is equal to}$$

- (1) π (2) $\frac{\pi}{2}$
 (3) 0 (4) none of these

64. The value of $\tan^{-1}\left(\frac{x \cos \theta}{1-x \sin \theta}\right) - \cot^{-1}\left(\frac{\cos \theta}{x - \sin \theta}\right)$ is

- (1) 2θ (2) θ
 (3) $\theta/2$ (4) independent of θ

65. If $\cot^{-1}(\sqrt{\cos \alpha}) - \tan^{-1}(\sqrt{\cos \alpha}) = x$, then $\sin x$ is

- (1) $\tan^2 \frac{\alpha}{2}$ (2) $\cot^2 \frac{\alpha}{2}$
 (3) $\tan \alpha$ (4) $\cot \frac{\alpha}{2}$

66. $\sum_{r=1}^n \sin^{-1}\left(\frac{\sqrt{r}-\sqrt{r-1}}{\sqrt{r(r+1)}}\right)$ is equal to

- (1) $\tan^{-1}(\sqrt{n}) - \frac{\pi}{4}$ (2) $\tan^{-1}(\sqrt{n+1}) - \frac{\pi}{4}$
 (3) $\tan^{-1}(\sqrt{n})$ (4) $\tan^{-1}(\sqrt{n+1})$

67. $\sum_{m=1}^n \tan^{-1}\left(\frac{2m}{m^4+m^2+2}\right)$ is equal to

- (1) $\tan^{-1}\left(\frac{n^2+n}{n^2+n+2}\right)$ (2) $\tan^{-1}\left(\frac{n^2-n}{n^2-n+2}\right)$
 (3) $\tan^{-1}\left(\frac{n^2+n+2}{n^2+n}\right)$ (4) none of these

68. The value of $\tan^{-1} \frac{4}{7} + \tan^{-1} \frac{4}{19} + \tan^{-1} \frac{4}{39} + \tan^{-1} \frac{4}{67} + \dots \infty$ equals

- (1) $\tan^{-1} 1 + \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3}$
 (2) $\tan^{-1} 1 + \cot^{-1} 3$
 (3) $\cot^{-1} 1 + \cot^{-1} \frac{1}{2} + \cot^{-1} \frac{1}{3}$
 (4) $\cot^{-1} 1 + \tan^{-1} 3$

69. The sum of series $\sec^{-1} \sqrt{2} + \sec^{-1} \frac{\sqrt{10}}{3} + \sec^{-1} \frac{\sqrt{50}}{7} + \dots$
 $+ \sec^{-1} \sqrt{\frac{(n^2+1)(n^2-2n+2)}{(n^2-n+1)^2}}$ is

- (1) $\tan^{-1} 1$ (2) $\tan^{-1} n$
 (3) $\tan^{-1}(n+1)$ (4) $\tan^{-1}(n-1)$

70. If $\frac{1}{2} \sin^{-1}\left[\frac{3 \sin 2\theta}{5+4 \cos 2\theta}\right] = \tan^{-1} x$, then $x =$

- (1) $\tan 3\theta$ (2) $3 \tan \theta$
 (3) $(1/3) \tan \theta$ (4) $3 \cot \theta$

71. The value $2 \tan^{-1}\left[\sqrt{\frac{a-b}{a+b}} \tan \frac{\theta}{2}\right]$ is equal to

- (1) $\cos^{-1}\left(\frac{a \cos \theta + b}{a + b \cos \theta}\right)$ (2) $\cos^{-1}\left(\frac{a + b \cos \theta}{a \cos \theta + b}\right)$
 (3) $\cos^{-1}\left(\frac{a \cos \theta}{a + b \cos \theta}\right)$ (4) $\cos^{-1}\left(\frac{b \cos \theta}{a \cos \theta + b}\right)$

Multiple Correct Answers Type

72. If $\sin^{-1}\left(\frac{2a}{1+a^2}\right) + \sin^{-1}\left(\frac{2b}{1+b^2}\right) = 2 \tan^{-1} x$, then x is equal to $[a, b \in (0, 1)]$

- (1) $\frac{a-b}{1+ab}$ (2) $\frac{b}{1+ab}$
(3) $\frac{b}{1-ab}$ (4) $\frac{a+b}{1-ab}$

73. If $3 \sin^{-1}\left(\frac{2x}{1+x^2}\right) - 4 \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) + 2 \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{3}$, where $|x| < 1$, then x is equal to

- (1) $\frac{1}{\sqrt{3}}$ (2) $-\frac{1}{\sqrt{3}}$ (3) $\sqrt{3}$ (4) $-\frac{\sqrt{3}}{4}$

74. If $x_1 = 2 \tan^{-1}\left(\frac{1+x}{1-x}\right)$, $x_2 = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, where $x \in (0, 1)$, then $x_1 + x_2$ is equal to

- (1) 0 (2) 2π
(3) π (4) none of these

75. If the equation $x^3 + bx^2 + cx + 1 = 0$, ($b < c$), has only one real root α , then the value of $2 \tan^{-1}(\operatorname{cosec} \alpha) + \tan^{-1}(2 \sin \alpha \sec^2 \alpha)$ is

- (1) $-\pi$ (2) $-\frac{\pi}{2}$ (3) $\frac{\pi}{2}$ (4) π

76. The value of $\sin^{-1}[x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}]$ is equal to

- (1) $\sin^{-1} x + \sin^{-1} \sqrt{x}$ (2) $\sin^{-1} x - \sin^{-1} \sqrt{x}$
(3) $\sin^{-1} \sqrt{x} - \sin^{-1} x$ (4) none of these

77. If $\cos^{-1} x - \cos^{-1} \frac{y}{2} = \alpha$, then $4x^2 - 4xy \cos \alpha + y^2$ is equal to

- (1) 4 (2) $2 \sin^2 \alpha$
(3) $-4 \sin^2 \alpha$ (4) $4 \sin^2 \alpha$

78. If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$, then $x^4 + y^4 + z^4 + 4x^2 y^2 z^2 = K(x^2 y^2 + y^2 z^2 + z^2 x^2)$, where K is equal to

- (1) 1 (2) 2
(3) 4 (4) none of these

79. If $f(x) = \sin^{-1}\left(\frac{\sqrt{3}}{2}x - \frac{1}{2}\sqrt{1-x^2}\right)$, $-\frac{1}{2} \leq x \leq 1$, then $f(x)$ is equal to

- (1) $\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(x)$ (2) $\sin^{-1} x - \frac{\pi}{6}$
(3) $\sin^{-1} x + \frac{\pi}{6}$ (4) none of these

80. If $2^{2\pi/\sin^{-1} x} - 2(a+2)2^{\pi/\sin^{-1} x} + 8a < 0$ for at least one real x , then

- (1) $\frac{1}{8} \leq a < 2$ (2) $a < 2$
(3) $a \in \mathbb{R} - \{2\}$ (4) $a \in \left[0, \frac{1}{8}\right) \cup (2, \infty)$

1. If α, β ($\alpha < \beta$) are the roots of the equation $6x^2 + 11x + 3 = 0$, then which of the following are real?

- (1) $\cos^{-1} \alpha$ (2) $\sin^{-1} \beta$
(3) $\operatorname{cosec}^{-1} \alpha$ (4) Both $\cot^{-1} \alpha$ and $\cot^{-1} \beta$

2. $2 \tan^{-1}(-2)$ is equal to

- (1) $-\cos^{-1}\left(\frac{-3}{5}\right)$ (2) $-\pi + \cos^{-1} \frac{3}{5}$
(3) $-\frac{\pi}{2} + \tan^{-1}\left(-\frac{3}{4}\right)$ (4) $-\pi + \cot^{-1}\left(-\frac{3}{4}\right)$

3. Which of the following is/are the value of

$$\cos\left[\frac{1}{2}\cos^{-1}\left(\cos\left(-\frac{14\pi}{5}\right)\right)\right]?$$

- (1) $\cos\left(-\frac{7\pi}{5}\right)$ (2) $\sin\left(\frac{\pi}{10}\right)$
(3) $\cos\left(\frac{2\pi}{5}\right)$ (4) $-\cos\left(\frac{3\pi}{5}\right)$

4. Which of the following is/are a rational number?

- (1) $\sin\left(\tan^{-1} 3 + \tan^{-1} \frac{1}{3}\right)$
(2) $\cos\left(\frac{\pi}{2} - \sin^{-1} \frac{3}{4}\right)$
(3) $\log_2\left(\sin\left(\frac{1}{4}\sin^{-1} \frac{\sqrt{63}}{8}\right)\right)$
(4) $\tan\left(\frac{1}{2}\cos^{-1} \frac{\sqrt{5}}{3}\right)$

5. Which of the following quantities is/are positive?

- (1) $\cos(\tan^{-1}(\tan 4))$ (2) $\sin(\cot^{-1}(\cot 4))$
(3) $\tan(\cos^{-1}(\cos 5))$ (4) $\cot(\sin^{-1}(\sin 4))$

6. If $x < 0$, then $\tan^{-1} x$ is equal to

- (1) $-\pi + \cot^{-1} \frac{1}{x}$ (2) $\sin^{-1} \frac{x}{\sqrt{1+x^2}}$
(3) $-\cos^{-1} \frac{1}{\sqrt{1+x^2}}$ (4) $-\operatorname{cosec}^{-1} \frac{\sqrt{1+x^2}}{x}$

7. If $-1 < x < 0$, then $\cos^{-1} x$ is equal to

- (1) $\sec^{-1} \frac{1}{x}$ (2) $\pi - \sin^{-1} \sqrt{1-x^2}$
(3) $\pi + \tan^{-1} \frac{\sqrt{1-x^2}}{x}$ (4) $\cot^{-1} \frac{x}{\sqrt{1-x^2}}$

8. If $(\sin^{-1} x + \sin^{-1} w)(\sin^{-1} y + \sin^{-1} z) = \pi^2$, then

$$D = \begin{vmatrix} x^{N_1} & y^{N_2} \\ z^{N_3} & w^{N_4} \end{vmatrix} \quad (N_1, N_2, N_3, N_4 \in \mathbb{N})$$

- (1) has a maximum value of 2
 (2) has a minimum value of 0
 (3) 16 different D are possible
 (4) has a minimum value of -2
9. Indicate the relation which can hold in their respective domain for infinite values of x .
 (1) $\tan|\tan^{-1}x| = |x|$ (2) $\cot|\cot^{-1}x| = |x|$
 (3) $\tan^{-1}|\tan x| = |x|$ (4) $\sin|\sin^{-1}x| = |x|$
10. If $\cot^{-1}\left(\frac{n^2 - 10n + 21.6}{\pi}\right) > \frac{\pi}{6}$, $n \in N$, then n can be
 (1) 3 (2) 2 (3) 4 (4) 8
11. If $z = \sec^{-1}\left(x + \frac{1}{x}\right) + \sec^{-1}\left(y + \frac{1}{y}\right)$, where $xy < 0$, then the possible values of z is (are)
 (1) $\frac{8\pi}{10}$ (2) $\frac{7\pi}{10}$ (3) $\frac{9\pi}{10}$ (4) $\frac{21\pi}{20}$
12. The value of k ($k > 0$) such that the length of the longest interval in which the function $f(x) = \sin^{-1}|\sin kx| + \cos^{-1}(\cos kx)$ is constant is $\pi/4$ is/are
 (1) 8 (2) 4 (3) 12 (4) 16
13. Which of the following pairs of function/functions has same graph?
 (1) $y = \tan(\cos^{-1}x)$; $y = \frac{\sqrt{1-x^2}}{x}$
 (2) $y = \tan(\cot^{-1}x)$; $y = \frac{1}{x}$
 (3) $y = \sin(\tan^{-1}x)$; $y = \frac{x}{\sqrt{1+x^2}}$
 (4) $y = \cos(\tan^{-1}x)$; $y = \sin(\cot^{-1}x)$
14. If $\sin^{-1}x + \sin^{-1}y = \frac{\pi}{2}$ and $\sin 2x = \cos 2y$, then
 (1) $x = \frac{\pi}{8} + \sqrt{\frac{1}{2} - \frac{\pi^2}{64}}$ (2) $y = \sqrt{\frac{1}{2} - \frac{\pi^2}{64}} - \frac{\pi}{12}$
 (3) $x = \frac{\pi}{12} + \sqrt{\frac{1}{2} - \frac{\pi^2}{64}}$ (4) $y = \sqrt{\frac{1}{2} - \frac{\pi^2}{64}} - \frac{\pi}{8}$
15. If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, then
 (1) $x^2 + y^2 + z^2 + 2xyz = 1$
 (2) $2(\sin^{-1}x + \sin^{-1}y + \sin^{-1}z) = \cos^{-1}x + \cos^{-1}y + \cos^{-1}z$
 (3) $xy + yz + zx = x + y + z - 1$
 (4) $\left(x + \frac{1}{x}\right) + \left(y + \frac{1}{y}\right) + \left(z + \frac{1}{z}\right) \geq 6$
16. If $\sin^{-1}\left(a - \frac{a^2}{3} + \frac{a^3}{9} + \dots\right) + \cos^{-1}(1 + b + b^2 + \dots) = \frac{\pi}{2}$, then
 (1) $b = \frac{2a-3}{3a}$ (2) $b = \frac{3a-2}{2a}$
- (3) $a = \frac{3}{2-3b}$ (4) $a = \frac{2}{3-2b}$
17. If $\tan^{-1}(x^2 + 3|x| - 4) + \cot^{-1}(4\pi + \sin^{-1}\sin 14) = \frac{\pi}{2}$, then the value of $\sin^{-1}\sin 2x$ is
 (1) $6 - 2\pi$ (2) $2\pi - 6$
 (3) $\pi - 3$ (4) $3 - \pi$
18. If $2\tan^{-1}x + \sin^{-1}\frac{2x}{1+x^2}$ is independent of x , then
 (1) $x > 1$ (2) $x < -1$
 (3) $0 < x < 1$ (4) $-1 < x < 0$
19. If $\alpha = \tan^{-1}\left(\frac{4x-4x^3}{1-6x^2+x^4}\right)$, $\beta = 2\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and $\tan \frac{\pi}{8} = k$, then
 (1) $\alpha + \beta = \pi$ for $x \in \left[1, \frac{1}{k}\right)$
 (2) $\alpha = \beta$ for $x \in (-k, k)$
 (3) $\alpha + \beta = -\pi$ for $x \in \left[1, \frac{1}{k}\right)$
 (4) $\alpha + \beta = 0$ for $x \in (-k, k)$
20. $2\tan(\tan^{-1}(x) + \tan^{-1}(x^3))$, where $x \in R - \{-1, 1\}$, is equal to
 (1) $\frac{2x}{1-x^2}$
 (2) $\tan(2\tan^{-1}x)$
 (3) $\tan(\cot^{-1}(-x) - \cot^{-1}(x))$
 (4) $\tan(2\cot^{-1}x)$
21. Let $\alpha = \sin^{-1}\left(\frac{36}{85}\right)$, $\beta = \cos^{-1}\left(\frac{4}{5}\right)$ and $\gamma = \tan^{-1}\left(\frac{8}{15}\right)$. Then
 (1) $\cot \alpha + \cot \beta + \cot \gamma = \cot \alpha \cot \beta \cot \gamma$
 (2) $\tan \alpha \tan \beta + \tan \beta \tan \gamma + \tan \alpha \tan \gamma = 1$
 (3) $\tan \alpha + \tan \beta + \tan \gamma = \tan \alpha \tan \beta \tan \gamma$
 (4) $\cot \alpha \cot \beta + \cot \beta \cot \gamma + \cot \alpha \cot \gamma = 1$
22. If $S_n = \cot^{-1}(3) + \cot^{-1}(7) + \cot^{-1}(13) + \cot^{-1}(21) + \dots$ terms, then
 (1) $S_{10} = \tan^{-1} \frac{5}{6}$ (2) $S_{\infty} = \frac{\pi}{4}$
 (3) $S_6 = \sin^{-1} \frac{4}{5}$ (4) $S_{20} = \cot^{-1} 1.1$
23. Equation $1 + x^2 + 2x \sin(\cos^{-1}y) = 0$ is satisfied by
 (1) exactly one value of x (2) exactly two values of x
 (3) exactly one value of y (4) exactly two values of y
24. To the equation $2^{2\pi/\cos^{-1}x} - \left(a + \frac{1}{2}\right)2^{\pi/\cos^{-1}x} - a^2 = 0$ has only one real root, then
 (1) $1 \leq a \leq 3$ (2) $a \geq 1$
 (3) $a \leq -3$ (4) $a \geq 3$

For Problems 1-3

For $x, y, z, t \in R$, $\sin^{-1}x + \cos^{-1}y + \sec^{-1}z \geq t^2 - \sqrt{2}\pi t + 3\pi$

1. The value of $x + y + z$ is equal to
(1) 1 (2) 0 (3) 2 (4) -1

2. The principal value of $\cos^{-1}(\cos 5t^2)$ is
(1) $\frac{3\pi}{2}$ (2) $\frac{\pi}{2}$

- (3) $\frac{\pi}{3}$ (4) $\frac{2\pi}{3}$

3. The value of $\cos^{-1}(\min\{x, y, z\})$ is

- (1) 0 (2) $\frac{\pi}{2}$

- (3) π (4) $\frac{\pi}{3}$

For Problems 4-6

$ax + b(\sec(\tan^{-1}x)) = c$ and $ay + b(\sec(\tan^{-1}y)) = c$

4. The value of xy is

- (1) $\frac{2ab}{a^2 - b^2}$ (2) $\frac{c^2 - b^2}{a^2 - b^2}$

- (3) $\frac{c^2 - b^2}{a^2 + b^2}$ (4) none of these

5. The value of $x + y$ is

- (1) $\frac{2ac}{a^2 - b^2}$ (2) $\frac{c^2 - b^2}{a^2 - b^2}$

- (3) $\frac{c^2 - b^2}{a^2 + b^2}$ (4) none of these

6. The value of $\frac{x+y}{1-xy}$ is

- (1) $\frac{2ab}{a^2 - c^2}$ (2) $\frac{2ac}{a^2 - c^2}$

- (3) $\frac{c^2 - b^2}{a^2 + b^2}$ (4) none of these

For Problems 7-9

Consider the system of equations $\cos^{-1}x + (\sin^{-1}y)^2 = \frac{p\pi^2}{4}$ and $(\cos^{-1}x)(\sin^{-1}y)^2 = \frac{\pi^4}{16}$, $p \in Z$.

7. The value of p for which system has a solution is

- (1) 1 (2) 2 (3) 0 (4) -1

8. The value of x which satisfies the system of equations is

- (1) $\cos \frac{\pi^2}{8}$ (2) $\sin \frac{\pi^2}{4}$

- (3) $\cos \frac{\pi^2}{2}$

- (4) none of these

9. Which of the following is not the value of y that satisfies the system of equations?

- (1) 1

- (2) -1

- (3) $\frac{1}{2}$

- (4) none of these

For Problems 10-12

Let $\cos^{-1}(4x^3 - 3x) = a + b \cos^{-1}x$.

10. If $x \in \left[-1, -\frac{1}{2}\right]$, then the value of $a + b\pi$ is

- (1) 2π (2) 3π (3) π (4) -2π

11. If $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, then the principal value of $\sin^{-1}\left(\sin \frac{a}{b}\right)$ is

- (1) $-\frac{\pi}{3}$ (2) $\frac{\pi}{3}$ (3) $-\frac{\pi}{6}$ (4) $\frac{\pi}{6}$

12. If $x \in \left[\frac{1}{2}, 1\right]$, then the value of $\lim_{y \rightarrow a} b \cos(y)$ is

- (1) $-1/3$ (2) -3 (3) $\frac{1}{3}$ (4) 3

For Problems 13 and 14

Let $a = \cos^{-1} \cos 20$, $b = \cos^{-1} \cos 30$ and $c = \sin^{-1} \sin(a + b)$ then

13. The largest integer x for which $\sin^{-1}(\sin x) \geq |x - (a + b + c)|$ is

- (1) 1 (2) 2 (3) 3 (4) 4

14. If $5 \sec^{-1}x + 10 \sin^{-1}y = 10(a + b + c)$ then the value of $\tan^{-1}x + \cos^{-1}(y - 1)$ is

- (1) $\frac{\pi}{2}$ (2) $\frac{\pi}{4}$ (3) π (4) 0

For Problems 15 and 16

Consider the functions $f(x) = \sin^{-1}x$, having principal value branch $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ and $g(x) = \cos^{-1}x$, having principal value branch $[0, \pi]$.

15. The value of $f(\sin 10)$ is

- (1) $10 - 3\pi$ (2) $10 - 2\pi$

- (3) $10 - \frac{5\pi}{2}$ (4) $\frac{7\pi}{2} - 10$

16. For $\sin^{-1}x < \frac{3\pi}{4}$, solution set of x is

- (1) $\left[\frac{1}{\sqrt{2}}, 1\right]$ (2) $\left[-\frac{1}{\sqrt{2}}, -1\right]$

- (3) $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$ (4) $[-1, 1]$

Matrix Match Type

1.

List I	List II
a. $\sin^{-1} \frac{4}{5} + 2 \tan^{-1} \frac{1}{3} =$	p. $\pi/6$
b. $\sin^{-1} \frac{12}{13} + \cos^{-1} \frac{4}{5} + \tan^{-1} \frac{63}{16} =$	q. $\pi/2$
c. If $A = \tan^{-1} \frac{x\sqrt{3}}{2\lambda - x}$ and $B = \tan^{-1} \left(\frac{2x - \lambda}{\lambda\sqrt{3}} \right)$ then the value of $A - B$ is,	r. $\pi/4$
d. $\tan^{-1} \frac{1}{7} + 2 \tan^{-1} \frac{1}{3} =$	s. π

2. Match the following lists:

List I	List II
a. Number of roots of the equation $\sin^{-1}(\sin x) = \frac{x}{4}$ is	p. 2
b. Number of roots of the equation $\cos^{-1}(\cos x) = \frac{x}{4}$ is	q. 3
c. Number of roots of the equation $\tan^{-1}(\tan x) = \frac{x}{4}$ is	r. 4
d. Number of roots of the equation $\cot^{-1}(\cot x) = \frac{x}{4}$ is	s. 5

Codes

- a b c d
 (1) s r q p
 (2) q s r p
 (3) s r p q
 (4) q r q p

3. Match the following lists:

List I	List II
a. If $3 \sin^{-1} x = \pi - \sin^{-1} (3x - 4x^3)$, then	p. $x \in \left[\frac{1}{\sqrt{2}}, 1 \right]$
b. If $2 \tan^{-1} x = \pi - \sin^{-1} \frac{2x}{1+x^2}$, then	q. $x \in \left[\frac{1}{2}, 1 \right]$
c. If $2 \tan^{-1} x = \pi + \tan^{-1} \frac{2x}{1-x^2}$, then	r. $x \in (1, \infty)$
d. If $2 \sin^{-1} x = \pi - \sin^{-1} (2x\sqrt{1-x^2})$, then	s. $x \in [1, \infty)$

Codes

- a b c d
 (1) s r q p
 (2) q s r p
 (3) s r p q
 (4) r p s q

4.

List I	List II
a. If $(\sin^{-1} x)^2 + (\sin^{-1} y)^2 = \frac{\pi^2}{2}$, then $x^3 + y^3$ can be	p. 0
b. $(\cos^{-1} x)^2 + (\cos^{-1} y)^2 = 2\pi^2$, then $x^5 + y^5$ can be	q. -2
c. $(\sin^{-1} x)^2 (\cos^{-1} y)^2 = \frac{\pi^4}{4}$, then $x - y$ can be	r. 2
d. $ \sin^{-1} x - \sin^{-1} y = \pi$, then $x - y$ can be	s. -1

Codes

- a b c d
 (1) r q p s
 (2) s r q p
 (3) q s p r
 (4) s r q p

5.

List I	List II
a. Range of $f(x) = \sin^{-1} x + \cos^{-1} x + \cot^{-1} x$ is	p. $\left[0, \frac{\pi}{2} \right] \cup \left(\frac{\pi}{2}, \pi \right]$
b. Range of $f(x) = \cot^{-1} x + \tan^{-1} x + \operatorname{cosec}^{-1} x$ is	q. $\left[\frac{\pi}{2}, \frac{3\pi}{2} \right]$
c. Range of $f(x) = \cot^{-1} x + \tan^{-1} x + \cos^{-1} x$ is	r. $\{0, \pi\}$
d. Range of $f(x) = \sec^{-1} x + \operatorname{cosec}^{-1} x + \sin^{-1} x$ is	s. $\left[\frac{3\pi}{4}, \frac{5\pi}{4} \right]$

Numerical Value Type

- The solution set of inequality $(\cot^{-1} x)(\tan^{-1} x) + \left(2 - \frac{\pi}{2} \right) \cot^{-1} x - 3 \tan^{-1} x - 3 \left(2 - \frac{\pi}{2} \right) > 0$ is (a, b) , then the value of $\cot^{-1} a + \cot^{-1} b$ is _____.
- If $x = \sin^{-1}(a^6 + 1) + \cos^{-1}(a^4 + 1) - \tan^{-1}(a^2 + 1)$, $a \in \mathbb{R}$, then the value of $\sec^2 x$ is _____.
- If the roots of the equation $x^3 - 10x + 11 = 0$ are u, v , and w , then the value of $3 \operatorname{cosec}^2(\tan^{-1} u + \tan^{-1} v + \tan^{-1} w)$ is _____.
- The number of values of x for which $\sin^{-1} \left(x^2 - \frac{x^4}{3} + \frac{x^6}{9} \dots \right) + \cos^{-1} \left(x^4 - \frac{x^8}{3} + \frac{x^{12}}{9} \dots \right) = \frac{\pi}{2}$ where $0 \leq |x| < \sqrt{3}$, is _____.

5. If the domain of the function $f(x) = \sqrt{3\cos^{-1}(4x) - \pi}$ is $[a, b]$, then the value of $(4a + 64b)$ is _____.
6. If $0 < \cos^{-1}x < 1$ and $1 + \sin(\cos^{-1}x) + \sin^2(\cos^{-1}x) + \sin^3(\cos^{-1}x) + \dots = 2$, then the value of $12x^2$ is _____.
7. If $\tan^{-1}\left(x + \frac{3}{x}\right) - \tan^{-1}\left(x - \frac{3}{x}\right) = \tan^{-1}\frac{6}{x}$, then the value of x^4 is _____.
8. If range of the function $f(x) = \sin^{-1}x + 2\tan^{-1}x + x^2 + 4x + 1$ is $[p, q]$, then the value of $(p + q)$ is _____.
9. If $\cos^{-1}(x) + \cos^{-1}(y) + \cos^{-1}(z) = \pi(\sec^2(u) + \sec^4(v) + \sec^6(w))$, where u, v, w are least non-negative angles such that $u < v < w$, then the value of $x^{2000} + y^{2002} + z^{2004} + \frac{36\pi}{u+v+w}$ is _____.
10. If the area enclosed by the curves $f(x) = \cos^{-1}(\cos x)$ and $g(x) = \sin^{-1}(\cos x)$ in $x \in [9\pi/4, 15\pi/4]$ is $9\pi^2/b$ (where a and b are coprime), then the value of b is _____.
11. Sum of all integers in the domain of $f(x) = \cot^{-1}\sqrt{(x+3)x} + \cos^{-1}\sqrt{x^2 + 3x + 1}$ is _____.

12. The least value of $(1 + \sec^{-1}x)(1 + \cos^{-1}x)$ is _____.
13. Let $\cos^{-1}(x) + \cos^{-1}(2x) + \cos^{-1}(3x)$ be π . If x satisfies the equation $ax^3 + bx^2 + c = 0$, then the value of $(b + a + c)$ is _____.
14. The number of integral values of x satisfying the equation $\tan^{-1}(3x) + \tan^{-1}(5x) = \tan^{-1}(7x) + \tan^{-1}(2x)$ is _____.
15. Number of solutions of equation $\sin(\cos^{-1}(\tan(\sec^{-1}x))) = \sqrt{1+x}$ is/are _____.
16. If the equation $\sin^{-1}(x^2 + x + 1) + \cos^{-1}(\lambda x + 1) = \frac{\pi}{2}$ has exactly two solutions for $\lambda \in [a, b]$, then the value of $a + b$ is _____.
17. $\sin\left\{2\left(\sin^{-1}\frac{\sqrt{5}}{3} - \cos^{-1}\frac{\sqrt{5}}{3}\right)\right\}$ is equal to $\frac{k\sqrt{5}}{81}$ then $k =$ _____.
18. The number of solutions of $\cos(2\sin^{-1}(\cot(\tan^{-1}(\sec(6\operatorname{cosec}^{-1}x)))) + 1 = 0$ where $x > 0$ is _____.

Archives

JEE MAIN

Single Correct Answer Type

1. If x, y, z are in A.P. and $\tan^{-1}x, \tan^{-1}y$ and $\tan^{-1}z$ are also in A.P., then
- (1) $x = y = z$ (2) $2x = 3y = 6z$
 (3) $6x = 3y = 2z$ (4) $6x = 4y = 3z$
- (JEE Main 2013)

2. Let $\tan^{-1}y = \tan^{-1}x + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, where $|x| < \frac{1}{\sqrt{3}}$. Then a value of y is

- (1) $\frac{3x-x^3}{1-3x^2}$ (2) $\frac{3x+x^3}{1-3x^2}$
 (3) $\frac{3x-x^3}{1+3x^2}$ (4) $\frac{3x+x^3}{1+3x^2}$
- (JEE Main 2015)

JEE ADVANCED

Single Correct Answer Type

1. The value of $\cot\left(\sum_{n=1}^{23}\cot^{-1}\left(1+\sum_{k=1}^n2k\right)\right)$ is
- (1) $\frac{23}{25}$ (2) $\frac{25}{23}$
 (3) $\frac{23}{24}$ (4) $\frac{24}{23}$
- (JEE Advanced 2013)

Multiple Correct Answers Type

1. If $\alpha = 3\sin^{-1}\left(\frac{6}{11}\right)$ and $\beta = 3\cos^{-1}\left(\frac{4}{9}\right)$, where the inverse trigonometric functions take only the principal values, then the correct option(s) is (are)
- (1) $\cos\beta > 0$ (2) $\sin\beta < 0$
 (3) $\cos(\alpha + \beta) > 0$ (4) $\cos\alpha < 0$
- (JEE Advanced 2015)
2. For any positive integer n , define $f_n : (0, \infty) \rightarrow \mathbb{R}$ as
- $$f_n(x) = \sum_{j=1}^n \tan^{-1}\left(\frac{1}{1+(x+j)(x+j-1)}\right) \text{ for all } x \in (0, \infty).$$
- (Here, the inverse trigonometric function $\tan^{-1}x$ assumes values in $(-\pi/2, \pi/2)$.)
- Then, which of the following statement(s) is (are) TRUE?
- (1) $\sum_{j=1}^5 \tan^2(f_j(0)) = 55$
 (2) $\sum_{j=1}^{10} (1 + f_j'(0)) \sec^2(f_j(0)) = 10$
 (3) For any fixed positive integer n , $\lim_{x \rightarrow \infty} \tan(f_n(x)) = \frac{1}{n}$
 (4) For any fixed positive integer n , $\lim_{x \rightarrow \infty} \sec^2(f_n(x)) = 1$
- (JEE Advanced 2018)

Matrix Match Type

1. Match the statements in List I with those in List II.

List I	List II
a. A line from the origin meets the lines $\frac{x-2}{1} = \frac{y-1}{-2} = \frac{z+1}{1}$ and $\frac{x-8}{2} = \frac{y+3}{-1} = \frac{z-1}{1}$ at P and Q respectively. If length PQ = d, then d^2 is	p. -4
b. The value of x satisfying $\tan^{-1}(x+3) - \tan^{-1}(x-3) = \sin^{-1}\left(\frac{3}{5}\right)$ are	q. 0
c. Non-zero vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ satisfy $\vec{a} \cdot \vec{b} = 0$, $(\vec{b} - \vec{a}) \cdot (\vec{b} + \vec{c}) = 0$ and $2 \vec{b} + \vec{c} = \vec{b} - \vec{a} $. If $\vec{a} = \mu\vec{b} + 4\vec{c}$, then the possible values of μ are	r. 4
d. Let f be the function on $[-\pi, \pi]$ given by $f(0) = 9$ and $f(x) = \sin\left(\frac{9x}{2}\right) / \sin\left(\frac{x}{2}\right)$ for $x \neq 0$. The value of $\frac{2}{\pi} \int_{-\pi}^{\pi} f(x) dx$ is	s. 5
	t. 6

(IIT-JEE 2010)

2. Match List I with List II and select the correct answer using the codes given below the lists:

List I	List II
a. $\left(\frac{1}{y^2} \left(\frac{\cos(\tan^{-1}y) + y \sin(\tan^{-1}y)}{\cot(\sin^{-1}y) + \tan(\sin^{-1}y)} \right)^2 + y^4 \right)^{1/2}$ takes value	p. $\frac{1}{2}\sqrt{\frac{5}{3}}$
b. If $\cos x + \cos y + \cos z = 0 = \sin x + \sin y + \sin z$ then possible value of $\cos \frac{x-y}{2}$ is	q. $\sqrt{2}$
c. If $\cos\left(\frac{\pi}{4} - x\right) \cos 2x + \sin x \sin 2x \sec x = \cos x \sin 2x \sec x + \cos\left(\frac{\pi}{4} + x\right) \cos 2x$ then possible value of $\sec x$ is	r. 1/2
d. If $\cot\left(\sin^{-1}\sqrt{1-x^2}\right) = \sin(\tan^{-1}(x\sqrt{6}))$, $x \neq 0$, then possible value of x is	s. 1

Codes:

- a b c d
- (1) s r p q
- (2) s r q p
- (3) r s q p
- (4) r s p q

(JEE Advanced 2013)

3. Match List I with List II and select the correct answer using the codes given below the lists:

List I	List II
a. Let $y(x) = \cos(3 \cos^{-1} x)$, $x \in [-1, 1]$, $x \neq \pm \frac{\sqrt{3}}{2}$. Then $\frac{1}{y(x)} \left\{ (x^2 - 1) \frac{d^2 y(x)}{dx^2} + x \frac{dy(x)}{dx} \right\}$ equals	p. 1
b. Let $A_1, A_2, \dots, A_n$ ($n > 2$) be the vertices of a regular polygon of n sides with its centre at the origin. Let $\vec{a}_k$ be the position vector of the point A_k , $k = 1, 2, \dots, n$. If $\left \sum_{k=1}^{n-1} (\vec{a}_k \times \vec{a}_{k+1}) \right = \left \sum_{k=1}^{n-1} (\vec{a}_k \cdot \vec{a}_{k+1}) \right $, then the minimum value of n is	q. 2
c. If the normal from the point $P(h, 1)$ on the ellipse $\frac{x^2}{6} + \frac{y^2}{3} = 1$ is perpendicular to the line $x + y = 8$, then the value of h is	r. 8
d. Number of positive solutions satisfying the equation $\tan^{-1}\left(\frac{1}{2x+1}\right) + \tan^{-1}\left(\frac{1}{4x+1}\right) = \tan^{-1}\left(\frac{2}{x^2}\right)$ is	s. 9

Codes:

- a b c d
- (1) s r q p
- (2) q s r p
- (3) s r p q
- (4) q s p r

(JEE Advanced 2014)

Numerical Value Type

1. Let $f: [0, 4\pi] \rightarrow [0, \pi]$ be defined by $f(x) = \cos^{-1}(\cos x)$. The number of points $x \in [0, 4\pi]$ satisfying the equation $f(x) = \frac{10-x}{10}$ is _____ (JEE Advanced 2014)

2. The number of real solutions of the equation

$$\sin^{-1} \left(\sum_{i=1}^{\infty} x^{i+1} - x \sum_{i=1}^{\infty} \left(\frac{x}{2} \right)^i \right) = \frac{\pi}{2} - \cos^{-1} \left(\sum_{i=1}^{\infty} \left(-\frac{x}{2} \right)^i - \sum_{i=1}^{\infty} (-x)^i \right)$$

lying in the interval $\left(-\frac{1}{2}, \frac{1}{2} \right)$ is _____.

(Here, the inverse trigonometric functions $\sin^{-1} x$ and $\cos^{-1} x$ assume values in $\left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$ and $[0, \pi]$, respectively.) (JEE Advanced 2018)

Answers Key

EXERCISES

Single Correct Answer Type

- | | | | | |
|---------|---------|---------|---------|---------|
| 1. (3) | 2. (1) | 3. (3) | 4. (1) | 5. (3) |
| 6. (4) | 7. (3) | 8. (3) | 9. (3) | 10. (3) |
| 11. (2) | 12. (3) | 13. (4) | 14. (1) | 15. (4) |
| 16. (2) | 17. (2) | 18. (2) | 19. (1) | 20. (3) |
| 21. (4) | 22. (2) | 23. (1) | 24. (1) | 25. (4) |
| 26. (3) | 27. (1) | 28. (4) | 29. (4) | 30. (1) |
| 31. (2) | 32. (3) | 33. (4) | 34. (3) | 35. (3) |
| 36. (2) | 37. (3) | 38. (1) | 39. (4) | 40. (3) |
| 41. (3) | 42. (3) | 43. (1) | 44. (2) | 45. (2) |
| 46. (4) | 47. (2) | 48. (3) | 49. (4) | 50. (3) |
| 51. (1) | 52. (3) | 53. (1) | 54. (3) | 55. (4) |
| 56. (1) | 57. (2) | 58. (4) | 59. (4) | 60. (3) |
| 61. (2) | 62. (2) | 63. (2) | 64. (2) | 65. (1) |
| 66. (3) | 67. (1) | 68. (2) | 69. (2) | 70. (3) |
| 71. (1) | 72. (4) | 73. (1) | 74. (3) | 75. (1) |
| 76. (2) | 77. (4) | 78. (2) | 79. (2) | 80. (4) |

Multiple Correct Answers Type

- | | |
|------------------------|-------------------|
| 1. (2), (3), (4) | 2. (1), (2), (3) |
| 3. (2), (3), (4) | 4. (1) (2) (3) |
| 5. (1), (2), (3) | 6. (1), (2), (3) |
| 7. (1), (2), (3), (4) | 8. (1), (3), (4) |
| 9. (1), (2), (3), (4) | 10. (1), (3) |
| 11. (3), (4) | 12. (2) |
| 13. (1), (2), (3), (4) | 14. (1) (4) |
| 15. (1), (2) | 16. (1), (3) |
| 17. (1), (2) | 18. (1), (2) |
| 19. (1), (2) | 20. (1), (2), (3) |
| 21. (1), (2) | 22. (1), (2), (4) |
| 23. (1), (3) | 24. (2), (3) |

Linked Comprehension Type

- | | | | | |
|---------|---------|---------|---------|---------|
| 1. (4) | 2. (2) | 3. (3) | 4. (2) | 5. (1) |
| 6. (2) | 7. (2) | 8. (4) | 9. (3) | 10. (3) |
| 11. (1) | 12. (4) | 13. (3) | 14. (2) | 15. (2) |
| 16. (1) | | | | |

Matrix Match Type

1. $a \rightarrow q; b \rightarrow s; c \rightarrow p; d \rightarrow r$
 2. (4)
 3. (2)
 4. (3)
 5. $a \rightarrow s; b \rightarrow p; c \rightarrow q; d \rightarrow r$

Numerical Value Type

- | | | | | |
|----------|----------|---------|---------|---------|
| 1. (5) | 2. (2) | 3. (6) | 4. (3) | 5. (7) |
| 6. (9) | 7. (9) | 8. (4) | 9. (9) | 10. (8) |
| 11. (-3) | 12. (25) | 13. (3) | 14. (1) | 15. (0) |
| 16. (1) | 17. (8) | 18. (3) | | |

ARCHIVES

JEE MAIN

Single Correct Answer Type

1. (1) 2. (1)

JEE ADVANCED

Single Correct Answer Type

1. (2)

Multiple Correct Answers Type

1. (2), (3), (4) 2. (1), (2), (4)

Matrix Match Type

1. $(b) \rightarrow (p), (r)$ 2. (2) 3. (1)

Numerical Value Type

1. (3) 2. (2)